The Logic of Truth in Paraconsistent Internal Realism

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The paper discusses which modal principles should hold for a truth operator answering to the truth theory of internal realism. It turns out that the logic of truth in internal realism is isomorphic to the modal system S4.

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1. Epistemic Conceptions of Truth

Versions of internal realism or so-called "anti-realism" claim *truth* to be not a non-epistemic concept, which means that it does not suffice for the truth of statement that it (merely) corresponds to the facts (or—more generally speaking—*reality*). These conceptions of truth claim rather that we have to have also some sort of justification for the statement being true (whether that is just an ordinary justification or some "canonical" justification in a corresponding theory of justification).

In some versions of anti-realism this amounts to the proposal that truth is something like ideal assertability or provability in some comprehensive (intuitionistic) formal system (cf. Tennant 1987). In some versions of internal realism truth has a double nature, requiring both correspondence to the facts *and* being justified as part of our best theories (cf. Bremer 2008).

We do not have to deal with the details what might be understood as an appropriate justification, the minimal consensus of theories which take truth as not non-epistemic—and thus at least as partially epistemic—is that from some fact obtaining—where again we have not to deal here with the details of an appropriate ontology or theory of truth makers—one *cannot* simply conclude to some corresponding statement being true (the statement

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made by using a corresponding sentence in some situation in an assertoric utterance).

2. Problems with Convention (T)

Given this further definitional requirement of justification/being justified, *Convention (T)*

(T) $T\ulcorner p\urcorner \equiv p$

becomes problematic in its right to left direction. Whereas

(T-LR) $T\ulcorner p\urcorner \rightarrow p$

is unproblematic since what is true has to be the case

(T-RL) $p \rightarrow T\ulcorner p\urcorner$

is now no longer acceptable. Even if $\ulcorner p\urcorner$ corresponds to the facts (i.e. obtains) that does not imply that we have a justification for believing $\ulcorner p\urcorner$.

Note: Any interesting theory of truth will possess enough expressive power to yield antinomies, at least if the truth operator are added to otherwise standard logics like PC or FOL (cf. Thomason 1980). Iterations of the truth operator then contain or simulate the ability to take about the truth of something being true. Given names for statements some statements speak about their own truth or non-truth. The logic of truth should thus be a paraconsistent logic and inconsistent contexts should be considered when discussing proposals for axioms of the truth operator.

Given that the system to be construed in rich enough to derive antinomies, on all accounts, there is no need to introduce 'is true' as a *predicate*. $\ulcorner T\urcorner$ may be introduced as an *operator*, i.e. not requiring the sentences in its scope to be quoted.

What then is the logic of *truth*—taking $\ulcorner T\urcorner$ to be an operator in one's paraconsistent theory of truth?

3. Truth as a Normal Modality

$\ulcorner T\urcorner$ behaves in several ways like a necessity operator $\ulcorner\Box\urcorner$. Thus we obviously should have

(T1) $Tp \rightarrow p$

If truth is not veridical what is? Further on *Modus Ponens* (deductive closure) should hold for truth:

(T2) $T(p \rightarrow q) \rightarrow (Tp \rightarrow Tq)$

Given that proofs *are* the best justification one can have the analogue to Necessitation: it should hold:

(T3) $\vdash\alpha \rightarrow \vdash T\alpha$

So far the logic for ⌜T⌝ is a normal modal logic of the strength of system T. Iterations of ⌜T⌝ are (syntactically) allowed and harmless.

4. Iterations of the Truth Operator

There is no dual operator to ⌜T⌝ like ⌜$\Diamond$⌝ is to ⌜$\Box$⌝. But one may consider:

(T4*) $p \rightarrow {\sim}T{\sim}p$

The operator ⌜$\sim T \sim$⌝ need not have an own name or symbol, but so would behave like ⌜$\Diamond$⌝.

Even if the theory under consideration is inconsistent (T4) may still hold for a *correspondence* conception of truth. Everything then, however, depends on whether the truth operator is taken as a *bivalent* operator or not.

Let us—for a moment—consider truth tables as means of a *realistic* (i.e. *not* internal realistic) conception of truth. If one wants to say that an inconsistent statement is true and is not true at the same time, then the truth operator is not bivalent: ⌜Tp⌝ and ⌜${\sim}Tp$⌝ are both accepted (cf. Priest 1979, 2006). If the truth operator is not bivalent, since inconsistent statements and therefore their negations are both true and not true, we can have at the same time: p, ${\sim}p$, Tp, $T{\sim}p$, ${\sim}Tp$, ${\sim}T{\sim}p$.

The truth operator then can be given by the following truth table:

p	${\sim}p$	Tp	${\sim}Tp$	$T{\sim}p$	${\sim}T{\sim}p$
1	0	1	0	0	1
0	1	0	1	1	0
0,1	0,1	0,1	0,1	0,1	0,1

An internal realist cannot accept this truth table, since it validates (T-RL), which was rejected above. It also validates (T4*).

If the truth operator is bivalent it is used to ascribe the semantic property of *being true* (cf. Bremer 2005). A statement can have that property even if it also has the property of *being false* or *being not true*. Saying of a statement that has this property that it has that property is simply true then, and not false at the same time. One then has even in case of an inconsistent statement ⌜p⌝: p, ${\sim}p$, Tp, $T{\sim}p$, but *not*: ${\sim}Tp$, ${\sim}T{\sim}p$.

The truth operator then can be given by the following truth table:

p	$\sim p$	Tp	$\sim Tp$	$T\sim p$	$\sim T\sim p$
1	0	1	0	0	1
0	1	0	1	1	0
0,1	0,1	1	0	1	0

Again, an internal realist cannot accept this truth table, since it also validates (T-RL).

Nevertheless, if one takes ⌜T⌝ to be a bivalent operator—as it is in consistent contexts anyway, and as one should rather do—then (T4*) can turn out false, namely in case that ⌜p⌝ is inconsistent. Thus even in a realistic setting (T4*) is not acceptable, once one takes ⌜T⌝ to be a bivalent operator.

As a *heuristic* the internal realist should at least reject all those supposed principles of truth that are rejected by a realist who takes truth to be a bivalent operator. Having a justification or having a statement being forced upon us by our best theories is something that is given or not, and not given and not-given at the same time, i.e. *being justified* is a bivalent property. It may happen that one has a justification for some statement ⌜p⌝ and its negation ⌜$\sim p$⌝, but this does not amount to having a justification for ⌜p⌝ and not having it. Rather one has two justification for contradictory statements.

If (T4*), however, is rejected and (T1) is accepted, then either Double Negation Elimination or Contraposition has to be rejected, since by these two principles (T4*) can be derived from (T1). Inasmuch as paraconsistent logics stick to a standard conception of negation—and thus to Double Negation Elimination—Contraposition has to be given up. Contraposition as a principle of inference is not valid in Priest's logic LP (cf. Priest 1979), and should not be valid even in a realistic paraconsistent conception of truth (cf. Priest 2006, 70–71, 79–80; Bremer 2005, 185–190).

An axiom expressing seriality in normal modal logics may claim:

(T5*) $Tp \rightarrow \sim T\sim p$

This again can hold for realists if the theory is inconsistent and the truth operator is not bivalent, otherwise it fails: In case of antinomies we have a justification for ⌜p⌝ and we have a justification for ⌜$\sim p$⌝, and we may accept that both contradictory facts obtain, thus invalidating (T5*).

Similar reasoning applies to the analogue to *Brouwer's Axiom*

(T6*) $p \rightarrow T\sim T\sim p$

Any antinomy ⌜p⌝ makes the antecedent true but not the consequent if the truth operator is taken as bivalent. In consistent contexts (T6*) may, of course, be accepted—but we are looking for general principles of truth.

So far then the logic of a bivalent truth operator corresponds just to modal system T (with the truth operator having no dual).

More difficult is the assessment of

(T7) $Tp \rightarrow TTp$

which corresponds to the modal S4 axiom. If some statement is true is it then also true that it is true? (T7) is an instance of the otherwise rejected (T-RL). Even if (T-RL) is not valid for simple (first level) truths, could it hold for second or higher level truths?

Remember that although

$$p \rightarrow \Box p$$

does not hold in S4 we nevertheless have

$$\Box p \rightarrow \Box\Box p$$

Accepting (T7) depends on whether in case that we have that Tp obtains we also have a justification that it obtains. This need not be so, it seems. Some statement may be true without our *recognizing* this. Internal realism makes truth epistemic, but that does not mean that we recognize all truths. Something can be true (i.e. there is a justification that is feasible and available for us) without us having hit on its justification. On the other hand if there is some justification in principle available to us that $\ulcorner p \urcorner$ is true, *then* there should also be a further justification in principle available to us which states that because $\ulcorner p \urcorner$ is true (i.e. we have already justified that p and so have that Tp obtains) we also have to assume as justified that $\ulcorner Tp \urcorner$ is true, i.e. $\ulcorner TTp \urcorner$. Iterating on our verdict that we not only believe *p* to obtain, but have justified $\ulcorner p \urcorner$ should be harmless. Thus (T7) should hold.

Remember that *Brouwer's Axiom* does not follow from the S4 axiom, thus one can have the S4 axiom without *Brouwer's Axiom*.

Given that *Brouwer's Axiom* does follow from the S5 axiom an internally realistic conception of the truth operator should not accept

(T8*) $\sim T\sim p \rightarrow T\sim T\sim p$

or respectively (substituting $\ulcorner \sim p \urcorner$ for $\ulcorner p \urcorner$ and applying double negation elimination):

(T8*) $\sim Tp \rightarrow T\sim Tp$

And one need not accept (T8*) as internal realist. If $\ulcorner \sim p \urcorner$ or $\ulcorner p \urcorner$ is not true that can be *either* because the corresponding fact does not obtain *or* because the required justification is missing (not available *in principle*). That the required justification is missing need not be available *in principle* to us, we may just be and stay unsure whether we come forward with such a justification.

The situation is different from the situation with respect to (T7), since the presence of a justification is something quite different from its absence. This difference resembles the difference between a sentence being provable and a sentence being not provable. If a sentence is provable a Turing machine setting forth all proofs will sooner or later hit upon it, and then after a finite time one knows the sentence to be provable. If, however, the sentence is not provable no amount of not coming about its proof does ascertain that it has no proof. Non-justifiability resembles non-provability. And so (T8*) demands too much. No internal realist should accept it.

(In epistemic modal logics (T8*) claims *negative introspection* for the knowledge operator. And although this may hold for closed technical systems like data bases it certainly does not hold for human knowledge.)

Therefore the analogue to the S5 axiom should be rejected. The logic of the truth operator has to be weaker than S5.

Where exactly between S4 and S5 has the logic of truth to reside? There are several logics (and characterising axioms) between S4 and S5. The first one is S4.1, the characteristic axiom of which

$$\Box((\Box(p \to \Box p) \to p) \to (\Diamond \Box p \to p))$$

translates into

$$\text{(T9?)} \quad T(T(p \to Tp) \to p) \to ({\sim}T{\sim}Tp \to p)$$

It is hard to make sense out of this in terms of truth. If it is true that the truth of obtaining p bringing forth the truth of $\ulcorner p \urcorner$ has p obtaining, then it being not true that $\ulcorner p \urcorner$ is not true gives us p. In a realistic understanding of the truth operator, even if the truth operator is taken as bivalent, (T9?) comes out valid, even for antinomic $\ulcorner p \urcorner$. So the negative heuristics used above does not apply here. On the other hand does the consequent of (T9?) look far too strong for any statement. If we consider the justification aspect of truth in internal realism, then the absence of a justification for not having a justification of something does not yield this something. For the consequent to be false (only) $\ulcorner p \urcorner$ has to be false (only). By the correspondence aspect of truth then ${\sim}Tp$ is the case. From this, however, we do not get $\ulcorner T{\sim}Tp \urcorner$ since we need not have a justification of $\ulcorner {\sim}Tp \urcorner$ being principally available to us. The status of the consequent remains then unresolved for the internal realist. The supposed truth of the antecedent may be of little help in this case. One might argue that in case of $\ulcorner p \urcorner$ being false (only) and given our knowledge of the irrelevant cases of the material conditional we should have $\ulcorner T(p \to Tp) \urcorner$. So $\ulcorner T(p \to Tp) \to p \urcorner$ would be false (only), and thus the whole antecedent would be false. This then would allow for (T9?) itself to be true. (T9?) might be vacuously valid, since its antecedent is always false (only) for internalistic truth. But none of these reflections provide the principle (T9?) with an acceptable reading/interpretation in terms of internalistic truth. So (T9?) should not be taken as valid for the truth operator.

And at this level our search for further axioms might then stop. There is, however, a bifurcation in extending S4: one can have S4.1 or one may move to S4.2, the latter *not* containing S4.1, although both extend S4 and both are contained in S4.3.1 (cf. Hughes and Cresswell 1968, 261–264). The alternative next stage then would be S4.2, the characteristic axiom of which

$$\Diamond \Box p \rightarrow \Box \Diamond p$$

translates into

(T10*) $\sim T \sim Tp \rightarrow T \sim T \sim p$

If it is not true that ⌜p⌝ is not true, then it is true that ⌜$\sim p$⌝ is not true. The acceptance of (T10) seems to depend on some principle of exhaustion like *tertium non datur*. Can we come to grips with (T10)? If it is not true that p is not true, then ⌜p⌝ may be true, it seems; thus *being consistent* one may be led to conclude that it is true that ⌜$\sim p$⌝ is not true. Now, even in a realistic understanding of truth: if ⌜p⌝ is inconsistent and the truth operator is bivalent, then ⌜Tp⌝ is true, thus ⌜$\sim Tp$⌝ false (only), thus ⌜$T\sim Tp$⌝ false (only), thus the antecedent is true, while the consequent is false, since ⌜$T\sim p$⌝ is true, thus ⌜$\sim T\sim p$⌝ false (only), thus ⌜$T\sim T\sim p$⌝ false (only). Thus, given our heuristics, the internal realist should also reject (T10*). The logic of the truth operator certainly has to be weaker than S4.2. Given the obscurity of (T9?) it should be weaker than S4.1 as well.

The logic of the truth operator in internal realism, therefore, is S4.

5. A Semantics for Semantics

Of course *truth* is as central to semantics as *reference* or *meaning*. And an understanding of truth was presupposed all along. Nevertheless one may ask—in the manner of ordinary modal semantics—what semantics goes with the truth operator.

As a first answer to that we can now outline the accessibility restrictions that come with the outlined behaviour of the truth operator. Accessibility is reflexive and transitive.

As a second answer one may interpret accessibility here as *preserving what is fixed according to our best theories and the facts* while allowing matters we have no justified opinion on to fluctuate in truth value. Since truth in internal realism possesses the double nature of a correspondence *and* a justification aspect not just justifiability has to be preserved. The accessible worlds are *possible as far as we know*. Accessibility here resembles—as to be expected for internal realism or versions of anti-realism—accessibility in epistemic modal logics (cf. Meyer and van der Hoek 1995). Any world should be compatible with what the best theories in that world say, thus should be accessible to itself: reflexivity. And inasmuch as truth is closed under consequence and a justification for $\ulcorner p \urcorner$ is a justification for $\ulcorner Tp \urcorner$ accessibility should be transitive.

As is well known, for standard logics like PC the modal extension to S4 is deductive complete and correct with respect to the reflexive and transitive frames. In a paraconsistent setting one may take a logic like LP as the extensional base logic and extend it with (T1), (T2), (T3) and (T7). If one provides a modal semantics for this system, then the truth operator has to have a necessity-like truth condition (analogue to knowledge in epistemic modal logics). The non-triviality of LP is preserved and the reasoning behind the standard correctness proof still applies here.

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