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On Horadam finite operator hybrid numbers

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Abstract. In this paper, we introduce and study a new hybrid number sequence with Horadam numbers and a finite operator, called Horadam finite operator hybrid numbers. We derive recurrence relation, Binet-like formula, ordinary generating function, exponential generating function, Poisson generating function, and summation formula for Horadam finite operator hybrid numbers. Moreover, we give matrix representation and Cassini's identities for these numbers.

1. Introduction

The Horadam sequence $\{W_n(W_0, W_1; p, q)\}_{n\geq 0}$, or briefly $\{W_n\}_{n\geq 0}$, introduced by Horadam [4, 5] is defined recursively by the relation

$$W_{n+2} = pW_{n+1} + qW_n, \quad n \geq 0,$$

where $W_0 = r$, $W_1 = s$, and p, q, r, s are integers.

The generating function of the Horadam sequence $\{W_n\}_{n\geq 0}$ is

$$g(t) = \frac{W_0 + (W_1 - pW_0)t}{1 - pt - qt^2},$$

and the nth term of the Horadam sequence $\{W_n\}_{n\geq 0}$ is given by

$$W_n = \frac{\mathsf{A}\gamma^n - \mathsf{B}\delta^n}{\gamma - \delta},$$

where $\mathsf{A} = s - r\delta$, $\mathsf{B} = s - r\gamma$, and $\gamma = \dfrac{p + \sqrt{p^2 + 4q}}{2}$, $\delta = \dfrac{p - \sqrt{p^2 + 4q}}{2}$ are the roots of the equation $t^2 - pt - q = 0$.

The Horadam sequence $\{W_n\}_{n\geq 0}$ is a generalization of the familiar second-order recurrent sequences. Some of the special cases of this sequence are the following.

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- If we take $r = 0$, $s = 1$, $p = 1$, $q = 1$, then we get the Fibonacci sequence $\{F_n\}_{n\geq 0}$ (see [13]).
- If we take $r = 2$, $s = 1$, $p = 1$, $q = 1$, then we get the Lucas sequence $\{L_n\}_{n\geq 0}$ (see [13]).
- If we take $r = 0$, $s = 1$, $p = 2$, $q = 1$, then we get the Pell sequence $\{P_n\}_{n\geq 0}$ (see [6, 14]).
- If we take $r = 2$, $s = 2$, $p = 2$, $q = 1$, then we get the Pell-Lucas sequence $\{Q_n\}_{n\geq 0}$ (see [7, 14]).
- If we take $r = 0$, $s = 1$, $p = 1$, $q = 2$, then we get the Jacobsthal sequence $\{J_n\}_{n\geq 0}$ (see [7, 8]).
- If we take $r = 2$, $s = 1$, $p = 1$, $q = 2$, then we get the Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$ (see [8]).

Finite difference operators [1, 32], among them the forward difference operator, the backward difference operator, the central difference operator, and the means operator are useful for various interesting applications in mathematics, and engineering. Recently, a new operator, introduced by Şimşek [27], is defined as

$$\mathbb{Y}_{\lambda,\beta}[f; a, b](x) = \lambda E^a[f](x) + \beta E^b[f](x),$$

where a, b are real parameters, λ, β are real or complex parameters, and $E^a[f](x) = f(x+a)$. Taking some special values of this new operator, Şimşek obtained different operators including the forward difference operator, the backward difference operator, the central difference operator, and the Gould operator.

Then for any sequence $\{x_n\}_{n\geq 0}$ and $i \geq 1$, Kızılateş [10] defined the i-th finite operator as

$$\Delta^{(i)}_{\lambda,\beta;a,b}(x_n) = \lambda\Delta^{(i-1)}_{\lambda,\beta;a,b}(x_{n+a}) + \beta\Delta^{(i-1)}_{\lambda,\beta;a,b}(x_{n+b}),$$

where a, b are integers, and λ, β are real parameters. The i-th finite operator $\Delta^{(i)}_{\lambda,\beta;a,b}(x_n)$ can be expressed shortly as $x^{(i)}_n$, and for $i = 1$, $\Delta^{(1)}_{\lambda,\beta;a,b}(x_n) = x^{(1)}_n = \lambda(x_{n+a}) + \beta(x_{n+b})$.

For some special values of λ, β, a and b, it was shown that the finite operator $\Delta^{(1)}_{\lambda,\beta;a,b}(x_n)$ turns into the identity operator, the forward difference operator, the backward difference operator, the means operator, and the Gould operator.

In the same study, Kızılateş applied the finite operator to the Horadam sequence. The i-th finite operator of $\{W_n\}_{n\geq 0}$, called Horadam finite operator sequence, is defined by

$$\Delta^{(i)}_{\lambda,\beta;a,b}(W_n) = W^{(i)}_n = \lambda\Delta^{(i-1)}_{\lambda,\beta;a,b}(W_{n+a}) + \beta\Delta^{(i-1)}_{\lambda,\beta;a,b}(W_{n+b}),$$

where a, b are integers, and λ, β are real parameters.

Furthermore, Kızılateş [10] obtained some special cases of $\Delta^{(1)}_{\lambda,\beta;a,b}(W_n) = W_n^{(1)} = \lambda(W_{n+a}) + \beta(W_{n+b})$ as follows.

- If we take $\lambda = 1$, $\beta = a = b = 0$, then we get the Horadam identity operator sequence $I(W_n^{(1)}) = W_n$.
- If we take $\lambda = 1$, $\beta = -1$, $a = 1$, $b = 0$, then we get the Horadam forward difference sequence $\Delta(W_n^{(1)}) = W_{n+1} - W_n$.
- If we take $\lambda = 1$, $\beta = -1$, $a = 0$, $b = -1$, then we get the Horadam backward difference sequence $\nabla(W_n^{(1)}) = W_n - W_{n-1}$.
- If we take $\lambda = \frac{1}{2}$, $\beta = \frac{-1}{2}$, $a = 1$, $b = 0$, then we get the Horadam means operator sequence $M(W_n^{(1)}) = \frac{1}{2}(W_{n+1} - W_n)$.
- If we take $\lambda = 1$, $\beta = -1$ and substitute $a \to a + b$, $b \to a$, $(ab \neq 0)$, then we get the Horadam-Gould operator sequence $G_{ab}(W_n^{(1)}) = W_{n+a+b} - W_{n+a}$.

Moreover, Kızılateş gave the recurrence relation, and the Binet-like formula for the Horadam finite operator sequences as

$$W_{n+2}{}^{(i)} = pW_{n+1}{}^{(i)} + qW_n{}^{(i)}, n \geq 0, \tag{1}$$

and for initial values $W_0{}^{(i)}, W_1{}^{(i)}$,

$$W_n{}^{(i)} = \frac{W_1{}^{(i)}(\gamma^n - \delta^n) + qW_0{}^{(i)}(\gamma^{n-1} - \delta^{n-1})}{\gamma - \delta}, \tag{2}$$

respectively. For further information on Horadam finite operator numbers, we refer to [10].

In [29], Terzioğlu et al. defined the Fibonacci finite operator quaternions with Fibonacci numbers and finite operators. In [19], Polatlı applied the finite operator to (p, q)-Fibonacci polynomials.

The complex, hyperbolic and dual numbers, which are the examples of two-dimensional number systems, have found many applications in science and engineering. Özdemir [17] defined and studied the hybrid number system which can be considered as a generalization of the complex, hyperbolic and dual number systems. The set of hybrid numbers is defined as

$$\mathbb{K} = \{z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} : a, b, c, d \in \mathbb{R}, \mathbf{i}^2 = -1, \varepsilon^2 = 0, \mathbf{h}^2 = 1, \\ \mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}\},$$

where a is called the scalar part and $b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is called the vector part.

From the relation $\mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}$, the multiplication table for the hybrid units $\mathbf{i}$, ε, $\mathbf{h}$ can be obtained as follows:

$\cdot$	1	$\mathbf{i}$	ε	$\mathbf{h}$
1	1	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	-1	$1-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	ε	$\mathbf{h}+\mathbf{1}$	0	$-\varepsilon$
$\mathbf{h}$	$\mathbf{h}$	$-\varepsilon-\mathbf{i}$	ε	1

The addition and multiplication of two hybrid numbers are defined in a natural way. Given $z_1 = a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}$ and $z_2 = a_2 + b_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h}$ in $\mathbb{K}$, we have

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\varepsilon + (d_1 + d_2)\mathbf{h},$$

$$\begin{aligned} z_1 \times z_2 = {} & (a_1a_2 - b_1b_2 + b_1c_2 + c_1b_2 + d_1d_2) + (a_1b_2 + b_1a_2 + b_1d_2 - d_1b_2)\mathbf{i} \\ & + (a_1c_2 + b_1d_2 - d_1b_2 + c_1a_2 - c_1d_2 + d_1c_2)\varepsilon \\ & + (a_1d_2 + d_1a_2 - b_1c_2 + c_1b_2)\mathbf{h}. \end{aligned}$$

Note that the addition operation in the hybrid numbers is commutative and associative, the multiplication operation in the hybrid numbers is associative but not commutative. For further information, we refer to [17].

Several papers related to the hybrid numbers and their generalizations with different integer sequences can be found in [2, 3, 9, 11, 12, 15, 18, 20-26, 28, 30], among others.

Motivated by the above papers, in the present study, a new hybrid number sequence with Horadam finite operator sequence, called Horadam finite operator hybrid numbers, is introduced. Furthermore, some properties including recurrence relation, Binet-like formula, ordinary generating function, exponential generating function, and Poisson generating function for these numbers are given. Finally, Cassini identities involving these hybrid numbers are derived with the help of a matrix representation.

2. The Horadam finite operator hybrid numbers

In this section, we first define Horadam finite operator hybrid numbers. Then we give some properties of these numbers.

Definition 1. The Horadam finite operator hybrid numbers ${HW_n}^{(i)}$ are defined by

$$ {HW_n}^{(i)} = {W_n}^{(i)} + {W_{n+1}}^{(i)}\mathbf{i} + {W_{n+2}}^{(i)}\varepsilon + {W_{n+3}}^{(i)}\mathbf{h}, \tag{3}$$

where ${W_n}^{(i)} = \Delta^{(i)}_{\lambda,\beta;a,b}(W_n)$ is the i-th Horadam finite operator number and $\mathbf{i}, \varepsilon, \mathbf{h}$ are hybrid units.

For $i = 1$, we get

$$\begin{aligned} {HW_n}^{(1)} &= {W_n}^{(1)} + {W_{n+1}}^{(1)}\mathbf{i} + {W_{n+2}}^{(1)}\varepsilon + {W_{n+3}}^{(1)}\mathbf{h} \\ &= \lambda W_{n+a} + \beta W_{n+b} + (\lambda W_{n+1+a} + \beta W_{n+1+b})\mathbf{i} \\ &\quad + (\lambda W_{n+2+a} + \beta W_{n+2+b})\varepsilon + (\lambda W_{n+3+a} + \beta W_{n+3+b})\mathbf{h}. \end{aligned}$$

Some special cases of ${HW_n}^{(1)}$ are the following.

- If we take $\lambda = 1$, $\beta = a = b = 0$, then we get the identity operator for Horadam hybrid numbers $I({HW_n}^{(1)}) = HW_n$.
- If we take $\lambda = 1$, $\beta = -1$, $a = 1$, $b = 0$, then we get the forward difference operator for Horadam hybrid numbers $\Delta({HW_n}^{(1)}) = HW_{n+1} - HW_n$.
- If we take $\lambda = 1$, $\beta = -1$, $a = 0$, $b = -1$, then we get the backward difference operator for Horadam hybrid numbers $\nabla({HW_n}^{(1)}) = HW_n - HW_{n-1}$.
- If we take $\lambda = \frac{1}{2}$, $\beta = \frac{-1}{2}$, $a = 1$, $b = 0$, then we get the means operator for Horadam hybrid numbers $M({HW_n}^{(1)}) = \frac{1}{2}(HW_{n+1} - HW_n)$.
- If we take $\lambda = 1$, $\beta = -1$ and substitute $a \to a + b$, $b \to a$, $(ab \neq 0)$ then we get the Gould operator for Horadam hybrid numbers $G_{ab}({HW_n}^{(1)}) = HW_{n+a+b} - HW_{n+a}$.

For further information on Horadam hybrid numbers, we refer to [21, 26].

Theorem 1. *The recurrence relation for the Horadam finite operator hybrid numbers is given by*

$$ {HW_{n+2}}^{(i)} = p{HW_{n+1}}^{(i)} + q{HW_n}^{(i)}, n \geq 0. \tag{4} $$

Proof. By virtue of the equations (1) and (3), we get

$$ \begin{aligned} p{HW_{n+1}}^{(i)} + q{HW_n}^{(i)} &= p({W_{n+1}}^{(i)} + {W_{n+2}}^{(i)}\mathbf{i} + {W_{n+3}}^{(i)}\varepsilon + {W_{n+4}}^{(i)}\mathbf{h}) \\ &\quad + q({W_n}^{(i)} + {W_{n+1}}^{(i)}\mathbf{i} + {W_{n+2}}^{(i)}\varepsilon + {W_{n+3}}^{(i)}\mathbf{h}) \\ &= (p{W_{n+1}}^{(i)} + q{W_n}^{(i)}) + (p{W_{n+2}}^{(i)} + q{W_{n+1}}^{(i)})\mathbf{i} \\ &\quad + (p{W_{n+3}}^{(i)} + q{W_{n+2}}^{(i)})\varepsilon + (p{W_{n+4}}^{(i)} + q{W_{n+3}}^{(i)})\mathbf{h} \\ &= {W_{n+2}}^{(i)} + {W_{n+3}}^{(i)}\mathbf{i} + {W_{n+4}}^{(i)}\varepsilon + {W_{n+5}}^{(i)}\mathbf{h} \\ &= {HW_{n+2}}^{(i)}. \end{aligned} $$

□

Theorem 2. *The Binet-like formula for the Horadam finite operator hybrid numbers is given by*

$$ HW_n^{(i)} = \frac{\gamma^{n-1}\gamma^*(q{W_0}^{(i)} + \gamma{W_1}^{(i)}) - \delta^{n-1}\delta^*(q{W_0}^{(i)} + \delta{W_1}^{(i)})}{\gamma - \delta}, \tag{5} $$

where $\gamma^* = 1 + \gamma\mathbf{i} + \gamma^2\varepsilon + \gamma^3\mathbf{h}$ *and* $\delta^* = 1 + \delta\mathbf{i} + \delta^2\varepsilon + \delta^3\mathbf{h}$.

Proof. By virtue of the equations (2) and (3), we have

$$\begin{aligned}
HW_n{}^{(i)} &= W_n{}^{(i)} + W_{n+1}{}^{(i)}\mathbf{i} + W_{n+2}{}^{(i)}\varepsilon + W_{n+3}{}^{(i)}\mathbf{h} \\
&= \frac{W_1{}^{(i)}(\gamma^n - \delta^n) + qW_0{}^{(i)}(\gamma^{n-1} - \delta^{n-1})}{\gamma - \delta} \\
&\quad + \frac{W_1{}^{(i)}(\gamma^{n+1} - \delta^{n+1}) + qW_0{}^{(i)}(\gamma^n - \delta^n)}{\gamma - \delta}\mathbf{i} \\
&\quad + \frac{W_1{}^{(i)}(\gamma^{n+2} - \delta^{n+2}) + qW_0{}^{(i)}(\gamma^{n+1} - \delta^{n+1})}{\gamma - \delta}\varepsilon \\
&\quad + \frac{W_1{}^{(i)}(\gamma^{n+3} - \delta^{n+3}) + qW_0{}^{(i)}(\gamma^{n+2} - \delta^{n+2})}{\gamma - \delta}\mathbf{h} \\
&= \frac{W_1{}^{(i)}\gamma^n(1 + \gamma\mathbf{i} + \gamma^2\varepsilon + \gamma^3\mathbf{h}) - W_1{}^{(i)}\delta^n(1 + \delta\mathbf{i} + \delta^2\varepsilon + \delta^3\mathbf{h})}{\gamma - \delta} \\
&\quad + \frac{qW_0{}^{(i)}\gamma^{n-1}(1+\gamma\mathbf{i}+\gamma^2\varepsilon+\gamma^3\mathbf{h}) - qW_0{}^{(i)}\delta^{n-1}(1+\delta\mathbf{i}+\delta^2\varepsilon+\delta^3\mathbf{h})}{\gamma - \delta} \\
&= \frac{W_1{}^{(i)}(\gamma^n\gamma^* - \delta^n\delta^*) + qW_0{}^{(i)}(\gamma^{n-1}\gamma^* - \delta^{n-1}\delta^*)}{\gamma - \delta} \\
&= \frac{\gamma^{n-1}\gamma^*(qW_0{}^{(i)} + \gamma W_1{}^{(i)}) - \delta^{n-1}\delta^*(qW_0{}^{(i)} + \delta W_1{}^{(i)})}{\gamma - \delta},
\end{aligned}$$

which completes the proof. □

Theorem 3. *The ordinary generating function for the Horadam finite operator hybrid numbers is given by*

$$g(t) = \frac{HW_0{}^{(i)} + (HW_1{}^{(i)} - pHW_0{}^{(i)})t}{1 - pt - qt^2}.$$

Proof. Let $g(t) = \sum_{n=0}^{\infty} HW_n{}^{(i)}t^n$ be the generating function of the Horadam finite operator hybrid numbers. Then we write

$$g(t) = HW_0{}^{(i)} + HW_1{}^{(i)}t + HW_2{}^{(i)}t^2 + ... + HW_n{}^{(i)}t^n + ... \quad . \tag{6}$$

Multiplying the equation (6) by $-pt$ and $-qt^2$ we have, respectively,

$$-ptg(t) = -pHW_0{}^{(i)}t - pHW_1{}^{(i)}t^2 - pHW_2{}^{(i)}t^3 - ... - pHW_{n-1}{}^{(i)}t^n - ... \tag{7}$$

and

$$-qt^2g(t) = -qHW_0{}^{(i)}t^2 - qHW_1{}^{(i)}t^3 - qHW_2{}^{(i)}t^4 - ... - qHW_{n-2}{}^{(i)}t^n - ... \quad . \tag{8}$$

From the equations (6), (7) and (8), we get

$$\begin{aligned}(1-pt-qt^2)g(t) &= HW_0{}^{(i)} + (HW_1{}^{(i)} - pHW_0{}^{(i)})t \\ &\quad + \sum_{n=2}^{\infty}(HW_n{}^{(i)} - pHW_{n-1}{}^{(i)} - qHW_{n-2}{}^{(i)})t^n.\end{aligned}$$

Using the equation (4), we obtain

$$(1-pt-qt^2)g(t) = HW_0{}^{(i)} + (HW_1{}^{(i)} - pHW_0{}^{(i)})t.$$

Thus, the proof is completed. □

Theorem 4. *The exponential generating function for the Horadam finite operator hybrid numbers is given by*

$$G(t) = \frac{\gamma^*(W_1{}^{(i)} - \delta W_0{}^{(i)})e^{\gamma t} - \delta^*(W_1{}^{(i)} - \gamma W_0{}^{(i)})e^{\delta t}}{\gamma - \delta}.$$

Proof. By virtue of the equation (5), we have

$$\begin{aligned}G(t) &= \sum_{n=0}^{\infty} HW_n{}^{(i)}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{\gamma^{n-1}\gamma^*(qW_0{}^{(i)} + \gamma W_1{}^{(i)}) - \delta^{n-1}\delta^*(qW_0{}^{(i)} + \delta W_1{}^{(i)})}{\gamma-\delta}\frac{t^n}{n!} \\ &= \frac{\gamma^*(qW_0{}^{(i)} + \gamma W_1{}^{(i)})}{\gamma(\gamma-\delta)}\sum_{n=0}^{\infty}\frac{(\gamma t)^n}{n!} - \frac{\delta^*(qW_0{}^{(i)} + \delta W_1{}^{(i)})}{\delta(\gamma-\delta)}\sum_{n=0}^{\infty}\frac{(\delta t)^n}{n!} \\ &= \frac{\gamma^*(\delta q W_0{}^{(i)} + \delta\gamma W_1{}^{(i)})e^{\gamma t} - \delta^*(\gamma q W_0{}^{(i)} + \gamma\delta W_1{}^{(i)})e^{\delta t}}{\gamma\delta(\gamma-\delta)}.\end{aligned}$$

Considering the fact that $\gamma\delta = \delta\gamma = -q$, we obtain the desired result. □

Since $\mathcal{G}(t) = e^{-t}G(t)$, where $\mathcal{G}(t)$ denotes the Poisson generating function, and $G(t)$ denotes the exponential generating function, we now give the following result without proof.

Corollary 1. *The Poisson generating function for the Horadam finite operator hybrid numbers is given by*

$$\mathcal{G}(t) = \frac{\gamma^*(W_1{}^{(i)} - \delta W_0{}^{(i)})e^{\gamma t} - \delta^*(W_1{}^{(i)} - \gamma W_0{}^{(i)})e^{\delta t}}{e^t(\gamma - \delta)}.$$

In [16], Liu obtained a summation formula for the generalized k-Horadam sequence (see [31] for the generalized k-Horadam sequence). In much the same way that Liu ([16], Lemma 5) did for the generalized k-Horadam sequence, we prove the summation formula for the Horadam finite operator sequences in the following lemma.

Lemma 1. *Let $\{W_n{}^{(i)}\}$ be the Horadam finite operator sequences. Then we have*

$$\sum_{k=0}^{n} W_k{}^{(i)} = \begin{cases} \frac{W_{n+1}{}^{(i)} + qW_n{}^{(i)} + (p-1)W_0{}^{(i)} - W_1{}^{(i)}}{p+q-1}, & \text{if } p+q \neq 1, \\ \frac{qW_n{}^{(i)} + n(W_1{}^{(i)} + qW_0{}^{(i)}) + W_0{}^{(i)}}{1+q}, & \text{if } p+q = 1,\ q \neq -1. \end{cases}$$

Proof. Let $p+q \neq 1$. From the equation (1), we have

$$\begin{aligned} \sum_{k=0}^{n} W_k{}^{(i)} &= p\sum_{k=0}^{n} W_{k-1}{}^{(i)} + q\sum_{k=0}^{n} W_{k-2}{}^{(i)} \\ &= p\sum_{k=0}^{n} W_k{}^{(i)} + q\sum_{k=0}^{n} W_k{}^{(i)} - W_{n+1}{}^{(i)} - qW_n{}^{(i)} + W_0{}^{(i)} + qW_{-1}{}^{(i)} \\ &= p\sum_{k=0}^{n} W_k{}^{(i)} + q\sum_{k=0}^{n} W_k{}^{(i)} - W_{n+1}{}^{(i)} - qW_n{}^{(i)} + W_1{}^{(i)} + (1-p)W_0{}^{(i)}. \end{aligned}$$

So, we get

$$(p+q-1)\sum_{k=0}^{n} W_k{}^{(i)} = W_{n+1}{}^{(i)} + qW_n{}^{(i)} + (p-1)W_0{}^{(i)} - W_1{}^{(i)},$$

which completes the proof of the first case.

Let $p+q = 1$ and $q \neq -1$. We first show that $W_{n+1}{}^{(i)} + qW_n{}^{(i)} \equiv W_1{}^{(i)} + qW_0{}^{(i)}$. Let $A_n{}^{(i)} = W_{n+1}{}^{(i)} + qW_n{}^{(i)}$. It is clear that $A_0{}^{(i)} = W_1{}^{(i)} + qW_0{}^{(i)}$. Using the equation (1), and considering the fact that $p+q = 1$, we get

$$\begin{aligned} A_n{}^{(i)} &= W_{n+1}{}^{(i)} + qW_n{}^{(i)} \\ &= (pW_n{}^{(i)} + qW_{n-1}{}^{(i)}) + qW_n{}^{(i)} \\ &= (p+q)W_n{}^{(i)} + qW_n{}^{(i)} \\ &= W_n{}^{(i)} + qW_{n-1}{}^{(i)} \\ &= A_{n-1}{}^{(i)}. \end{aligned}$$

This shows that the sequence $\{A_n{}^{(i)}\}$ is a constant sequence. Hence $A_n{}^{(i)} = A_{n-1}{}^{(i)} = A_0{}^{(i)}$. Thus we have

$$\sum_{k=0}^{n} W_{k+1}{}^{(i)} + q\sum_{k=0}^{n} W_k{}^{(i)} = (n+1)(W_1{}^{(i)} + qW_0{}^{(i)}).$$

Then we get

$$(\sum_{k=0}^{n} W_k{}^{(i)} + W_{n+1}{}^{(i)} - W_0{}^{(i)}) + q\sum_{k=0}^{n} W_k{}^{(i)} = (n+1)(W_1{}^{(i)} + qW_0{}^{(i)}).$$

Therefore, we obtain

$$
\begin{aligned}
(1+q)\sum_{k=0}^{n} W_k{}^{(i)} &= -W_{n+1}{}^{(i)} + (n+1)(W_1{}^{(i)} + qW_0{}^{(i)}) + W_0{}^{(i)} \\
&= qW_n{}^{(i)} + n(W_1{}^{(i)} + qW_0{}^{(i)}) + W_0{}^{(i)}.
\end{aligned}
$$

So, we have

$$
\sum_{k=0}^{n} W_k{}^{(i)} = \frac{qW_n{}^{(i)} + n(W_1{}^{(i)} + qW_0{}^{(i)}) + W_0{}^{(i)}}{1+q}.
$$

This completes the proof of the second case. □

Theorem 5. *Let $\{HW_n{}^{(i)}\}$ be the Horadam finite operator hybrid sequences. Then we have*

$$
\sum_{k=0}^{n} HW_k{}^{(i)} = \begin{cases}
\frac{W_{n+1}{}^{(i)} + qW_n{}^{(i)} + (p-1)W_0{}^{(i)} - W_1{}^{(i)}}{p+q-1}(1+\mathbf{i}+\varepsilon+\mathbf{h}) \\
\quad +(W_{n+1}{}^{(i)} - W_0{}^{(i)})(\mathbf{i}+\varepsilon+\mathbf{h}) \\
\quad +(W_{n+2}{}^{(i)} - W_1{}^{(i)})(\varepsilon+\mathbf{h}) + (W_{n+3}{}^{(i)} - W_2{}^{(i)})\mathbf{h}, \\
\\
\qquad\qquad \textit{if } p+q \neq 1, \\
\\
\frac{qW_n{}^{(i)} + n(W_1{}^{(i)} + qW_0{}^{(i)}) + W_0{}^{(i)}}{1+q}(1+\mathbf{i}+\varepsilon+\mathbf{h}) \\
\quad +(W_{n+1}{}^{(i)} - W_0{}^{(i)})(\mathbf{i}+\varepsilon+\mathbf{h}) \\
\quad +(W_{n+2}{}^{(i)} - W_1{}^{(i)})(\varepsilon+\mathbf{h}) + (W_{n+3}{}^{(i)} - W_2{}^{(i)})\mathbf{h}, \\
\\
\qquad\qquad \textit{if } p+q = 1,\ q \neq -1.
\end{cases}
$$

Proof. We will just prove the first case, since the second case can be handled in a similar manner. Let $p+q \neq 1$. By virtue of the equation (3), and the summation formula for the Horadam finite operator numbers, we have

$$\begin{aligned}
&\sum_{k=0}^{n} HW_k{}^{(i)} \\
&= \sum_{k=0}^{n} W_k{}^{(i)} + (\sum_{k=0}^{n} W_{k+1}{}^{(i)})\mathbf{i} + (\sum_{k=0}^{n} W_{k+2}{}^{(i)})\varepsilon + (\sum_{k=0}^{n} W_{k+3}{}^{(i)})\mathbf{h} \\
&= \sum_{k=0}^{n} W_k{}^{(i)} + (\sum_{k=0}^{n} W_k{}^{(i)} + W_{n+1}{}^{(i)} - W_0{}^{(i)})\mathbf{i} \\
&\quad + (\sum_{k=0}^{n} W_k{}^{(i)} + W_{n+1}{}^{(i)} + W_{n+2}{}^{(i)} - W_0{}^{(i)} - W_1{}^{(i)})\varepsilon \\
&\quad + (\sum_{k=0}^{n} W_k{}^{(i)} + W_{n+1}{}^{(i)} + W_{n+2}{}^{(i)} + W_{n+3}{}^{(i)} - W_0{}^{(i)} - W_1{}^{(i)} - W_2{}^{(i)})\mathbf{h} \\
&= \frac{W_{n+1}{}^{(i)} + qW_n{}^{(i)} + (p-1)W_0{}^{(i)} - W_1{}^{(i)}}{p+q-1}(1+\mathbf{i}+\varepsilon+\mathbf{h}) \\
&\quad + (W_{n+1}{}^{(i)} - W_0{}^{(i)})(\mathbf{i}+\varepsilon+\mathbf{h}) + (W_{n+2}{}^{(i)} - W_1{}^{(i)})(\varepsilon+\mathbf{h}) \\
&\quad + (W_{n+3}{}^{(i)} - W_2{}^{(i)})\mathbf{h}.
\end{aligned}$$

The proof is completed. □

Let us define two 2×2 matrices $\mathbf{A}$ and $\mathbf{H}$ as follows:

$$\mathbf{A} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} HW_2{}^{(i)} & qHW_1{}^{(i)} \\ HW_1{}^{(i)} & qHW_0{}^{(i)} \end{pmatrix}.$$

Theorem 6. *For $n \geq 1$, we have*

$$\mathbf{HA^{n-1}} = \begin{pmatrix} HW_{n+1}{}^{(i)} & qHW_n{}^{(i)} \\ HW_n{}^{(i)} & qHW_{n-1}{}^{(i)} \end{pmatrix}.$$

Proof. For the proof, we use the induction method on n. For $n = 1$, the proof is trivial. We assume that our assertion is true for n. For $n + 1$, by the aid of matrix multiplication and by using the equation (4), we have

$$\mathbf{HA^{n}} = \begin{pmatrix} HW_{n+1}{}^{(i)} & qHW_n{}^{(i)} \\ HW_n{}^{(i)} & qHW_{n-1}{}^{(i)} \end{pmatrix} \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} HW_{n+2}{}^{(i)} & qHW_{n+1}{}^{(i)} \\ HW_{n+1}{}^{(i)} & qHW_n{}^{(i)} \end{pmatrix}.$$

This completes the proof. □

Since the multiplication operation in the hybrid numbers is not commutative, we give two Cassini's identities for the Horadam finite operator hybrid numbers in the following result.

Theorem 7. *Let n be a positive integer. Then we have*

$$HW_{n+1}{}^{(i)}HW_{n-1}{}^{(i)} - (HW_n{}^{(i)})^2 = (-q)^{n-1}(HW_2{}^{(i)}HW_0{}^{(i)} - (HW_1{}^{(i)})^2),$$
$$HW_{n-1}{}^{(i)}HW_{n+1}{}^{(i)} - (HW_n{}^{(i)})^2 = (-q)^{n-1}(HW_0{}^{(i)}HW_2{}^{(i)} - (HW_1{}^{(i)})^2).$$

Proof. The proof of the theorem becomes straightforward by taking determinants of both sides of the equation in Theorem 6, respectively. □

3. Conclusion

In this study, the sequence of Horadam finite operator hybrid numbers is introduced. Horadam finite operator hybrid numbers are hybrid numbers with Horadam finite operator coefficients. Several properties involving these hybrid numbers including the recurrence relation, Binet-like formula, and generating functions are investigated. Furthermore, the summation formulas of the Horadam finite operator numbers and Horadam finite operator hybrid numbers are presented, respectively. With the help of a matrix representation of Horadam finite operator hybrid numbers, Cassini's identities involving these numbers are obtained.

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