Constructive sets of real trigonometric series

JAAK SIKK

Abstract. In the present paper we consider the abstract classes of trigono-
metric series with complementary characteristics to $L^p$. The concept used
will generalize the method of complementary spaces introduced by Goes [2]
and Tönnov [6] and the concept of $T^\lambda$-constructive spaces considered by
the present author [4,5].

1. Introduction

We have introduced a concept of $T^\lambda$-constructive spaces into theory of
trigonometric series [4] and [5]. Constructive spaces created a possibility to
use the $\lambda$-summability method of Kangro [3] for the investigation of Fourier
series. In the present paper we will generalize the concept of constructive
spaces by introducing the constructive-type classes of abstract trigonometric
series $L^p_{T^\lambda}$. The results will link the classes of Fourier series and the classes
$L^p_{T^\lambda}$ of trigonometric series with Fourier coefficients of $L^p$.

Throughout the paper the integral is considered taken over any interval
of length $2\pi$. So, for any real number $p \geq 1$, we denote the set of equivalence
classes of real-valued measurable functions $f$ by $L^p$, where

$$
\| f \|_p = \left[ \frac{1}{2\pi} \int | f(x) |^p dx \right]^{1/p}
$$

is finite and the integral being extended over any interval of length $2\pi$.

Lemma 1 (Hölder inequality (see [1], Vol. I, p. 28)). Let $p, q \geq 1$ for
$p^{-1} + q^{-1} = 1$ and $f \in L^p$, $g \in L^q$, then $f \cdot g \in L^1$ and

$$
\| f \cdot g \|_1 \leq \| f \|_p \cdot \| g \|_q
$$

Received May 14, 1998; revised October 28, 1998.
1991 Mathematics Subject Classification. Primary 40A30, 42A24.
This work was supported by Estonian Science Foundation Grant 2416.
Lemma 2 (see [1], Vol. I, p. 190). Let $1 \leq q < \infty$ and let $F$ be any continuous linear functional on $L^q$. Then there exists an essentially unique function $f \in L^p$, where $p^{-1} + q^{-1} = 1$, such that

$$F(g) = \frac{1}{2\pi} \int f(x)g(x)dx$$

for all $g \in L^q$. For any such $f$ one has

$$\|f\|_p = \|F\| \equiv \sup \left\{ |F(g)| : g \in L^q, \|g\|_q \leq 1 \right\}.$$

Let $a = (a_k)$ and $b = (b_k)$ be the real sequences and $T = (\tau_{nk})$ be triangular method in the series-to-sequence form. We determine a formal trigonometric series $f^0(x) = \sum_k (a_k \cos kx + b_k \sin kx) = (n_k, b_k)$ and use notation

$$\tau_n(f(x)) := \sum_{k=1}^n \tau_{nk}(a_k \cos kx + b_k \sin kx).$$

The set of all Fourier series of functions of $L^p$ will be denoted by $\hat{L}^p$. The notion $f^0(x) = (a_k, b_k)$ will be also used for the Fourier series of function $f \in L^p$.

We call a positive sequence a rate. Thus, $\lambda = (\lambda_n)$ is a rate, if $\lambda_n > 0$ for all $n \in \mathbb{N}$. Rates are denoted by $\lambda$ and $\mu$. The rate is called monotonic if $\lambda_{n+1} \geq \lambda_n$ for all $n$.

2. The constructive spaces $L_{T,\lambda}^p$ and $\mathcal{L}_{T,\lambda}^p$

**Definition 1.** Let $\lambda = (\lambda_n)$ be a rate and let $p \geq 1$. The set of all $f \in L^p$ for which

$$\lambda_n \| \tau_n f - f \|_p = O(1)$$

is called the $T^\lambda$-constructive space $L_{T,\lambda}^p$.

**Definition 2.** Let $\lambda$ be a rate and let $p \geq 1$. The set of all formal trigonometric series $(a_k, b_k)$ for which

$$\lambda_n \| \tau_n f \|_p = O(1),$$

is called the $T^\lambda$-constructive space $\mathcal{L}_{T,\lambda}^p$.

The space $L_{T,\lambda}^p$ were introduced by the author in [4]. These spaces together with the $\lambda$-summability method of Kangro [3] were the tools applied for the investigation of the multipliers of classes $(\operatorname{Lip}(\alpha, p), \operatorname{Lip}(\beta, p)), (L^p, \operatorname{Lip}(\alpha, p))$ etc. for $\alpha, \beta \in (0, 1)$ (see [4]).
**Theorem 1.** Let $p^{-1} + q^{-1} = 1$ for $p, q > 1$ and $g \in L^q$ with the Fourier series $(c_k, d_k) = \sum_{k \in \mathbb{N}} (c_k \cos kx + d_k \sin kx)$. Then $(a_k, b_k) \in L^p_{T, \lambda}$ if and only if for every $g \in L^q$ the condition

$$\lambda_n \left| \sum_{k=1}^{n} \tau_{nk}(a_k c_k + b_k d_k) \right| = O(1)$$

is satisfied.

**Proof.** Let $(a_k, b_k) \in L^p_{T, \lambda}$, then by the Parseval formula we have

$$\sum_{k=1}^{n} \tau_{nk}(a_k c_k + b_k d_k) = \frac{1}{2\pi} \int \tau_n f(x) g(x) dx$$

for every $n$. By Lemma 1 (Hölder inequality) now

$$\left| \frac{1}{2\pi} \int \tau_n f(x) g(x) dx \right| \leq \| \tau_n f \|_p \cdot \| g \|_q,$$

and by (2) since $f^0 \in \hat{L}^p_{T, \lambda}$, we have $\| \tau_n f \|_p = O(\lambda_n^{-1})$. So for any $f^0 \in \hat{L}^p_{T, \lambda}$ the condition (3) is fulfilled.

Let now (3) be fulfilled for every $g^0 = (c_k, d_k) \in \hat{L}^q$. Then by the Parseval formula the condition

$$\lambda_n \left| \int \tau_n f(x) g(x) dx \right| = O(1)$$

is satisfied. By Lemma 2 we have

$$\| \tau_n(f) \|_p = \sup_{\|g\|_q \leq 1} \frac{1}{2\pi} \left| \int \tau_n f(x) g(x) dx \right|,$$

and by (5) the $f^0 \in \hat{L}^p_{T, \lambda}$, which completes the proof.

**Theorem 2.** Let $\lambda = (\lambda_n)$ be a monotonic rate and $p^{-1} + q^{-1} = 1$ for $p > 1$. Let $T = (\tau_{nk})$ be series-to-sequence matrix which for all $f \in L^p$ satisfies

$$\lim_n \| \tau_n f - f \|_p = 0.$$  

Then $f^0 = (a_k, b_k) \in \hat{L}^p_{T, \lambda}$ if and only if for every $g^0 = (c_k, d_k) \in \hat{L}^q$ the

$$\lim_n \sum_{k=1}^{n} \tau_{nk}(a_k c_k + b_k d_k) = s$$

(7)
exists, and
\[ \lambda_n \left| \sum_{k=1}^{n} \tau_{nk}(a_k c_k + b_k d_k) - s \right| = O(1). \] (8)

Proof. We will show that for \( f \in L^p_{T_\lambda} \) and for arbitrary \( g \in L^q \) the conditions (7) and (8) are satisfied. If \( f \in L^p_{T_\lambda} \) then also \( f \in L^p \) and by Lemma 1 we have
\[ \frac{1}{2\pi} \int \tau_n f(x)g(x)dx - \frac{1}{2\pi} \int f(x)g(x)dx \leq \|g\|_q \|\tau_n f - f\|_p. \] (9)

Since \( f \in L^p_{T_\lambda} \) and (1) is satisfied,
\[ \lim_n \sum_{k=1}^{n} \tau_{nk}(a_k c_k + b_k d_k) = \lim_n \frac{1}{2\pi} \int \tau_n f(x)g(x)dx = s \]
exists, and from (9) it follows that (7) and (8) are satisfied.

Let the conditions (7) and (8) be fulfilled for every \( g \in L^q \). For every fixed \( n \) we form a polynomial
\[ \tau_n(x) := \sum_{k=1}^{n} \tau_{nk}(a_k \cos kx + b_k \sin kx) \] (10)
and calculate
\[ \frac{1}{2\pi} \int \tau_n(x)g(x)dx = \sum_{k=1}^{n} \tau_{nk}(a_k c_k + b_k d_k). \] (11)

By Lemma 2 we have
\[ \| \tau_n(x) - \tau_{n+i}(x) \|_p \leq \sup_{\|g\|_q \leq 1} \frac{1}{2\pi} \int |(\tau_n(x) - \tau_{n+i}(x))g(x)dx|. \]

By (7) and (11) from the last inequality it follows that \( \tau_n \) is a Cauchy sequence in \( L^p \). Since \( L^p \) is complete so by \( \lim \tau_n(x) = f(x) \) in \( L^p \) exists. Therefore, from (11) it follows that
\[ \| \tau_n f - f \|_p \leq \sup_{\|g\|_q \leq 1} \frac{1}{2\pi} \int |(\tau_n f(x) - f(x))g(x)dx|. \]

Now from (7), (8) and from last equality we conclude that \( f \in L^p_{T_\lambda} \). This completes the proof.

The condition (6) was used only in the second part of the proof. Therefore we have the following
Corollary 1. Let $f \in L^p_{T^\lambda}$, then the conditions (7) and (8) are satisfied for every $g \in L^q$.

Let $E$ be identity method, $\lambda = (n^{\frac{1}{8}})$ and

$$\frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{1}{2} D_n(x),$$

then $\|D_n(x)\|_p = O(\lambda_n)$ and $D_n(x) \in L^p_{E^{\lambda}}$ (see [1], Vol. I, p. 115). Let $g^0(x) = (c_k, 0) \in L^q$. By Theorem 1 we have the following

Corollary 2. For series $\sum_{k=1}^{n} c_k \cos kx \in L^q$ the condition

$$\left| \sum_{k=1}^{n} c_k \right| = O(n^{\frac{1}{4}})$$

is satisfied.

Last corollary gives a simple asymptotic condition for Fourier coefficients, complementing so the Paley’s $L^p$ Fourier coefficients theorem ([7], Chap.12).

References

5. J. Sikk, *The $T^\lambda$-constructive spaces and multipliers of class $(X_{T^\lambda}, X_{T^\mu})$*, Tartu Ül. Toimetised, No. 374 (1975), 163–179. (in Russian)

Institute of Mathematics, Estonian Agricultural University, Tartu, Estonia

E-mail address: jaak@eau.ee