Multipliers and $L^1$-convergence of cosine series

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1. We will consider the integrability and $L^1$-convergence of the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$$

where $(a_k)$ is a sequence of real numbers. The problem of integrability of (1) is to decide $(a_k) \in \{(f_k) : f \in L^1\} = \hat{L}^1$, where

$$\hat{f}_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx.$$

The other question investigated in this paper is the $L^1$-convergence of (1). It is known that $(a_k) \in \hat{L}^1$ alone does not guarantee the $L^1$-convergence of (1). Connecting the two problems we introduce the following concept. A class of sequences is said to be an integrability and $L^1$-convergence class if it is an integrability class and for each $(a_n)$ of it the corresponding series converges in $L^1$-norm if and only if

$$\lim_{n \to \infty} a_n \ln n = 0.$$

Kolmogorov [6] showed that the set of quasiconvex null sequences,

$$\sum_{k=0}^{\infty} (k + 1) \mid \Delta^2 a_k \mid < \infty, \quad a_k \to 0 \ (k \to \infty),$$

form an $L^1$-convergence class.

In this paper we will show that the $L^1$-convergence class can be replaced by a class of multipliers.

2. Let $\omega$ denote the set of all sequences of real numbers. Let $T = (\tau_{nk})$ be a regular triangular matrix of real numbers:

$$\lim_{n \to \infty} \tau_{nk} = 1; \sup_n \sum_{k=0}^{n} | \tau_{nk} - \tau_{n,k+1} | < \infty.$$

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Throughout, $T$ will be a reversible matrix $(\tau_{nn} \neq 0)$. We denote the summability fields of $T$ by
\[
c_T = \{ (x_k) \in \omega : \lim_{n \to \infty} \sum_{k=0}^{n} \tau_{nk}x_k \text{ exist}, \}
\]
the boundedness domain of $T$ by
\[
m_T = \{ (x_k) \in \omega : \sup_n \left| \sum_{k=0}^{n} \tau_{nk}x_k \right| < \infty \}
\]
and the class of multipliers by
\[
(m_T, c_T) = \{ (a_k) \in \omega : (a_kx_k) \in c_T \text{ for every } (x_k) \in m_T \}.
\]
If $(a_k) \in (m_T, c_T)$, then $\lim_{k \to \infty} a_k = 0$ and
\[
\sum_{n=0}^{\infty} \left| \sum_{k=n}^{\infty} \tau_{kn}^{-1}a_k \right| < \infty,
\]
where $(\tau_{kn}^{-1})$ denotes the matrix $T^{-1}$. (See [2], Theorem 3 and [10], 6.4., p. 92).

3. **Theorem.** Let $T$ be reversible regular triangular matrix with
\[
\sup_n \int_{0}^{\pi} \left| \frac{\tau_{n0}}{2} + \sum_{k=1}^{n} \tau_{nk} \cos kx \right| dx = K < \infty. \tag{4}
\]
Then the cosine series (1) converges, except possibly at $x = 0$, to an integrable function $f(x)$, is the Fourier series of $f(x)$, and the partial sums converge in $L^1$-norm to $f$ if and only if $\lim_{n \to \infty} a_n \ln n = 0$.

**Proof.** Clearly,
\[
S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx = \sum_{k=0}^{n} \left( \sum_{j=k}^{n} \tau_{jk}^{-1}a_j \right) K_k(x) =
\]
\[
= \sum_{k=0}^{n-1} \left( \sum_{j=k}^{n-1} \tau_{jk}^{-1}a_j \right) K_k(x) + a_n D_n(x), \tag{5}
\]
where
\[
K_k(x) = \frac{\tau_{k0}}{2} + \sum_{j=1}^{k} \tau_{kj} \cos jx = \sum_{j=0}^{k} t_{kj} D_j(x),
\]
\[
t_{kj} = \tau_{kj} - \tau_{k,j+1},
\]
\[
D_j(x) = \frac{1}{2} + \sum_{\nu=1}^{j} \cos \nu x = \frac{\sin(j + \frac{1}{2})x}{2\sin \frac{x}{2}}.
\]

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By (2) we have for some \( C \) and \( M \)

\[
|K_k(x)| \leq \sum_{j=0}^{k} |t_{kj}| |D_j(x)| \leq \frac{C}{|x|} \sum_{j=0}^{k} |t_{kj}| \leq \frac{CM}{|x|} \quad (x \neq 0) \text{ for } k = 0, 1, 2, \ldots.
\]

Since \( (a_k) \in (mT, cT) \), then by (3) we have the pointwise convergence of

\[
\sum_{k=0}^{\infty} \left( \sum_{j=k}^{\infty} \tau_{jk}^{-1} a_j \right) K_k(x) = f(x) \quad (x \neq 0)
\]

and \( \lim_{n \to \infty} S_n(x) = f(x) \) in \((0, \pi]\). Clearly \( f(x) \) is an integrable function and by [4] and [8] series (1) is the Fourier series of \( f(x) \). By (5) we have

\[
\int_{0}^{\pi} |S_n(x) - f(x)| \, dx = \int_{0}^{\pi} |a_n D_n(x) - \sum_{k=n}^{\infty} \left( \sum_{j=k}^{\infty} \tau_{jk}^{-1} a_j \right) K_k(x) | \, dx \leq |a_n| \int_{0}^{\pi} |D_n(x)| \, dx + \sum_{k=n}^{\infty} \left| \sum_{j=k}^{\infty} \tau_{jk}^{-1} a_j \right| \int_{0}^{\pi} |K_k(x)| \, dx \leq |a_n| \int_{0}^{\pi} |D_n(x)| \, dx + K \sum_{k=n}^{\infty} \left| \sum_{j=k}^{\infty} \tau_{jk}^{-1} a_j \right|
\]

and by

\[
\int_{0}^{\pi} |a_n D_n(x)| \, dx - \int_{0}^{\pi} \left| \sum_{k=n}^{\infty} \left( \sum_{j=k}^{\infty} \tau_{jk}^{-1} a_j \right) K_k(x) \right| \, dx \leq \int_{0}^{\pi} |S_n(x) - f(x)| \, dx,
\]

then

\[
|a_n| \int_{0}^{\pi} |D_n(x)| \, dx \leq \int_{0}^{\pi} |S_n(x) - f(x)| \, dx + K \sum_{k=n}^{\infty} \left| \sum_{j=k}^{\infty} \tau_{jk}^{-1} a_j \right|.
\]

If \( (a_k) \in (mT, cT) \), then by (3) and (4)

\[
\lim_{n \to \infty} \int_{0}^{\pi} |S_n(x) - f(x)| \, dx = 0
\]
if and only if
\[ \lim_{n \to \infty} |a_n| \int_0^\pi |D_n(x)| \, dx = 0, \]
it is (see [3], 5.1.1) if and only if \( \lim a_n \ln n = 0 \).

4. For the Cesàro matrix \( T = C^n (\alpha > 0) \),
\[ \tau_{nk} = \frac{A^\alpha_{n-k}}{A^\alpha_n}, \quad A^\alpha_n = \frac{(n+\alpha)(n+\alpha-1)\ldots(\alpha+1)}{n!}, \]
is by [1] and [7]
\[ (m_{C^\alpha}, e_{C^\alpha}) = \{ (a_k) \in \omega : \sum_{k=0}^\infty (k+1)^\alpha |\Delta^{\alpha+1}a_k| < \infty; \lim_{k \to \infty} a_k = 0 \}. \]

It is well known ([5]) that the Riesz matrix \( P \),
\[ \tau_{nk} = \begin{cases} 1 - \frac{P_{k-1}}{P_n}, & 0 \leq k \leq n \\ 0, & n < k, \end{cases} \]
\( P_{-1} = 0; \quad P_n = p_0 + p_1 + \ldots + p_n, \) is the regular matrix if and only if the following conditions hold:
\[ \lim_{n \to \infty} |P_n| = \infty, \quad \sup_n \frac{1}{|P_n|} \sum_{k=0}^n |p_k| < \infty. \]

Then by [5]
\[ (m_p, e_p) = \]
\[ = \{ (a_k) \in \omega : \sum_{k=0}^\infty |P_k \Delta a_k| \leq \infty; \quad \lim_{k \to \infty} \frac{P_k}{p_k} \Delta a_k = 0; \quad \lim_{k \to \infty} a_k = 0 \}. \]

For the case when \( T = C^n \) or \( T = P \) this Theorem is proved in [9].

References


Received December 18, 1995

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