Counterexamples concerning topologization of spaces of strongly almost convergent sequences

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ABSTRACT. Let $\lambda$ be a sequence space, $f$ a modulus function, and $\mathcal{F} = (f_k)$ a sequence of moduli. We characterize the $F$-normability of the sequence space $\lambda(\mathcal{F})$ for $\lambda \subseteq \ell_\infty$. In the special case if $\lambda$ is the space $sac_0$ of strongly almost convergent to zero sequences, we give two counterexamples concerning the topologization of various extensions of $sac_0(\mathcal{F})$ and $sac_0(f)$ considered by Nanda and others. We also correct a similar inaccuracy in a previous paper of the author.

1. Introduction

First, let us fix some terminology. By the term sequence space, we shall mean, as usual, any linear subspace of the vector space $\omega$ of all (real or complex) sequences $x = (x_k) = (x_k)_{k \in \mathbb{N}}$, where $\mathbb{N} = \{1, 2, \cdots\}$.

A function $f : [0, \infty) \to [0, \infty)$ is called a modulus function (or simply a modulus) if

(i) $f(t) = 0$ if and only if $t = 0$,
(ii) $f(t + u) \leq f(t) + f(u)$,
(iii) $f$ is increasing,
(iv) $f$ is continuous from the right at 0.

Provided a modulus $f$ and a sequence space $\lambda$, Ruckle [13], Maddox [10], and some other authors define a new sequence space $\lambda(f)$ by

$$\lambda(f) = \{(x_k) : (f(|x_k|)) \in \lambda\}.$$

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As an extension of the space $\lambda(f)$, the author [5, 7] considers, for a sequence of moduli $\mathcal{F} = (f_k)$, the set

$$
\lambda(\mathcal{F}) = \{ x = (x_k) : \mathcal{F}(|x|) \in \lambda \},
$$

where $\mathcal{F}(|x|) = (f_k(|x_k|))$. It is not difficult to see that $\lambda(\mathcal{F})$ is a solid sequence space whenever the sequence space $\lambda$ is solid (i.e. $(x_k) \in \lambda$ and $|y_k| \leq |x_k|$ ($k \in \mathbb{N}$) yield $(y_k) \in \lambda$).

Recall that an F-seminorm $g$ on a vector space $X$ is a functional $g : X \to \mathbb{R}$ satisfying, for all $x, y \in X$, the axioms

(N1) $g(0) = 0,$
(N2) $g(x + y) \leq g(x) + g(y),$
(N3) $g(\alpha x) \leq g(x)$ for all scalars $\alpha$ with $|\alpha| \leq 1,$
(N4) $\lim_n g(\alpha_n x) = 0$ for every scalar sequence $(\alpha_n)$ with $\lim_n \alpha_n = 0.$

An F-seminorm $g$ is called an F-norm if

(N5) $g(x) = 0 \implies x = 0.$

A paranorm on $X$ is a functional $g : X \to \mathbb{R}$ satisfying (N1), (N2) and

(N6) $g(-x) = g(x),$
(N7) $\lim_n g(\alpha_n x_n - \alpha x) = 0$ for every scalar sequence $(\alpha_n)$ with $\lim_n \alpha_n = \alpha$ and every sequence $(x_n)$ with $\lim_n g(x_n - x) = 0$ ($x_n, x \in X$).

An F-seminorm (paranorm) $g$ on a solid sequence space $\lambda$ is said to be absolutely monotone if $g(x) \leq g(y)$ for all $x = (x_k), y = (y_k) \in \lambda$ with $|x_k| \leq |y_k|$ ($k \in \mathbb{N}$). An F-seminormed solid sequence space $(\lambda, g)$ is called an AK-space if $x = \lim_n \sum_{k=1}^{\infty} x_k e^k$ for all $x = (x_k) \in \lambda$ (here $e^k = (\delta_{ik})_{i \in \mathbb{N}}$, where $\delta_{ik} = 1$ if $i = k$ and $\delta_{ik} = 0$ otherwise).

If the sequence space $\lambda$ is topologized by an F-seminorm (or paranorm) $g$, then, for the topologization of $\lambda(\mathcal{F})$, it is natural to consider the functional $g_{\mathcal{F}}$ defined by

$$
g_{\mathcal{F}}(x) = g(\mathcal{F}(|x|)) \quad (x \in \lambda(\mathcal{F})).
$$

It is known (cf. [10], Theorem 8) that, in general, $g_{\mathcal{F}}$ may fail to be an F-seminorm on $\lambda(\mathcal{F})$. The author ([8], Theorem 2) proved

**Theorem 1.** Let $\mathcal{F} = (f_k)$ be a sequence of moduli and let $g$ be an absolutely monotone F-seminorm on a solid sequence space $\lambda$. If $(\lambda, g)$ is an AK-space, then the functional $g_{\mathcal{F}}$ is an absolutely monotone F-seminorm on $\lambda(\mathcal{F})$. Moreover, $(\lambda(\mathcal{F}), g_{\mathcal{F}})$ is an AK-space.

In Section 2, the F-normability of the space $\lambda(\mathcal{F})$ is characterized in the case $\lambda \subset \ell_\infty$ with some restrictions on $\lambda$ and $\mathcal{F} = (f_k)$ (Theorem 2). If $\lambda$ is the space $s_{ac0}$ of strongly almost convergent to zero sequences, we
give, in Section 3, two counterexamples (Corollaries 1 and 2) concerning the topologization of various extensions of the spaces $\text{sac}_0(f)$ and $\text{sac}_0(F)$, which have been considered by Nanda [11], Nuray and Savaş [12], Esi [3, 4], and Bilgin [2]. We also correct a similar inaccuracy in the paper [8] of the author.

2. On the topologization of $\lambda(f)$ and $\lambda(F)$

Our main theorem deals with the topologization of $\lambda(F)$ if $\lambda$ is a subspace of the Banach space $\ell_\infty$ of all bounded sequences equipped with the norm $\|x\|_\infty = \sup_k |x_k|$. For $g = \| \cdot \|_\infty$, we shall write $g_F^\infty$ instead of $g_F$, i.e.

$$g_F^\infty(x) = \sup_k f_k(|x_k|) \quad (x \in \lambda(F)).$$

**Theorem 2.** Let $\lambda$ be a solid subspace of the Banach space $\ell_\infty$ and let $F = (f_k)$ be a sequence of moduli.

(a) If $\lambda(F) \subset \ell_\infty$ and

$$\lim_{t \to 0^+} \sup_k f_k(t) = 0,$$

then $g_F^\infty$ is an F-norm on $\lambda(F)$.

(b) If $\lambda(F) \not\subset \ell_\infty$ and

$$\phi(t) = \inf_k f_k(t) > 0 \quad (t > 0),$$

then $g_F^\infty$ is not an F-norm on $\lambda(F)$.

**Proof.** (a). Suppose $\lambda(F) \subset \ell_\infty$ and $f$ satisfies (1). It is straightforward to verify that the functional $g_F^\infty$ satisfies the axioms (N1)–(N3) and (N5). To prove the axiom (N4), let $x = (x_k) \in \lambda(F)$ and $\lim_n \alpha_n = 0$. The inclusion $\lambda(F) \subset \ell_\infty$ yields the existence of a natural number $N$ such that $|x_k| \leq N$ ($k \in \mathbb{N}$). Thus, for all $n \in \mathbb{N}$,

$$g_F^\infty(\alpha_n x) = \sup_k f_k(|\alpha_n x_k|) \leq N \sup_k f_k(|\alpha_n|),$$

and from (1) it follows that $\lim_n g_F^\infty(\alpha_n x) = 0$.

(b). Let $F$ satisfy (2) and let $\lambda(F)$ contain an unbounded sequence $y = (y_k)$. We can choose an index sequence $(n_i)$ so that $\lim_i |y_{n_i}| = \infty$. Defining

$$\alpha_n = \begin{cases} |y_{n_i}|^{-1} & \text{if } n = n_i \text{ for some } i \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases},$$

we have

$$g_F^\infty(\alpha_n x) = \sup_k f_k(|\alpha_n x_k|) \geq \sup_k f_k(|y_{n_i}|) = \infty.$$
we have \( \lim_n \alpha_n = 0 \). But since
\[
\sup_k f_k(|\alpha_n y_k|) \geq \phi(1) > 0 \quad (i \in \mathbb{N}),
\]
we get that \( \lim_n g_{\mathcal{F}}^p(\alpha_n y) \neq 0 \). Thus \( g_{\mathcal{F}}^p \) fails the axiom (N4).

\textbf{Remark 1.} It is easy to see (cf. [7], Theorem 3) that \( \ell_\infty(\mathcal{F}) \subset \ell_\infty \) (and thus also \( \lambda(\mathcal{F}) \subset \ell_\infty \) for any solid subspace \( \lambda \subset \ell_\infty \)) whenever
\[
\lim_{t \to \infty} \inf_k f_k(t) = \infty.
\]

A simple argument shows (cf. [6], Lemma 1) that (3) also implies (2).

Let \( f \) be a modulus function and let \( p = (p_k) \) be a sequence with \( 0 < p_k \leq 1 \). For a solid sequence space \( \lambda \), denote
\[
\lambda^p(f) = \{ x = (x_k) : (f(|x_k|)^{p_k}) \in \lambda \}.
\]
The function \( f_k^p \) defined by
\[
f_k^p(t) = (f(t))^{p_k}
\]
is clearly a modulus for every \( k \in \mathbb{N} \). Denoting \( \mathcal{F}^p = (f_k^p) \), we may write
\[
\lambda^p(f) = \lambda(\mathcal{F}^p).
\]
Thus, provided an \( \mathcal{F} \)-seminorm \( g \) on \( \lambda \), it is natural to consider the functional \( g_{\mathcal{F}^p} \) for the topologization of \( \lambda^p(f) \). If \( \lambda \) is a subspace of \( \ell_\infty \) and \( g = \| \cdot \|_\infty \), then \( g_{\mathcal{F}^p} \) reduces to the functional \( g_{f,p}^\infty \) where
\[
g_{f,p}^\infty(x) = \sup_k (f(|x_k|))^{p_k} \quad (x \in \lambda^p(f)).
\]
In the sequel, we shall use the following characteristic for \( \lambda \):

(C) Every infinite sequence of indices \( (k_i) \) has a subsequence \( (l_i) \) such that \( \lambda \) contains the sequence \( (h_k) \) where
\[
h_k = \begin{cases} 1 & \text{if } k = l_i \text{ for some } i \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}
\]
Lemma 1. Let \( f \) be a modulus, \( \mathbf{p} = (p_k) \) a sequence with \( 0 < p_k \leq 1 \), and \( \lambda \) a solid sequence space with property (C). Then \( \lambda^\mathbf{p}(f) \not\subseteq \ell_\infty \) whenever

(a) \( f \) is bounded

or

(b) \( \inf_k p_k = 0 \).

Proof. (a). Suppose \( f \) is bounded. Then there exists a constant \( M \geq 1 \) such that \( f(t) \leq M \) \( (t \geq 0) \). By property (C), there exists an index sequence \( (l_i) \) such that \( \lambda \) contains the sequence \( (h_k) \) defined as in (4). Defining

\[
y_k = \begin{cases} 
  i & \text{if } k = l_i \text{ for some } i \in \mathbb{N} \\
  0 & \text{otherwise},
\end{cases}
\]

we obtain an unbounded sequence \( (y_k) \in \lambda^\mathbf{p}(f) \) since

\[
f(|y_k|)^{p_k} \leq M h_k \quad (k \in \mathbb{N})
\]

and \( \lambda \) is solid.

(b). Suppose \( \inf_k p_k = 0 \). Then there exists an index sequence \( (k_i) \) such that \( (p_{k_i}) \) is decreasing and

\[
f(i)^{p_{k_i}} \leq 2.
\]

By property (C), there exists a subsequence \( (l_i) \) of \( (k_i) \) such that \( \lambda \) contains the sequence \( (h_k) \) defined as in (4). The unbounded sequence \( y = (y_k) \) defined by (5) belongs to \( \lambda^\mathbf{p}(f) \) because

\[
f(|y_k|)^{p_k} \leq 2 h_k \quad (k \in \mathbb{N})
\]

and \( \lambda \) is solid. \( \square \)

Applying Theorem 2 to \( \lambda^\mathbf{p}(f) \), we get

Proposition 1. Let \( f \) be a modulus, \( \mathbf{p} = (p_k) \) a sequence with \( 0 < p_k \leq 1 \), and \( \lambda \) a solid subspace of \( \ell_\infty \) with the property (C). Then the functional \( g_{f,\mathbf{p}}^\infty \) is an F-norm on \( \lambda^\mathbf{p}(f) \) if and only if \( f \) is unbounded and \( \inf_k p_k > 0 \).

Proof. Necessity. Note that the sequence \( \mathcal{F}^\mathbf{p} \) of moduli satisfies (2) for any \( \mathbf{p} = (p_k) \) with \( 0 < p_k \leq 1 \) because

\[
f(t)^{p_k} \geq \min\{1, f(t)\} \quad (t > 0).
\]

Thus, Theorem 2(b) together with Lemma 1 yield that \( g_{f,\mathbf{p}}^\infty \) is not an F-norm on \( \lambda^\mathbf{p}(f) \) whenever \( f \) is bounded or \( \inf_k p_k = 0 \).

Sufficiency. Suppose that \( f \) is unbounded and \( \inf_k p_k > 0 \). It is straightforward to verify that the sequence \( \mathcal{F}^\mathbf{p} \) of moduli satisfies (1) and (3). Theorem 2(a) with an appeal to Remark 1 now yield that \( g_{f,\mathbf{p}}^\infty \) is an F-norm on \( \lambda^\mathbf{p}(f) \). \( \square \)
3. Consequences and counterexamples

Let \( c_0 \) be the space of all convergent to zero sequences. In the sequel, we shall consider the space of strongly almost convergent to zero sequences (cf. [9, 14])

\[
\text{sac}_0^p = \{ x = (x_k) : \lim \frac{1}{n} \sum_{k=i}^{i+n-1} |x_k|^p = 0 \text{ uniformly in } i \},
\]

where \( p > 0 \). We shall write \( \text{sac}_0 \) instead of \( \text{sac}_0^1 \).

It is essential to note that, for \( p \geq 1 \), the natural norm

\[
\|x\| = \sup_{n,i} \left( \frac{1}{n} \sum_{k=i}^{i+n-1} |x_k|^p \right)^{1/p}
\]
on \( \text{sac}_0^p \) coincides with the supremum-norm \( \| \cdot \|_\infty \) since

\[
|x_i| \leq \sup_{n} \left( \frac{1}{n} \sum_{k=i}^{i+n-1} |x_k|^p \right)^{1/p} \leq \sup_{k} |x_k| \quad (i \in \mathbb{N}).
\]

Thus, for \( p \geq 1 \), we may consider \( \text{sac}_0^p \) as a solid subspace of the Banach space \( \ell_\infty \). Since \( c_0 \) is the largest AK-subspace of \( (\ell_\infty, \| \cdot \|_\infty) \) and

\[
c_0 \subseteq \text{sac}_0^p,
\]

then we have that \( (\text{sac}_0^p, \| \cdot \|_\infty) \) is not an AK-space in case \( p \geq 1 \). Hence Theorem 1 is inapplicable in the case \( \lambda = \text{sac}_0 \).

Our approach to the study of \( \text{sac}_0(f) \) and also of the more general space

\[
\text{sac}_0^p(f) = \{ x = (x_k) : \lim \frac{1}{n} \sum_{k=i}^{i+n-1} (f(|x_k|))^{p_k} = 0 \text{ uniformly in } i \}
\]
is grounded on Proposition 1 since \( \text{sac}_0^p \) has property (C). Indeed, if \( (k(i)) \) is an index sequence, then \( \text{sac}_0^p \) contains, for example, the sequence \( (h_n) \) where \( h_n = 1 \) if \( n = k(2^i) \) for some \( i \in \mathbb{N} \), and \( h_n = 0 \) otherwise.

If, in the definition of \( \text{sac}_0^p(f) \), we allow the sequence \( p = (p_k) \) to be an arbitrary bounded sequence of positive numbers, then, denoting \( r = \max \{1, \sup_k p_k \} \) and \( q = (p_k/r) \), we may write

\[
\text{sac}_0^p(f) = \text{sac}_0^r(\mathcal{F}^q),
\]

where \( \mathcal{F}^q \) is the sequence of moduli \( f^q \) defined by \( f^q_k(t) = (f(t))^{p_k/r} \). Thus from Proposition 1, for \( \lambda = \text{sac}_0^r \), we get the following two corollaries.
Corollary 1. Let $f$ be a modulus and let $p = (p_k)$ be a bounded sequence of positive numbers. Then the functional $g_{f,q}^{\infty}$ defined by

$$g_{f,q}^{\infty}(x) = \sup_k (f(|x_k|))^{p_k/r} \quad (x \in sac_0^p(f))$$

is an $F$-norm on $sac_0^p(f)$ if and only if $f$ is unbounded and $\inf_k p_k > 0$.

Corollary 2. The functional $g_{f}^{\infty}$ is an $F$-norm on $sac_0(f)$ if and only if $f$ is unbounded.

Subsequently, we shall interpret Corollaries 1 and 2 as counterexamples concerning the topologization of the various extensions of the spaces $sac_0(f)$ and $sac_0^p(f)$ considered in [2, 3, 4, 8, 11, 12]. Several of these extensions are spaces of the type (cf. [2])

$$w_0(B, f, p) = \{x = (x_k) : \lim_n \sum_k b_{nk}(i)(f(|x_k|))^{p_k} = 0 \text{ uniformly in } i\},$$

where $p = (p_k)$ is a bounded sequence of positive numbers and $B$ is a sequence of infinite non-negative matrices $B_i = (b_{nk}(i))$. The space $w_0(B, f, p)$ reduces to $sac_0^p(f)$ if $B = B_1$ where $B_1 = (b_{nk}^{1}(i))$ with

$$b_{nk}^{1}(i) = \begin{cases} 1/n & \text{if } i \leq k < i+n \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = (a_{nk})$ be an infinite non-negative matrix. Proposition 3 of Nanda [11] asserts that, for every bounded sequence $p = (p_k)$ of positive numbers, the space $w_0(B, f, p)$ with

$$b_{nk}(i) = \frac{1}{n+1} \sum_{j=0}^{n} a_{i+j,k}$$

can be paranormed by the functional $g_{B,f}^{p}$ where

$$g_{B,f}^{p}(x) = \sup_n \left( \sum_k b_{nk}(i)(f(|x_k|))^{p_k} \right)^{1/r}.$$

Since, for $A$ being the unit matrix, this space is exactly the space $sac_0^p(f)$ with $g_{f,q}^{\infty} = g_{B,f}^{p}$, Corollary 1 shows that Proposition 3 of [11] is not true for every $p = (p_k)$ and every $f$. The same inaccuracy is contained in Theorem 2 of Bilgin [2].

A generalization of the space $sac_0(f)$ is related to invariant means (or $\sigma$-means). Consider a one-to-one mapping $\sigma$ of $\mathbb{N}$ into itself such that
\[ \sigma^k(n) \neq n \text{ for all } n, k \in \mathbb{N}, \text{ where } \sigma^k(n) \text{ denotes the iterate of order } k \text{ of the mapping } \sigma \text{ at } n. \text{ Nuray and Savas [12] introduced the space} \]
\[ w_0(A_\sigma, f) = \{ x = (x_k) : \lim_{n} \sum_{k} a_{nk} f(|x_{\sigma^k(i)}|) = 0 \text{ uniformly in } i \}. \]

Theorem 3 of [12] claims that \( w_0(A_\sigma, f) \) can be topologized by the paranorm
\[ g(x) = \sup_{n, i} \sum_{k} a_{nk} f(|x_{\sigma^k(i)}|) \]
for an arbitrary modulus function \( f \). But our Corollary 2 shows that this is not true for all modulus functions, since \( w_0(A_\sigma, f) \) reduces to \( sa_0(f) \) if \( \sigma(n) = n + 1 \) and \( A = C_1 \) is the matrix of arithmetical means. A similar correction is needed in some results of Esi [3, 4].

It should be noted that an inaccuracy of the same type is contained also in Proposition 6 of the author [8] that deals with the F-seminormability of the more general space
\[ w_0^p(B, F) = \{ x = (x_k) : \lim_{n} \sum_{k} b_{nk}(i)(f_k(|x_k|))^p = 0 \text{ uniformly in } i \}, \]
where \( p \geq 1 \). This proposition asserts that the space \( w_0^p(B, F) \) is a complete F-seminormed AK-space with the F-seminorm
\[ g_{B, F}^p(x) = \sup_{n, i} \left( \sum_{k} b_{nk}(i)(f_k(|x_k|))^p \right)^{1/p} \]
for an arbitrary sequence of modulus functions \( F = (f_k) \). But this is not true in general. For example, if a bounded sequence \( p = (p_k) \) of positive numbers is such that \( \inf_k p_k = 0 \), then, in view of the equality
\[ w_0^p(B_1, F^q) = sa_0^p(f), \]
the functional \( g_{B_1, F^q}^p \) is not an F-seminorm on \( w_0^p(B_1, F^q) \) by Corollary 1.

The mentioned inaccuracy arises from Proposition 2 of [8] which asserts that, for an arbitrary matrix sequence \( B \), the space
\[ w_0^p(B) = \{ x = (x_k) : \lim_{n} \sum_{k} b_{nk}(i)|x_k|^p = 0 \text{ uniformly in } i \} \]
with \( p \geq 1 \) is a complete seminormed AK-space with respect to the seminorm
\[ g_{B}^p(x) = \sup_{n, i} \left( \sum_{k} b_{nk}(i)|x_k|^p \right)^{1/p}. \]
In fact, \((w_0^p(\mathcal{B}), g_0^p)\) is not an AK-space in general, since it reduces to \((sac_0^p, \|\|_\infty)\) if \(\mathcal{B} = \mathcal{B}_1\). A simple argument shows that \((w_0^p(\mathcal{B}), g_0^p)\) is a seminormed AK-space whenever

\[
\limsup_{m} \sum_{n,i} b_{nk}(i)|x_k|^p = 0 \quad (x \in w_0^p(\mathcal{B})).
\]  

Thus, in [8], Propositions 2 and 6 must be reworded, respectively, in the following way.

**Proposition 2.** Let \(\mathcal{B}\) be a sequence of non-negative matrices. Then \((w_0^p(\mathcal{B}), g_0^p)\) is a complete seminormed space, it is a BK-space if \(\mathcal{B}\) is column-positive. If (6) holds then \((w_0^p(\mathcal{B}), g_0^p)\) is an AK-space.

**Proposition 3.** Let \(\mathcal{F} = (f_k)\) be a sequence of modulus functions. Then \((w_0^p(\mathcal{B}, \mathcal{F}), g_0^p(\mathcal{B}, \mathcal{F}))\) is a complete \(\mathcal{F}\)-seminormed space, where \(g_0^p(\mathcal{B}, \mathcal{F})\) is absolutely monotone and has the property (K). If \(\mathcal{B}\) is column-positive then \((w_0^p(\mathcal{B}, \mathcal{F}), g_0^p(\mathcal{B}, \mathcal{F}))\) is an FK-space. It is an AK-space if (6) is satisfied.

Analogous corrections are needed in Corollaries 3 and 8 of [8].

Finally, using that condition (6) holds for any constant sequence \(\mathcal{B} = (A)\), from Proposition 3 we immediately get

**Corollary 3.** Let \(p \geq 1\), \(A = (a_{nk})\) a non-negative matrix, and \(\mathcal{F} = (f_k)\) a sequence of modulus functions. Then the space

\[
w_0^p(A, \mathcal{F}) = \{x = (x_k) : \lim_n \sum_k a_{nk}(f_k(|x_k|))^p = 0\}
\]

is a complete \(\mathcal{F}\)-seminormed AK-space with respect to the absolutely monotone \(\mathcal{F}\)-seminorm

\[
g_{A, \mathcal{F}}^p(x) = \sup_n \left(\sum_k a_{nk}(f_k(|x_k|))^p\right)^{1/p}.
\]

Corollary 3 partially extends Theorem 1 of Bilgin [1].

**References**


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