Spaces of strongly $A$-summable sequences

Tunay Bilgin

1. Introduction

The class of sequences which are strongly summable with respect to a modulus was introduced by Maddox [6] and extended by Connor [3]. In [2,4,5,8] a further extension of these definitions was given by using a sequence of positive real numbers $p = (p_k)$ or a sequence of moduli $F = (f_k)$.

We first recall the notion of modulus.

**Definition 1.** A function $f : [0, \infty) \to [0, \infty)$ is called a modulus if
1) $f(t) = 0$ if and only if $t = 0$ ;
2) $f(t + s) \leq f(t) + f(s)$ for all $t \geq 0, s \geq 0$ ;
3) $f$ is increasing;
4) $f$ is continuous from the right at $0$.

The notion of strong $A$-summability with respect to a modulus was given in [1,2].

Let $A = (a_{nk})$ be an infinite matrix of nonnegative real numbers, $p = (p_k)$ be a sequence of positive real numbers and $f$ be a modulus. A sequence $x = (x_k)$ is called strongly $A$-summable to $L$ with respect to the modulus $f$ if (see [2])

$$\lim_{n \to \infty} \sum_{k} a_{nk} f(|x_k - L|)^{p_k} = 0.$$ 

Here and henceforth we write $f(t)^{p_k}$ instead of $[f(t)]^{p_k}$.

Let $A$ denote the sequences of infinite matrices $A' = (a_{nk}(i))$ of nonnegative real numbers. A sequence $x = (x_k)$ is called strongly $A$-summable to $L$ with respect to the modulus $f$ if

$$\lim_{n \to \infty} \sum_{k} a_{nk}(i) f(|x_k - L|)^{p_k} = 0 \text{ uniformly in } i$$

(notation $[A, f, p]-\lim x = L$). The sets of strongly $A$-summable sequences with respect to the modulus, and strongly $A$-summable to zero sequences

75
with respect to the modulus are denoted, respectively, by \([A, f, p]\) and \([A, f, p]_0 \).

A sequence \(x = (x_k)\) is called strongly \(A\)-bounded with respect to the modulus \(f\) if
\[
\sup_{n, i} \sum_k a_{nk}(i)f(|x_k|)^{p_k} < \infty.
\]

The set of strongly \(A\)-bounded sequences with respect to the modulus is denoted by \([A, f, p]_{\infty}\). If \(A = (A)\), \(A = (a_{nk})\), then in the notations we write \(A\) instead of \(A\). If \(f(t) = t\), then we omit \(f\) in the notations.

The various special cases of the spaces \([A, f, p], [A, f, p]_0\) and \([A, f, p]_{\infty}\) are considered earlier by Bilgin [1,2] and Connor [3] (in the case \(A = (A)\)), Soomer [9] (in the case \(f(t) = t\)) and Kolk [5] (in the case \(A = (A)\) and \(p_k = p\) \((k \in \mathbb{N})\), where one modulus \(f\) is replaced with a sequence of moduli \((f_k)\)).

In the present paper we examine some properties of the sequence spaces \([A, f, p]_0\), \([A, f, p]\) and \([A, f, p]_{\infty}\).

2. Fundamental and inclusion theorems

The following theorem gives inclusion relations among the spaces \([A, f, p], [A, f, p]_0\), and \([A, f, p]_{\infty}\). This is a routine verification and therefore we omit the proof. We have

**Theorem 1.** \([A, f, p]_0 \subset [A, f, p], [A, f, p]_0 \subset [A, f, p]_{\infty}\) and \([A, f, p]_{\infty}\) if
\[
\|A\| = \sup_{n, i} \sum_k a_{nk}(i) < \infty.
\] (1)

**Theorem 2.** Let \(0 < p_k \leq \sup \{p_k\} = H < \infty\). Then \([A, f, p]_0\) is complete linear topological spaces paranormed by \(h\) defined by
\[
h(x) = \sup_{n, i} \left( \sum_k a_{nk}(i)f(|x_k|)^{p_k}\right)^{1/M}
\]
where \(M = \max\{1, H\}\). If (1) holds and \(\inf p_k > 0\), then \([A, f, p]\) is paranormed with the same paranorm \(h\). The space \([A, f, p]\) is complete if
\[
\lim_n \sum_k a_{nk}(i) = 0 \text{ uniformly in } i.
\] (2)

**Proof.** By using standard techniques we can prove that \([A, f, p]_0\) and \([A, f, p]\) (if (1) holds and \(\inf p_k > 0\)) have the paranorm \(h\) and that \([A, f, p]_0\) is complete.
If $H = \sup p_k$ and $K = \max\{1, 2^H - 1\}$, we have (see, Maddox [7])

$$| a_k + b_k |^p \leq K (| a_k |^p + | b_k |^p)$$

(3)

and for all $\lambda \in \mathbb{C}$,

$$| \lambda |^p \leq \max\{1, | \lambda |^H\}.$$  

(4)

Now by the inequalities (3) and (4)

$$T_{p, r, s} = \sum_k a_{nk}(i) f(| x_k |)^p = \sum_k a_{nk}(i) f(| x_k - L + L |)^p$$

$$\leq K \sum_k a_{nk}(i) f(| x_k - L |)^p$$

$$+ K \max\{1, f(| L |)^H\} \sum_k a_{nk}(i).$$

From this inequality, (2) and Theorem 1, it is easy to see that $[A, f, p] = [A, f, p]_r$ and therefore the completeness of $[A, f, p]$ follows from the completeness of $[A, f, p]_r$.

We now characterize the class of strongly regular methods $A$. The summability method $A$ is said to be strongly regular if $x_k \to L$ implies that $[A, f, p]$-lim $x_k = L$.

Let $X$ and $Y$ be two nonempty subsets of the space $w$ of all sequences. If $x \in X$ implies that $(\sum_k a_{nk} x_k) \in Y$, we say that $A$ defines a matrix transformation from $X$ into $Y$ and we write $A : X \to Y$. The symbol $(X, Y)$ denotes the class of matrices $A$ such that $A : X \to Y$. It is known that $A \in (c_0, c_0)$ if and only if $\| A \| = \sup_n \sum_k | a_{nk} | < \infty$ and $\lim_n a_{nk} = 0$ for all $k$, where $c_0$ denotes the Banach spaces of null sequences $x = (x_k)$.

By $A \in (c_0, c_0)$, we mean that for every $x \in c_0$,

$$\lim_{n \to \infty} \sum_k a_{nk}(i) x_k = 0 \text{ uniformly in } i.$$  

(5)

Theorem 3. Let $0 < r = \inf p_k \leq p_k \leq \sup p_k = H < \infty$ and

$$\lim_{t \to 0} \frac{f(t)}{t} = \beta > 0.$$

Then $A$ is strongly regular if and only if $A \in (c_0, c_0)$.

For $A = (A)$ this result is proved by Bilgin in [1].

Theorem 4. Suppose that $A \in (c_0, c_0)$ and $p = (p_k)$ converges to a positive limit. Then $x_k \to L$, $[A, f, p]$-lim $x = L$, $[A, f, p]$-lim $x = L'$ imply $L = L'$ if and only if

$$\lim_n \sum_k a_{nk}(i) \neq 0 \text{ uniformly in } i.$$  

(6)
Proof. Let \( A \in (c_0, c_0) \) and \((p_k)\) be bounded. Suppose that \( x_k \to L \) imply \([A, f, p]_\text{-lim} x = L\) uniquely. By Definition 1 we get \([A, f, p]_\text{-lim} \epsilon = 1\), where \( \epsilon = (1, 1, 1, \ldots) \). Hence, we must have (6), for otherwise \([A, f, p]_\text{-lim} \epsilon = 0\) which contradicts the uniqueness of \( L \).

The rest of the claim can be proved by using the techniques similar to those used in Theorem 2 of Bilgin [1]. Using the same technique as in Theorem 1 in [1], it is easy to prove the following theorem.

**Theorem 5.** Suppose that \( 0 < p_k \leq q_k \) (for all \( k \)). \((q_k/p_k)\) is bounded and (1) holds. Then \([A, f, q] \subset [A, f, p]\).

**Theorem 6.** If (1) holds and \( 0 < r = \inf p_k \leq p_k \leq \sup p_k = H < \infty \), then \([A, p]_0 \subset [A, f, p]_0\) and \([A, p] \subset [A, f, p]\).

Proof. We consider \([A, p]_0 \subset [A, f, p]_0\) only. Let \( \epsilon > 0 \) and choose \( \delta \) with \( 0 < \delta < 1 \) such that \( f(t) < \epsilon \) for \( 0 \leq t \leq \delta \). For a sequence \((x_k) \in [A, p]_0\), let

\[
T_n^i = \sum_k a_{nk}(i) x_k \| p^i,
\]

so that \( \lim_n T_n^i = 0 \) uniformly in \( i \). We split the sum \( T_n^i(x) \) into two sums \( \sum_1 \) and \( \sum_2 \) over \( \{ k : |x_k| \leq \delta \} \) and \( \{ k : |x_k| > \delta \} \), respectively. Then

\[
\sum_1 < \max\{\epsilon, \epsilon^*\} \| A \|
\]

Further, for \( |x_k| > \delta \) we have by Definition 1 that \( f(|x_k|) \leq \frac{2f(1)}{\delta} |x_k| \).

Thus \( \sum_2 \leq \max\{1, \left(\frac{2f(1)}{\delta}\right)^H\} T_n^i \), which together with (7) yields \((x_k) \in [A, f, p]_0\).

**Corollary 7.** If \( \| A \| = \sum k a_{nk} < \infty \) and \( 0 < r = \inf p_k \leq p_k \leq \sup p_k = H < \infty \), then \([A, p]_0 \subset [A, f, p]_0\) and \([A, p] \subset [A, f, p]\).

Oztürk and Bilgin ([8], Theorem 5) proved Corollary 7 in the case \( A = (C, 1) \). Note that in this case if \( p_k = 1 \) for all \( k \). Maddox ([6], Theorem 1) proved Corollary 7.

**Theorem 8.** If (1) and (5) hold and \( 0 < r = \inf p_k \leq p_k \leq \sup p_k = H < \infty \), then \([A, p] = [A, f, p]\).

Proof. In Theorem 6, it was shown that \([A, f, p] \supset [A, p]\). We must show that \([A, f, p] \subset [A, p]\). This inclusion can be proved by using the techniques similar to those used in Theorem 4 of Bilgin [2].
Let $B$ denote the sequence of infinite matrices $B^i = (b_{nk}(i))$ of nonnegative real numbers. We write $[A, f, p] \subset [B, f, q]$ (reg) if $[A, f, p]$ \subset $[B, f, q]$ and $[A, f, p]$-lim $x = [B, f, q]$-lim $x$ for every $x \in [A, f, p]$.

We now establish an inclusion relation between the spaces $[A, f, p]$ and $[B, f, q]$.

**Theorem 9.** Suppose that $0 < \rho_k < p_k$, $r = \sup \frac{q_k}{\rho_k} < 1$, $\lambda = \inf \frac{\rho_k}{\rho_k} > 0$ and $b_{nk}(i) \neq 0$ implies $a_{nk}(i) \neq 0$. If the conditions

$$\sup_{n,j} \sum_k \left[ b_{nk}(i) \right]^{1/r} \left[ a_{nk}(i) \right]^{r/r-1} < \infty$$

and

$$\sup_{n,j} \sum_k \left[ b_{nk}(i) \right]^{1/\lambda} \left[ a_{nk}(i) \right]^{\lambda/\lambda-1} < \infty$$

are fulfilled, then $[A, f, p] \subset [B, f, q]$ (reg).

**Proof.** Let $x = (x_k) \in [A, f, p]$ and $[A, f, p]$-lim $x = L$. We write $t_k = f(|x_k - L|^p_k)$ and $\lambda_k = \frac{q_k}{p_k}$, so that $0 < \lambda < \lambda_k < 1$, and

$$\lim_{k} \sum \frac{\rho_k a_{nk}(i) t_k}{\rho_k} = 0 \text{ uniformly in } i.$$  \hfill (10)

Define

$$U_k \equiv \begin{cases} t_k, & t_k \geq 1 \\ 0, & t_k < 1 \end{cases} \text{ and } V_k \equiv \begin{cases} t_k, & t_k \geq 1 \\ 0, & t_k < 1 \end{cases}.$$

So $t_k = U_k + V_k$, $\lambda_k = U_k^\lambda_k + V_k^\lambda_k$, $U_k \leq t_k$, $V_k \leq t_k$, $U_k^\lambda_k \leq U_k^\lambda_k$ and $V_k^\lambda_k \leq V_k^\lambda_k$. By Hölder's inequality we obtain

$$\sum_k b_{nk}(i) f(|x_k - L|^p_k) = \sum_k b_{nk}(i) U_k^\lambda_k$$

$$= \sum_k b_{nk}(i) U_k^\lambda_k + \sum_k b_{nk}(i) V_k^\lambda_k$$

$$\leq \sum_k b_{nk}(i) U_k^\lambda_k + \sum_k b_{nk}(i) V_k^\lambda_k$$

$$= \sum_k \left[ a_{nk}(i) U_k \right]^{\lambda_k} \frac{\rho_k a_{nk}(i)}{\rho_k a_{nk}(i)}$$

$$+ \sum_k \left[ a_{nk}(i) V_k \right]^{\lambda_k} \frac{\rho_k a_{nk}(i)}{\rho_k a_{nk}(i)}$$

$$\leq \left( \sum_k a_{nk}(i) f(t_k) \right) \left( \sum_k b_{nk}(i) \frac{\lambda_k}{\lambda_k} a_{nk}(i) \right)$$

$$+ \left( \sum_k a_{nk}(i) \right) \left( \sum_k b_{nk}(i) \frac{\lambda_k}{\lambda_k} a_{nk}(i) \right)$$

$$= \left( \sum_k a_{nk}(i) f(t_k) \right) \left( \sum_k b_{nk}(i) \frac{\lambda_k}{\lambda_k} a_{nk}(i) \right)$$

$$+ \left( \sum_k a_{nk}(i) \right) \left( \sum_k b_{nk}(i) \frac{\lambda_k}{\lambda_k} a_{nk}(i) \right).$$
The result follows from (8), (9) and (10).

It is essential to note that for $A=B$ Theorem 9 follows from Theorem 5.

Sooner ([9], Theorem 1) proved Theorem 9 in the case $f(t) = t$.

Acknowledgement

The author wishes to thank Professor E. Kolk for valuable suggestions and the referee for some useful comments which improved the presentation of the paper.

References


Received May 8, 1995, revised December 14, 1995

Department of Mathematics
Yüzyüncü Yıl University
Van Turkey