On sequence spaces defined by a regularly varying modulus

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Abstract. Let $\lambda$ be an F-seminormed solid sequence space. We characterize the F-seminormability of the sequence space $\lambda(f) = \{(x_k) : (f(|x_k|)) \in \lambda\}$ for a regularly varying modulus function $f$.

1. Introduction

By the term sequence space we shall mean, as usual, any linear subspace of the vector space $\omega$ of all (real or complex) sequences $x = (x_k) = (x_k)_{k \in \mathbb{N}}$, where $\mathbb{N} = \{1, 2, \ldots\}$. A sequence space $\lambda$ is called solid if $(x_k) \in \lambda$ and $|y_k| \leq |x_k|$ ($k \in \mathbb{N}$) yield $(y_k) \in \lambda$.

A function $f : [0, \infty) \to (0, \infty)$ is called a modulus function (or simply a modulus) if

(M1) $f(t) = 0$ if and only if $t = 0$,
(M2) $f(t + u) \leq f(t) + f(u)$,
(M3) $f$ is increasing,
(M4) $f$ is continuous from the right at $0$.

Provided a modulus $f$ and a sequence spaces $\lambda$, Ruckle [4], Maddox [3] and some other authors define a new sequence space $\lambda(f)$ by

$$\lambda(f) = \{(x_k) \in \omega : (f(|x_k|)) \in \lambda\}.$$ 

It is not difficult to see that $\lambda(f)$ is a solid sequence space whenever the sequence space $\lambda$ is solid.

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A positive, finite and measurable function $f$, defined on $[a, \infty)$ for some $a > 0$, is said to be regularly varying at infinity (see [1]) if the limit

$$\lim_{t \to \infty} \frac{f(ut)}{f(t)} = \mu(u)$$

is positive and finite for each $u > 0$. The function $\mu(u)$ is called index function of regularly varying function $f$. It is known ([1], Theorem 1.4.1) that the index function of a regularly varying function $f$ is necessarily in the form

$$\mu(u) = u^\rho$$

for some $\rho \in \mathbb{R}$ and for each $u > 0$. Here the number $\rho$ is called index of $f$. Thus $f$ varies regularly with index $\rho$ at infinity if for each $u > 0$,

$$\lim_{t \to \infty} \frac{f(ut)}{f(t)} = u^\rho. \quad (1)$$

A positive, finite and measurable function $f$, defined on $(0, b]$ for some $b > 0$, is said to be regularly varying of index $\sigma \in \mathbb{R}$ at the origin if

$$\lim_{t \to 0^+} \frac{f(ut)}{f(t)} = u^\sigma \quad (2)$$

for each $u > 0$. This is equivalent to saying that the function $f(1/t)$ varies regularly with index $-\sigma$ at infinity.

Regularly varying function of index $\rho = 0$ ($\sigma = 0$) is said to be slowly varying at infinity (at the origin).

In this note we describe the F-semimormability of $\lambda(f)$ assuming that $\lambda$ is an F-semimormed solid sequence space and the modulus function $f$ is regularly varying at infinity and at the origin.

2. On topologization of $\lambda(f)$

Recall that an F-semimorm $g$ on a vector space $X$ is a functional $g : X \to \mathbb{R}$ satisfying for all $x, y \in X$ the axioms

(N1) $g(0) = 0$,
(N2) $g(x + y) \leq g(x) + g(y)$,
(N3) $g(\alpha x) \leq |\alpha| g(x)$ for all scalars $\alpha$ with $|\alpha| \leq 1$,
(N4) $\lim_{n} g(\alpha_n x) = 0$ for every scalar sequence $(\alpha_n)$ with $\lim_n \alpha_n = 0$.

An F-semimorm on a solid sequence space $\lambda$ is said to be absolutely monotone if $g(x) \leq g(y)$ for all $x = (x_k), y = (y_k)$ from $\lambda$ with $|x_k| \leq |y_k| \quad (k \in \mathbb{N})$.  

If the sequence space $\lambda$ is topologized by an $F$-seminorm (or paranorm) $g$ then for the topologization of $\lambda(f)$ it is natural to consider the functional $g_f$ defined by

$$g_f(x) = g(f(|x|)) \quad (x \in \lambda(f)).$$

It is known (cf. [3], Theorem 8) that $g_f$ may not be an $F$-seminorm on $\lambda(f)$ in general. From some results of Soomer ([5], Theorem 3) and the author ([2], Theorem 1) we immediately get

**Proposition 1.** Let $f$ be a modulus and let $g$ be an absolutely monotone $F$-seminorm on a solid sequence space $\lambda$. The functional $g_f$ is an absolutely monotone $F$-seminorm on $\lambda(f)$ if $f$ satisfies one of following two equivalent conditions:

(M5) There exists a function $\nu$ with $f(ut) \leq \nu(u)f(t)$ \quad ($0 < u \leq 1$, \quad $t > 0$)

and $\lim_{u \to 0^+}\nu(u) = 0$;

(M6) $\lim_{u \to 0^+}\sup_{t > 0}\frac{f(ut)}{f(t)} = 0$.

First we characterize indices of regularly varying moduli.

**Proposition 2.** Any modulus function which is regularly varying at infinity (or at the origin) has index from the interval $[0,1]$.

**Proof.** Let $f$ be a modulus function. By monotonicity and subadditivity of $f$ we have

$$f(ut) \leq f([u] + 1)t \leq ([u] + 1)f(t),$$

where $[u]$ denotes the integer part of $u$. Hence for all $u, t > 0$,

$$0 < \frac{f(ut)}{f(t)} \leq u + 1.$$  \quad (3)

If $f$ is also regularly varying at infinity of index $\rho$, then (1) holds, and by (3) we see that $0 \leq \rho \leq 1$.

If $f$ is regularly varying at the origin of index $\sigma$, then from (2) and (3) we similarly get $0 \leq \sigma \leq 1$. \quad \Box

In the following we give some examples of regularly and slowly varying modulus functions.

**Example 1.** Every bounded modulus $f$ is slowly varying at infinity by

$$\lim_{t \to \infty} \frac{f(ut)}{f(t)} = \frac{\sup_{t > 0} f(ut)}{\sup_{t > 0} f(t)} = 1.$$
Example 2. The unbounded modulus \( f(t) = t^p \) \((0 < p \leq 1)\) is a regularly varying function of index \( p \) because of \( f(ut)/f(t) = u^p \).

Example 3. The unbounded modulus \( f(t) = \ln(1 + t) \) is slowly varying at infinity by \( \lim_{t \to \infty} \ln(1 + ut)/\ln(1 + t) = 1 \) and regularly varying of index 1 at the origin by \( \lim_{t \to 0} \ln(1 + ut)/\ln(1 + t) = u \).

Example 4. The unbounded modulus \( f(t) = t/\ln(t + e^2) \), considered by Maddox (see [3], p. 164), varies regularly with index 1 at infinity and at the origin.

Now we are ready to prove our main result.

**Theorem.** Let \( g \) be an absolutely monotone \( F \)-seminorm on a solid sequence space \( \lambda \). If a modulus \( f \) varies regularly with index \( \rho \) at infinity and with index \( \sigma \) at the origin, then the functional \( g_f \) is an absolutely monotone \( F \)-seminorm on \( \lambda(f) \) whenever \( \min\{\rho, \sigma\} > 0 \). If \( f \) is slowly varying at infinity or at the origin, then conditions (M5) and (M6) are not satisfied.

**Proof.** Let \( f \) be a modulus which varies regularly with index \( \rho > 0 \) at infinity and with index \( \sigma > 0 \) at the origin. It is known ([1], Theorem 1.5.2) that the equalities (1) and (2) hold uniformly in \( u \) on each interval \([0, b]\) with \( b > 0 \). Thus for an arbitrary number \( \varepsilon > 0 \) we can find a number \( t_1 > 0 \) and a natural number \( t_2 > t_1 \) such that

\[
\frac{f(ut)}{f(t)} < u^\sigma + \frac{\varepsilon}{2} \quad (0 < t < t_1), \quad \frac{f(ut)}{f(t)} < u^\rho + \frac{\varepsilon}{2} \quad (t_2 < t < \infty)
\]

for all \( u \in (0, 1] \). If \( t \in [t_1, t_2] \), then by monotonicity and subadditivity of \( f \) we have

\[
\frac{f(ut)}{f(t)} \leq \frac{t}{t_1} f(u).
\]

Since \( u^\sigma \), \( u^\rho \) and \( f(u) \) tend to zero as \( u \to 0^+ \), there exists \( \delta > 0 \) such that

\[
\max\{u^\sigma, u^\rho\} < \frac{\varepsilon}{2}, \quad f(u) < \frac{f(t_1)}{t_2} \varepsilon
\]

for \( 0 < u < \delta \). Consequently, we get

\[
\sup_{t > 0} \frac{f(ut)}{f(t)} \leq \varepsilon \quad (0 < u < \delta),
\]

which shows that the condition (M6), and hence the equivalent condition (M5), of Proposition 1 hold.
If the modulus $f$ varies slowly at infinity or at the origin, then
\[
\lim_{t \to \infty} \frac{f(ut)}{f(t)} = 1 \quad \text{or} \quad \lim_{t \to 0^+} \frac{f(ut)}{f(t)} = 1.
\]
Since for $0 < u \leq 1$ we have
\[
0 < \frac{f(ut)}{f(t)} \leq 1,
\]
then clearly
\[
\sup_{t > 0} \frac{f(ut)}{f(t)} = 1.
\]
Thus (M6), and hence (M5), are not satisfied in this case. □

Examples 1–4 show that the conditions (M5) and (M6) of Proposition 1 hold for moduli $f(t) = t^p$ $(0 < p \leq 1)$ and $f(t) = t/\ln(t + e^2)$, but are not satisfied for bounded moduli and for unbounded modulus $f(t) = \ln(1 + t)$.

References


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