Sequence spaces defined by a sequence of modulus functions and $\mathcal{X}$-nearly convergence

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Abstract. Let $E$ be a sequence space and let $\mathcal{F} = (f_k)$ be a sequence of modulus functions. For $x = (\xi_k) \in E$ define $\mathcal{F}(x) = (f_k(|\xi_k|))$. The purpose of this paper is to investigate relations between the inclusion $\mathcal{F}(x) \in E$ and the $\mathcal{X}$-nearly convergence of the sequences $x$ and $\mathcal{F}(x)$.

1. Introduction

The notions of zero-classes and $\mathcal{X}$-nearly convergence were introduced by Freedman and Sember [3].

Let $\mathbb{N}$ denote the set of positive integers. A class $\mathcal{X}$ of the subsets of $\mathbb{N}$ is called a zero-class if the following conditions hold:

1. $A$ finite $\Rightarrow A \in \mathcal{X}$,
2. $A, B \in \mathcal{X} \Rightarrow A \cup B \in \mathcal{X}$,
3. $A \subset B$, $B \in \mathcal{X} \Rightarrow A \in \mathcal{X}$,
4. $\mathbb{N} \notin \mathcal{X}$.

Definition. Let $\mathcal{X}$ be a zero-class. A number sequence $x = (\xi_k)$ is called $\mathcal{X}$-nearly convergent to $l$ if there exists a set $Z \in \mathcal{X}$ such that

$$\lim_{k \in \mathbb{N} \setminus Z} \xi_k = l.$$ 

The sets of all bounded real $\mathcal{X}$-nearly convergent and $\mathcal{X}$-nearly convergent to zero sequences are denoted respectively by $\omega_X$ and $\omega^0_X$. Let $c$, $c_0$ and $\ell_\infty$ be respectively the spaces of convergent, convergent to zero and bounded real sequences. Then $c \subset \omega_X$, $c_0 \subset \omega^0_X$ and $\omega_X$, $\omega^0_X$ are linear subspaces of $\ell_\infty$ (cf. [3]).

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**Definition.** A function \( f : [0, \infty) \to [0, \infty) \) is called a *modulus function* if \( f \) is strictly increasing and continuous on \([0, \infty)\), \( f(t + u) \leq f(t) + f(u) \) and \( f(0) = 0 \).

Let \( E \) be a sequence space of real sequences and \( \mathcal{F} = (f_k) \) be a sequence of modulus functions. The space \( E(\mathcal{F}) \) is defined as follows:

\[
E(\mathcal{F}) = \{ x = (\xi_k) : \mathcal{F}(x) = (f_k(\left|\xi_k\right|)) \in E \}.
\]

The spaces of this kind were introduced by Ruckle [12] and Maddox [10] for \( \mathcal{F} = (f) \), this definition was extended by Kolk [5] to non-constant sequences \( \mathcal{F} = (f_k) \).

In this paper we investigate relations between \( E(\mathcal{F}) \cap \ell_\infty \) and \( \omega_{\mathcal{X}}^0 \) for certain zero-classes \( \mathcal{X} \).

2. Some preliminary results

Let \( E \) be a sequence space such that

1. \( c_0 \subset E \),
2. \( E \) is solid, i.e. \( \ell_\infty \cdot E \subset E \),
3. \( e \notin E \), where \( e = (1, 1, ...) \).

We denote the characteristic sequence of a set \( Z \subset \mathbb{N} \) by \( \varphi_Z = (\varphi^Z_k) \), i.e.

\[
\varphi^Z_k = \begin{cases} 
1 & \text{if } k \in Z, \\
0 & \text{if } k \notin Z.
\end{cases}
\]

It is easy to check that if a sequence space \( E \) satisfies conditions (5)-(7), then the system

\[
\mathcal{X}_E = \{ Z \subset \mathbb{N} : \varphi_Z \in E \}
\]

is a zero-class. For \( \mathcal{X} = \mathcal{X}_E \) we further denote \( \omega_{\mathcal{X}_E} = \omega_E \) and \( \omega_{\mathcal{X}_E}^0 = \omega_E^0 \).

**Proposition 1.** Let \( E \) be a sequence space which satisfies conditions (5)-(7). Then

\[
\omega_E^0 \subset E \cap \ell_\infty \subset \overline{\omega_E^0},
\]

where the closure is taken with respect to the norm on \( \ell_\infty \).

**Proof.** 1) Let \( x = (\xi_k) \in \omega_E^0 \), i.e. there exits a set \( Z \subset \mathbb{N} \) such that \( \lim_{k \in \mathbb{N} \setminus Z} \xi_k = 0 \) and \( \varphi_Z \in E \). We may write

\[
x = \varphi_{\mathbb{N} \setminus Z} x + \varphi_Z x.
\]

Then \( \varphi_{\mathbb{N} \setminus Z} x \in c_0 \subset E \). Since \( E \) is solid, \( x \in \omega_E^0 \subset \ell_\infty \) and \( \varphi_Z \in E \), we also have \( \varphi_Z x \in E \cap \ell_\infty \). Therefore \( x \in E \cap \ell_\infty \) for each \( x \in \omega_E^0 \).
2) Let now \( x = (\xi_k) \in E \cap \ell_{\infty} \). Define
\[
Z_n = \{ k \in \mathbb{N} : |\xi_k| > n^{-1} \}
\]
and take \( y = (\eta_k) \) such that
\[
\eta_k = \begin{cases} 
\frac{1}{\xi_k} & \text{for } k \in Z_n, \\
0 & \text{otherwise.}
\end{cases}
\]
Then \( y \in \ell_{\infty} \) for each fixed \( n \) and so \( \phi_{Z_n} = y \cdot x \in E \), because \( E \) is solid. Now define \( y_n = (\eta_k^n) \) by
\[
\eta_k^n = \begin{cases} 
\xi_k & \text{for } k < n, \\
\xi_k & \text{for } k > n \text{ and } k \in Z_n, \\
0 & \text{otherwise.}
\end{cases}
\]
Then it follows from the representation of \( y_n \) that \( y_n \in \omega^0_E \) for each \( n \in \mathbb{N} \). Since \( |\eta_k^n - \xi_k| < 1/n \) for each \( k \in \mathbb{N} \), the sequence \( y_n \) converges to \( x \) in \( \ell_{\infty} \) and therefore \( x \in \omega^0_E \).

3. The space \( E(\mathcal{F}) \cap \ell_{\infty} \) and \( X \)-nearly convergence

Further we use the following characteristics for a sequence \( \mathcal{F} = (f_k) \) of modulus functions:

8) \( \sup_k f_k(t) < \infty \) for each \( t > 0 \),
9) \( \lim_{t \to 0^+} \sup_k f_k(t) = 0 \),
10) \( \inf_k f_k(t) > 0 \) for each \( t > 0 \).

Kolk (cf. [6] and [7]) proved that (8) \( \iff \ell_{\infty} \subset \ell_{\infty}(\mathcal{F}) \), (9) \( \iff c_0 \subset c_0(\mathcal{F}) \) and (10) \( \iff c_0(\mathcal{F}) \subset c_0 \).

**Proposition 2.** If a sequence space \( E \) satisfies conditions (5)-(7) and a sequence \( \mathcal{F} = (f_k) \) of modulus functions satisfies conditions (8)-(10), then
\[
\omega^0_E = \omega^0_E(\mathcal{F}) \cap \ell_{\infty}.
\]

**Proof.** Let \( x \in \omega^0_E \), then \( \varphi_{\mathbb{N}\setminus Z} \cdot x \in c_0 \) and \( \varphi_Z \in E \) for a certain set \( Z \subset \mathbb{N} \). Then it follows from (9) that also \( \varphi_{\mathbb{N}\setminus Z} \cdot \mathcal{F}(x) \in c_0 \), i.e. \( \mathcal{F}(x) \) is \( X \)-nearly convergent to zero. As \( \omega^0_E \subset \ell_{\infty} \), we have \( x \in \ell_{\infty} \) and by (8) we have \( \mathcal{F}(x) \in \ell_{\infty} \). Therefore, \( \mathcal{F}(x) \in \omega^0_E \) and \( x \in \omega^0_E(\mathcal{F}) \cap \ell_{\infty} \). In the same manner we can show that if (10) holds, then \( \omega^0_E(\mathcal{F}) \cap \ell_{\infty} \subset \omega^0_E \).
Lemma 1. Let $B \subset \ell_\infty$ be a solid sequence space and let $\mathcal{F} = (f_k)$ be a sequence of modulus functions. If

\[ |f_k^{-1}(u_2) - f_k^{-1}(u_1)| \leq H |u_2 - u_1|, \]

then $\overline{\mathcal{B}(\mathcal{F})} \subset \overline{\mathcal{B}}$.

Proof. Take $x = (\xi_k) \in \overline{\mathcal{B}(\mathcal{F})}$, i.e. $\mathcal{F}(x) = (f_k(\xi_k)) \in \overline{\mathcal{B}}$. Then there exists a sequence $y_n = (\eta_k^n) \in B$, $\eta_k^n \geq 0$ so that $y_n$ converges to $\mathcal{F}(x)$ in $\ell_\infty$. It means that for arbitrary $\varepsilon > 0$ there exists $N_\varepsilon > 0$ such that

\[ |\eta_k^n - f_k(\xi_k)| < \varepsilon H^{-1} \tag{3.1} \]

for all $n > N_\varepsilon$ and $k \in \mathbb{N}$.

Let $x_n = (\xi_k^n)$, $\xi_k^n \geq 0$, be the sequence such that $f_k(\xi_k^n) = \eta_k^n$. Take

\[ \overline{\xi_k^n} = \begin{cases} \xi_k^n (\text{sgn} \xi_k)^{-1} & \text{if } \xi_k \neq 0, \\ 0 & \text{if } \xi_k = 0. \end{cases} \]

Then we have by assumption (11) and by (3.1) that the following estimations are true (on a certain finite interval):

\[
|\overline{\xi_k^n} - \xi_k| = |\xi_k^n - \overline{\xi_k^n}| \\
= |f_k^{-1}[f_k(\xi_k^n)] - f_k^{-1}[f_k(\xi_k)|] \\
\leq H|\eta_k^n - f_k(\xi_k)| < \varepsilon
\]

for all $n > N_\varepsilon$, $k \in \mathbb{N}$. Hence the sequence $x_n = (\overline{\xi_k^n})$ converges to $x = (\xi_k)$ in $\ell_\infty$.

We show now that $x_n \in B$, $n \in \mathbb{N}$. Take in (11) $u_2 = \eta_k^n$ and $u_1 = 0$. We have $0 \leq \xi_k^n = f_k^{-1}(\eta_k^n) < H \eta_k^n$ and, as $B$ is solid, $x_n \in B$ and also $x_n \in B$. $\square$

Corollary 1. If a sequence space $E$ satisfies conditions (5)–(7) and a sequence $\mathcal{F} = (f_k)$ of modulus functions satisfies condition (11), then

\[ \overline{\omega_E^0(\mathcal{F})} \subset \overline{\omega_E^0}. \]

Proof. Take $B = \omega_E^0$ in Lemma 1. $\square$
Example 1. If the functions \( f_k^{-1} \) are differentiable on \([0, \infty)\) and the sequence of derivatives of \( f_k^{-1} \) is uniformly bounded on each interval \([0, b]\), then applying Lagrangian theorem it is easy to show that condition (11) holds. In the particular case when \( f_k(t) = t^{p_k}, 0 < p_k \leq 1\), we have that

\[
\frac{d}{du} f_k^{-1}(u) = \frac{1}{p_k} u^{1/p_k - 1}
\]

and condition (11) holds if and only if \( \inf_k p_k > 0 \).

Example 2. Let \( \mathcal{F} = (f) \), where \( f \) is an unbounded modulus function. Then condition (11) is the Lipschitz condition for the inverse function \( f^{-1} \). As the inverse function \( f^{-1} \) of an unbounded modulus function \( f \) is convex in each interval \((0, b), b > 0\), it satisfies Lipschitz condition (cf. [8], Lemma 1.3) and (11) is fulfilled.

Lemma 2. If condition (11) holds, then \( \ell_\infty(\mathcal{F}) \subset \ell_\infty \) and \( c_0(\mathcal{F}) \subset c_0 \) (i.e. (10) is fulfilled).

Proof. Let \( (f_k(|\xi_k|)) = (\eta_k) \in \ell_\infty \), then \( 0 \leq \eta_k \leq b \) for a certain \( b > 0 \).

Take \( u_2 = \eta_k, u_1 = 0 \) in (11), then we have:

\[
|\xi_k| \leq Hb,
\]

i.e. \( x = (\xi_k) \in \ell_\infty \).

In the same manner we can show that \( c_0(\mathcal{F}) \subset c_0 \). □

Proposition 3. Let a sequence space \( E \) satisfy conditions (5)–(7) and a sequence \( \mathcal{F} = (f_k) \) of modulus functions satisfy conditions (8), (9) and (11). Then

\[
\omega_E^0 \subset E(\mathcal{F}) \cap \ell_\infty \subset \overline{\omega_E^0}.
\] (3.2)

Proof. It follows immediately from Proposition 1 that

\[
\omega_E^0(\mathcal{F}) \subset (E \cap \ell_\infty)(\mathcal{F}) \subset \overline{\omega_E^0}(\mathcal{F}).
\]

Then, applying Lemma 2 and (8), it is not difficult to show that \( (E \cap \ell_\infty)(\mathcal{F}) = E(\mathcal{F}) \cap \ell_\infty \) and (3.2) follows immediately from Proposition 2 (by Lemma 2, (10) is satisfied) and Corollary 1. □

Proposition 4. Let a sequence space \( E \) satisfy conditions (5)–(7) and \( E \cap \ell_\infty = \omega_E^0 \). If a sequence \( \mathcal{F} = (f_k) \) of modulus functions satisfies conditions (8)–(10), then

\[
E(\mathcal{F}) \cap \ell_\infty = E \cap \ell_\infty.
\]

Proof. It follows immediately from \( E \cap \ell_\infty = \omega_E^0 \) that \( E(\mathcal{F}) \cap \ell_\infty(\mathcal{F}) = \omega_E^0(\mathcal{F}) \). Since \( \ell_\infty \subset \ell_\infty(\mathcal{F}) \) by (8), we may write \( E(\mathcal{F}) \cap \ell_\infty = \omega_E^0(\mathcal{F}) \cap \ell_\infty \). Applying also Proposition 2 we have \( E(\mathcal{F}) \cap \ell_\infty = \omega_E^0 = E \cap \ell_\infty \). □

Remark. If \( \mathcal{F} = (f) \), then conditions (8)–(10) are fulfilled.
4. Some sequence spaces related to $\mathcal{X}$-nearly convergence

4.1. The space $[c_A]_0(\mathcal{F})$. Let $A = (a_{nk})$ be a regular matrix method with $a_{nk} \geq 0$ and

$$[c_A]_0 = \{ x = (\xi_k) : \lim_n \sum_{k} a_{nk} |\xi_k| = 0 \},$$

i.e. $[c_A]_0$ is the set of strongly $A$-summable to zero sequences.

Take $E = [c_A]_0$, then it is easy to check that conditions (5)–(7) are fulfilled. By results of Hill and Sledd [4] and Sember and Freedman [3] we have

$$E \cap \ell_\infty = \omega^0_E = \overline{\omega^0_E}. \tag{4.1}$$

Therefore, by Proposition 4, we can state that

$$[c_A]_0(\mathcal{F}) \cap \ell_\infty = [c_A]_0 \cap \ell_\infty \tag{4.2}$$

for each sequence $\mathcal{F} = (f_k)$ of modulus functions which satisfies conditions (8)–(10).

**Example 3.** Let $E = [c_A]_0$ and $f_k(t) = t^{p_k}$, $0 < p_k \leq 1$. In this case 

$$[c_A]_0(\mathcal{F}) = [c_A(p)]_0,$$

the space of sequences that are strongly $A$ summable to zero with exponent $p = (p_k)$. Then conditions (8) and (9) are fulfilled and condition (10) is fulfilled if and only if $\inf p_k > 0$ (cf. [7]). Hence for $0 < \inf p_k \leq p_k \leq 1$

$$[c_A(p)]_0 \cap \ell_\infty = [c_A]_0 \cap \ell_\infty.$$

This result is well known if $A = (C, 1)$ and $p_k = \tilde{p}$, $0 < \tilde{p} < 1$ (cf. [9]).

Further (cf. Proposition 5) we show that a stronger result than (4.2), namely the equality $[c_A]_0(\mathcal{F}) = [c_A]_0$ is not true in general.

Let $\bar{f}(t) = \sup_k f_k(t)$, then condition (9) guarantees that $\bar{f}$ is also a modulus function (cf. [5]). Let

$$w_0 = \{ x = (\xi_k) : \lim_n (n+1)^{-1} \sum_{k=0}^{n} |\xi_k| = 0 \},$$

i.e. $w_0 = [c_A]_0$ for $A = (C, 1)$. It is clear that $E(\bar{f}) \subset E(\mathcal{F})$ for each solid space $E$ and thus applying also (4.2) for $\bar{f}$ we have

$$w_0 \cap \ell_\infty = w_0(\bar{f}) \cap \ell_\infty \subset w_0(\mathcal{F}) \cap \ell_\infty. \tag{4.3}$$
Proposition 5. If the sequence $F = (f_k)$ of modulus functions satisfies condition (9) and $\lim_{t \to \infty} \tilde{f}(t)/t = 0$, then there exists an unbounded sequence $z \in w_0(F) \setminus w_0$.

Proof. By the version of Kuttner's theorem proved by Maddox [11] the condition $\lim_{t \to \infty} \tilde{f}(t)/t = 0$ implies that for each locally convex FK-space $X$

$$X \supset w_0(\tilde{f}) \Rightarrow X \supset \ell_\infty$$  \hspace{1cm} (4.4)

(for an arbitrary modulus function $\tilde{f}$). Let us take $X = w_0$, then $\ell_\infty \not\subseteq w_0$ and by (4.4) we have that $w_0 \not\supset w_0(\tilde{f})$. Therefore, by (4.3), there exists an unbounded sequence $z \in w_0(\tilde{f}) \setminus w_0$ and also $z \in w_0(F) \setminus w_0$. \hfill \Box

Remark. The condition $\lim_{t \to \infty} f(t)/t = 0$ is fulfilled if the modulus function $f$ is the inverse function of any Orlicz function (cf. [8]).

Remark. Proposition 5 has an extension if we consider the strong summability field determined by a lacunary sequence (cf. [14], Theorem 9) instead of $w_0$.

4.2. The space $[c_\alpha]_0(F)$. Let $\alpha = (A_i)$ be a sequence of matrices $A_i = (a_{nik})$ with $a_{nik} \geq 0$. Define

$$[c_\alpha]_0 = \{ x = (\xi_k) : \lim_{n} \sum_k a_{nik} |\xi_k| = 0 \text{ uniformly in } i \},$$

i.e. $[c_\alpha]_0$ is the set of sequences which are strongly summable to zero by a sequential method $\alpha$.

Take $E = [c_\alpha]_0$. Then it is clear that $E$ is solid, i.e. condition (6) holds. Let the sequential method $\alpha$ be regular, i.e. $\lim_{n} \sum_k a_{nik} \xi_k = \lim_k \xi_k$ (uniformly in $i$) for each $x = (\xi_k) \in c$. It is not difficult to show that in this case $c_\alpha \subseteq E$ and $e \not\subseteq E$. Therefore conditions (5) and (7) also hold.

Let now $f_k(t) = t$, $k \in \mathbb{N}$. Then $E(F) = E$ and by Proposition 1 we have

$$\omega_E^0 \subseteq E \cap \ell_\infty \subset \overline{\omega_E^0}.$$

If $\alpha$ is regular, then $A_i$ are regular matrix methods. Then, by Section 4.1, the sets $[c_{A_i}]_0 \cap \ell_\infty$ are closed in $\ell_\infty$. Moreover (cf. [13])

$$E = \cap_{T \in U} [c_T]_0,$$

where $U$ is the family of matrices $T = (t_{nk})$ such that $t_{nk} = a_{nik}$ for some $i$. Therefore the set $E \cap \ell_\infty$ is also closed and

$$E \cap \ell_\infty = \overline{\omega_E^0}.$$  \hspace{1cm} (4.5)

The following example shows that there exists a sequential method $\alpha$ so that $\omega_E^0 \supsetneq E \cap \ell_\infty$ (cf. (4.1)).
Example 4. Take

\[ a_{nk} = \begin{cases} \frac{(n+1)^{-1}}{a_{nk}} & \text{if } i \leq k \leq i + n, \\ 0 & \text{otherwise.} \end{cases} \]

Then \([c_{\alpha}]_{0} = sac_{0}\), the space of strongly almost convergent to zero sequences, and (cf. [3])

\[ \omega_{E}^{0} \subseteq sac_{0} = \overline{\omega_{E}^{0}}. \]

4.3. \( \mathcal{K} \)-nearly convergence and statistical convergence. Let \( \omega \) denote the space of all real sequences. For any \( x = (\xi_{k}) \in \omega \) and \( \varepsilon > 0 \) define

\[ N_{\varepsilon}(x) = \{ y = (\eta_{k}) \in \omega : \sup_{k} |\xi_{k} - \eta_{k}| < \varepsilon \}. \]

Then the class \( \{ N_{\varepsilon}(x) : x \in \omega, \varepsilon > 0 \} \) forms a base for the topology \( \tau_{\infty} \) of uniform convergence on \( \omega \). On the space \( \ell_{\infty} \) this is the usual "sup-norm" topology.

For an arbitrary zero class \( \mathcal{X} \) Chun and Freedman [1] defined

\[ V_{A}^{\theta} = \{ x = (\xi_{k}) \in \omega : \forall \alpha > 0, Z_{\alpha} = \{ k : |\xi_{k}| > \alpha \} \in \mathcal{X} \} \]

and proved that \( V_{A}^{\theta} \) is the closure (with respect to the topology \( \tau_{\infty} \)) of the space of \( \mathcal{X} \)-nearly convergent to zero sequences.

If \( A = (a_{nk}) \) is a regular matrix method with \( a_{nk} \geq 0 \) and

\[ \mathcal{X} = \{ Z \subseteq \mathbb{N} : \varphi_{Z} \in [c_{\alpha}]_{0} \}, \]

then \( V_{A}^{\theta} = st_{A}^{\theta} \), where \( st_{A}^{\theta} \) denotes the set of \( A \)-statistically convergent to zero sequences. About \( \theta \)-statistically convergence see, for example, [2] and [7]. If \( E = [c_{\alpha}]_{0} \), then \( \omega_{E}^{0} = st_{A} \cap \ell_{\infty} \) and therefore (4.1) and (4.5) also follow from Theorem 4.4 of [7].

References


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