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A generalization of mg-closed sets in hereditary m-spaces

Ahmad Al-Omari and Takashi Noiri

Abstract. In this paper, we introduce the notion of $mg\mathcal{H}$-closed sets in a hereditary m-space $(X, m, \mathcal{H})$ and obtain a further generalization of mg-closed sets. We investigate basic properties, characterizations and preservation properties of $mg\mathcal{H}$-closed sets.

1. Introduction

In 1970, Levine [16] introduced the notion of generalized closed (briefly g-closed) sets in topological spaces. Since then, many variations of g-closed sets have been introduced and investigated. As an application of these sets, many low separation axioms have been introduced. Among them, $T_{3/4}$-spaces due to Dontchev and Ganster [13] are useful. They showed that the digital line lies between a T_1-space and a $T_{3/4}$-space.

The notion of ideals in topological spaces was introduced by Kuratowski [15]. Janković and Hamlett [14] defined the local function on an ideal topological space $(X, \tau, \mathcal{I})$. By using it they obtained a new topology $\tau^\star$ for X and investigated relations between τ and $\tau^\star$. In [14, 15, 23], further properties of ideals on a topological space are obtained.

A subfamily μ of the power set $\mathcal{P}(X)$ on a nonempty set X is called a generalized topology (briefly GT) [12] if $\emptyset \in \mu$ and any union of elements of μ belongs to μ. Császár [11] defined a hereditary class $\mathcal{H}$ which is weaker than an ideal and constructed a new GT $\mu^\star$ from a GT μ and a hereditary class $\mathcal{H}$. Moreover he showed that many properties related to τ and $\tau^\star$ remain valid (possibly with small modifications) for μ and $\mu^\star$.

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Corresponding author: Ahmad Al-Omari

In [18], the author introduced and investigated a unified notion of many generalizations of g-closed sets by using mg-closed sets in an m-space (X, m).

In [19], Noiri and Popa introduced the minimal local function on a minimal space (X, m) with a hereditary class $\mathcal{H}$ and constructed a minimal structure $m^\star_H$ which contains m. They showed that many properties related to τ and $\tau^\star$ (or μ and $\mu^\star$) remain similarly valid on m and $m^\star_H$.

In [5], Al-Omari and Noiri investigated relationships between a minimal structure m and a hereditary class $\mathcal{H}$. They defined an operator, called $\Gamma^\star_{mH}$, on a heredatary minimal space $(X, m, \mathcal{H})$. Also they investigated a minimal structure m which is said to be m-compatible with a hereditary class $\mathcal{H}$. Several characterizations of minimal structures with the notion of a hereditary class were provided in [1, 2, 3, 4, 6, 7, 8, 11].

In this paper, we introduce the notion of $mg\mathcal{H}$-closed sets in a hereditary m-space $(X, m, \mathcal{H})$ and obtain a further generalization of mg-closed sets. We investigate basic properties, characterizations and preservation properties of $mg\mathcal{H}$-closed sets.

2. m-structures

Definition 1. A subfamily m of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) [21] on X if $\emptyset \in m$ and $X \in m$.

By (X, m), we denote a nonempty set X with a minimal structure m on X and call it an *m-space*. Each member of m is said to be *m-open* and the complement of an m-open set is said to be *m-closed*.

Definition 2. Let (X, m) be an m-space. For a subset A of X, the *m-closure* of A and the *m-interior* of A are defined in [17] as follows:
(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X \setminus F \in m\}$,
(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m\}$.

Lemma 1 ([17]). *Let (X, m) be an m-space. For subsets A and B of X, the following properties hold:*
(1) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mCl}(A) = A$ *if A is m-closed,*
(2) $\mathrm{mInt}(A) \subset A$ *and* $\mathrm{mInt}(A) = A$ *if A is m-open,*
(3) *if $A \subset B$, then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$,
(4) $\mathrm{mCl}(A) \cup \mathrm{mCl}(B) \subset \mathrm{mCl}(A \cup B)$,
(5) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$.

Remark 1. The converse of (2) in Lemma 1 is not true as the following simple example shows.

Example 1. Let $X = \{a, b, c\}$, $m = \{\emptyset, X, \{a\}, \{b\}\}$ and $A = \{a, b\}$. Then $\mathrm{mInt}(A) = \cup\{U \in m : U \subset A\} = \{a, b\}$. Hence $\mathrm{mInt}(A) = A$ but A is not m-open.

Definition 3. A minimal structure m on a nonempty set X is said to have the *property* $(\mathcal{B})$ [17] if the union of any family of subsets belonging to m belongs to m.

Lemma 2 ([22])**.** *Let X be a nonempty set and m a minimal structure on X satisfying property $(\mathcal{B})$. For a subset A of X, the following properties hold:*
(1) $A \in m$ *if and only if* $\mathrm{mInt}(A) = A$,
(2) A *is m-closed if and only if* $\mathrm{mCl}(A) = A$,
(3) $\mathrm{mInt}(A) \in m$ *and* $\mathrm{mCl}(A)$ *is m-closed.*

3. $mg\mathcal{H}$-closed sets

A nonempty subfamily $\mathcal{H}$ of $\mathcal{P}(X)$ is called a *hereditary class* on X [11] if it satisfies the following property: $A \in \mathcal{H}$ and $B \subset A$ implies $B \in \mathcal{H}$. A hereditary class $\mathcal{H}$ is called an *ideal* if it satisfies the additional condition: $A \in \mathcal{H}$ and $B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$. The notion of ideals has been introduced in [15] and [23] and further investigated in [14]. An m-space (X, m) with a hereditary class $\mathcal{H}$ on X is called a *hereditary m-space* and is denoted by $(X, m, \mathcal{H})$.

Definition 4. Let $(X, m, \mathcal{H})$ be a hereditary m-space. A subset A of X is said to be *$mg\mathcal{H}$-closed* (resp. *mg-closed* [18]) if $\mathrm{mCl}(A) \setminus U \in \mathcal{H}$ (resp. $\mathrm{mCl}(A) \subset U$) whenever $A \subset U$ and $U \in m$.

Remark 2. Let $(X, m, \mathcal{H})$ be a hereditary m-space.
(1) Let $\mathcal{H} = \{\emptyset\}$, then every $mg\mathcal{H}$-closed set is mg-closed.
(2) We have the following implications:
m-closed $\Rightarrow$ mg-closed $\Rightarrow$ $mg\mathcal{H}$-closed.
The converses of the above implications are not necessary true as shown by the following examples.

Example 2. Let $X = \{a, b, c\}$, $m = \{\emptyset, X, \{a\}\}$, and $A = \{b\}$. Then only X is an m-open set containing A. Hence $\mathrm{mCl}(A) = \{b, c\} \subset X$ but $\mathrm{mCl}(A) \neq A$. Therefore A is mg-closed but not m-closed.

Example 3. Let $X = \{a, b, c\}$, $m = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $A = \{a\}$ and $\mathcal{H} = \{\emptyset, \{c\}\}$. Then $(X, m, \mathcal{H})$ is a hereditary m-space.
(1) A is not mg-closed. Let $U = \{a\}$, then $A \subset U$ and $\mathrm{mCl}(\{a\}) = \{a, c\}$. Therefore $\mathrm{mCl}(A)$ is not contained in U and A is not mg-closed.
(2) A is $mg\mathcal{H}$-closed. Let $A \subset U$ and $U \in m$.
(i) Let $U = \{a\}$, then $A \subset U$ and $\mathrm{mCl}(A) \setminus U = \{a, c\} \setminus \{a\} = \{c\} \in \mathcal{H}$.
(ii) Let $U = \{a, b\}$, then $A \subset U$ and $\mathrm{mCl}(A) \setminus U = \{a, c\} \setminus \{a, b\} = \{c\} \in \mathcal{H}$.
(iii) Let $U = X$, then $A \subset U$ and $\mathrm{mCl}(A) \setminus U = \{a, c\} \setminus X = \emptyset \in \mathcal{H}$. Therefore A is an $mg\mathcal{H}$-closed set.

Proposition 1. *Let $(X, m\ \mathcal{H})$ be a hereditary m-space. Then a subset A of X is $mg\mathcal{H}$-closed if* $\mathrm{mCl}(\{x\}) \cap A \notin \mathcal{H}$ *holds for any* $x \in \mathrm{mCl}(A)$.

Proof. Suppose that A is not $mg\mathcal{H}$-closed. We show that there exists $x \in \mathrm{mCl}(A)$ such that $\mathrm{mCl}(\{x\}) \cap A \in \mathcal{H}$. By assumption, there exists an m-open set U such that $A \subset U$ and $\mathrm{mCl}(A) \setminus U \notin \mathcal{H}$. Then $\mathrm{mCl}(A) \setminus U \neq \emptyset$ and there exists $x \in \mathrm{mCl}(A)$ such that $x \notin U$. But U is m-open and $X \setminus U$ is m-closed. Since $x \in X \setminus U$, $\mathrm{mCl}(\{x\}) \subset X \setminus U$ and hence $\mathrm{mCl}(\{x\}) \cap A \subset (X \setminus U) \cap A = \emptyset \in \mathcal{H}$. Therefore $\mathrm{mCl}(\{x\}) \cap A \in \mathcal{H}$. □

The following example shows that the converse of the above theorem is not true.

Example 4. Let $X = \{a, b, c\}$, $m = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $A = \{a\}$ and $\mathcal{H} = \{\emptyset, \{c\}\}$. Then, by Example 3.2, A is an $mg\mathcal{H}$-closed set. There exists $x = c \in \mathrm{mCl}(A) = \{a, c\}$ such that $\mathrm{mCl}(\{x\}) \cap A = \{c\} \cap \{a\} = \emptyset \in \mathcal{H}$.

Proposition 2. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. Then, for each $x \in X$, either $\{x\}$ is m-closed or $X \setminus \{x\}$ is an $mg\mathcal{H}$-closed set.*

Proof. Suppose that $\{x\}$ is not m-closed. Then $X \setminus \{x\}$ is not m-open. Let U be any m-open set such that $X \setminus \{x\} \subset U$. Hence $U = X$. Thus $\mathrm{mCl}(X \setminus \{x\}) \setminus U = \mathrm{mCl}(X \setminus \{x\}) \setminus X = \emptyset \in \mathcal{H}$ and hence $X \setminus \{x\}$ is an $mg\mathcal{H}$-closed set. □

Proposition 3. *Let $(X, m, \mathcal{H})$ be an ideal m-space and A, B be subsets of X. If A and B are $mg\mathcal{H}$-closed, then $A \cup B$ is $mg\mathcal{H}$-closed.*

Proof. Suppose A and B are $mg\mathcal{H}$-closed sets in $(X, m, \mathcal{H})$. Let $A \cup B \subset U$ and let U be m-open, then $A \subset U$ and $B \subset U$. By assumption, $\mathrm{mCl}(A) \setminus U \in \mathcal{H}$ and $\mathrm{mCl}(B) \setminus U \in \mathcal{H}$. Hence $\mathrm{mCl}(A \cup B) \setminus U = [\mathrm{mCl}(A) \cup \mathrm{mCl}(B)] \setminus U = [\mathrm{mCl}(A) \setminus U] \cup [\mathrm{mCl}(B) \setminus U] \in \mathcal{H}$. Therefore $A \cup B$ is $mg\mathcal{H}$-closed. □

Proposition 4. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and A, B be subsets of X. If A is $mg\mathcal{H}$-closed and $A \subset B \subset \mathrm{mCl}(A)$, then B is $mg\mathcal{H}$-closed.*

Proof. Let $B \subset U$ and U be m-open. Then $A \subset U$ and A is $mg\mathcal{H}$-closed and hence $\mathrm{mCl}(A) \setminus U \in \mathcal{H}$. Since $\mathrm{mCl}(A) \subset \mathrm{mCl}(B) \subset \mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$, $\mathrm{mCl}(B) \setminus U \in \mathcal{H}$ and B is $mg\mathcal{H}$-closed. □

Proposition 5. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and let m have the property $(\mathcal{B})$. If A is $mg\mathcal{H}$-closed and F is m-closed, then $A \cap F$ is $mg\mathcal{H}$-closed.*

Proof. Let $A \cap F \subset U$ and U be any m-open set. Then $A \subset U \cup (X \setminus F)$. Since m has the property $(\mathcal{B})$, $U \cup (X \setminus F)$ is m-open and hence $\mathrm{mCl}(A) \setminus (U \cup (X \setminus F)) \in \mathcal{H}$. We have the following:

$\mathrm{mCl}(A\cap F)\setminus U\subset(\mathrm{mCl}(A)\cap F)\setminus(X\setminus F)\setminus U\subset\mathrm{mCl}(A)\setminus(U\cup(X\setminus F))\in\mathcal{H}$. Therefore, $A\cap F$ is $mg\mathcal{H}$-closed. □

Let $(X,m,\mathcal{H})$ be a hereditary m-space. If, for each $H_1\in\mathcal{H}$, there exists $H_2\in\mathcal{H}\cap m$ such that $H_1\subseteq H_2$, then m is said to be saturated by $\mathcal{H}$.

Theorem 1. *Let $(X,m,\mathcal{H})$ be an ideal m-space. Let $B\subseteq A\subseteq X$, B be $m\mathcal{H}g$-closed relative to A and A be an $m\mathcal{H}g$-closed subset of X, where m has the property $(\mathcal{B})$. If m is saturated by $\mathcal{H}$, then B is $m\mathcal{H}g$-closed relative to X.*

Proof. Let m be saturated by $\mathcal{H}$. Let $B\subseteq U$ and U be m-open in X. Then $B\subseteq U\cap A$. Since B is $m\mathcal{H}g$-closed relative to A, we have $m\mathrm{Cl}_{\mathrm{A}}(B)\subseteq(U\cap A)\cup H_1$ for some $H_1\in\mathcal{H}$. By assumption, there exists $H_2\in\mathcal{H}\cap m$ such that $A\cap m\mathrm{Cl}(B)\subseteq(U\cap A)\cup H_2$. So $A\subseteq(U\cup H_2)\cup[X\setminus m\mathrm{Cl}(B)]$. Since A is $m\mathcal{H}g$-closed and $(U\cup H_2)\cup[X\setminus m\mathrm{Cl}(B)]\in m$, $m\mathrm{Cl}(A)\subseteq(U\cup H_2)\cup[X\setminus m\mathrm{Cl}(B)]\cup H_3$ for some $H_3\in\mathcal{H}$. By assumption, there exists $H_4\in\mathcal{H}\cap m$ such that $m\mathrm{Cl}(A)\subseteq(U\cup H_2)\cup[X\setminus m\mathrm{Cl}(B)]\cup H_4$. Since $B\subseteq A$, we have $m\mathrm{Cl}(B)\subseteq m\mathrm{Cl}(A)\subseteq(U\cup H_2)\cup[X\setminus m\mathrm{Cl}(B)]\cup H_4$. Hence $m\mathrm{Cl}(B)\subseteq U\cup(H_2\cup H_4)$ for some $H_2,H_4\in\mathcal{H}$. Therefore $mCl(B)\setminus U\subseteq(H_2\cup H_4)$. This shows that B is $m\mathcal{H}g$-closed relative to X. □

Definition 5. Let (X,m) be an m-space. For a subset A of X, $\Lambda_m(A)$ [9] is defined as follows: $\Lambda_m(A)=\cap\{U:A\subseteq U\in m\}$.

Theorem 2. *Let $(X,m,\mathcal{H})$ be an ideal m-space. If $m\mathrm{Cl}(A)\setminus\Lambda_m(A)\in\mathcal{H}$, then A is $m\mathcal{H}g$-closed.*

Proof. Let $m\mathrm{Cl}(A)\setminus\Lambda_m(A)\in\mathcal{H}$ and V be any m-open set containing A. Then

$$\begin{aligned}m\mathrm{Cl}(A)\setminus V\subseteq&\bigcup_{U\in m}\{m\mathrm{Cl}(A)\setminus U:A\subseteq U\}\\ =&m\mathrm{Cl}(A)\setminus\bigcap_{U\in m}\{U:A\subseteq U\}\\ =&m\mathrm{Cl}(A)\setminus\Lambda_m(A)\in\mathcal{H}.\end{aligned}$$

Thus, $m\mathrm{Cl}(A)\setminus V\in\mathcal{H}$ and hence A is $m\mathcal{H}g$-closed set. □

Definition 6. Let $(X,m,\ \mathcal{H})$ be a hereditary m-space. A subset A of X is said to be *$mg\mathcal{H}$-open* if $X\setminus A$ is $mg\mathcal{H}$-closed.

Theorem 3. *Let $(X,m,\ \mathcal{H})$ be a hereditary m-space and A be a subset of X. Then, A is $mg\mathcal{H}$-open if and only if $F\setminus\mathrm{mInt}(A)\in\mathcal{H}$ whenever $F\subset A$ and F is m-closed.*

Proof. $(\Rightarrow)$ Let $F\subset A$ and F be any m-closed set. Then $X\setminus A\subset X\setminus F$ and $X\setminus F$ is m-open. Since $X\setminus A$ is $mg\mathcal{H}$-closed, $\mathrm{mCl}(X\setminus A)\setminus(X\setminus F)\in$

$\mathcal{H}$ and hence $(X \setminus \mathrm{mInt}(A)) \cap F = F \setminus \mathrm{mInt}(A)) \in \mathcal{H}$.

($\Leftarrow$) Let $X \setminus A \subset U$ and U be any m-open set. Then $X \setminus U \subset A$ and $X \setminus U$ is m-closed and, by assumption, $(X \setminus U) \setminus \mathrm{mInt}(A) \in \mathcal{H}$. We have $(X \setminus U) \setminus \mathrm{mInt}(A) = (X \setminus U) \cap (X \setminus \mathrm{mInt}(A)) = (X \setminus \mathrm{mInt}(A)) \setminus U = \mathrm{mCl}(X \setminus A) \setminus U$. Hence we obtain $\mathrm{mCl}(X \setminus A) \setminus U \in \mathcal{H}$ and $X \setminus A$ is *$mg\mathcal{H}$-closed*. Therefore A is *$mg\mathcal{H}$-open*. □

Proposition 6. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and A, B be subsets of X. If A is $mg\mathcal{H}$-open and $\mathrm{mInt}(A) \subset B \subset A$, then B is $mg\mathcal{H}$-open.*

Proof. Since $X \setminus A \subset X \setminus B \subset (X \setminus \mathrm{mInt}(A)) = \mathrm{mCl}(X \setminus A)$ and $X \setminus A$ is $mg\mathcal{H}$-closed, by Proposition 3.4, $X \setminus B$ is $mg\mathcal{H}$-closed. Hence B is $mg\mathcal{H}$-open. □

Proposition 7. *Let $(X, m, \mathcal{H})$ be an ideal m-space and A, B be subsets of X. If A and B are $mg\mathcal{H}$-open, then $A \cap B$ is $mg\mathcal{H}$-open.*

Proof. Since $X \setminus A$ and $X \setminus B$ are $mg\mathcal{H}$-closed, by Proposition 3.3 $(X \setminus A) \cup (X \setminus B) = X \setminus (A \cap B)$ is $mg\mathcal{H}$-closed. Hence $A \cap B$ is $mg\mathcal{H}$-open. □

Definition 7. Let (X, m) be an m-space. Then subsets A and B of X are said to be *m-separated* if $\mathrm{mCl}(A) \cap B = \emptyset = A \cap \mathrm{mCl}(B)$.

Proposition 8. *Let $(X, m, \mathcal{H})$ be an ideal m-space, m have the property $(\mathcal{B})$ and A, B be subsets of X. If A and B are m-separated and $mg\mathcal{H}$-open, then $A \cup B$ is $mg\mathcal{H}$-open.*

Proof. Let $F \subseteq A \cup B$ and F be any m-closed set. Since A and B are m-separated, $F \cap \mathrm{mCl}(A) \subseteq A$ and $F \cap \mathrm{mCl}(A)$ is m-closed because m has property $\mathcal{B}$. Hence, by Theorem 1, $(F \cap \mathrm{mCl}(A)) \setminus \mathrm{mInt}(A) \in \mathcal{H}$. Similarly, we obtain $(F \cap \mathrm{mCl}(B)) \setminus \mathrm{mInt}(B) \in \mathcal{H}$. Therefore $F \cap \mathrm{mCl}(A) \subseteq \mathrm{mInt}(A) \cup H_A$ and $F \cap \mathrm{mCl}(B) \subseteq \mathrm{mInt}(B) \cup H_B$ for some $H_A, H_B \in \mathcal{H}$. Since $F \subseteq A \cup B \subset \mathrm{mCl}(A) \cup \mathrm{mCl}(B)$, we obtain $F \subseteq (F \cap \mathrm{mCl}(A)) \cup (F \cap \mathrm{mCl}(B)) \subseteq \mathrm{mInt}(A) \cup \mathrm{mInt}(B) \cup (H_A \cup H_B) \subseteq \mathrm{mInt}(A \cup B) \cup (H_A \cup H_B)$. Hence, $F \setminus \mathrm{mInt}(A \cup B) \subseteq (H_A \cup H_B) \in \mathcal{H}$. Therefore $A \cup B$ is $mg\mathcal{H}$-open. □

Corollary 1. *Let $(X, m, \mathcal{H})$ be a hereditary m-space, m have the property $(\mathcal{B})$ and A, B be subsets of X. If $X \setminus A$ and $X \setminus B$ are m-separated and $mg\mathcal{H}$-open, then $A \cap B$ is $mg\mathcal{H}$-closed.*

Proof. The proof is obvious from Proposition 8.

□

4. Characterizations of $mg\mathcal{H}$-closed sets

In this section, we obtain some characterizations of $mg\mathcal{H}$-closed sets and $mg\mathcal{H}$-open sets.

Theorem 4. *Let $(X, m, \mathcal{H})$ be a hereditary m-space, m have the property $(\mathcal{B})$ and A be a subset of X. Then the following properties are equivalent, where (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) hold without the assumption that m has property $(\mathcal{B})$:*
(1) *A is $mg\mathcal{H}$-closed;*
(2) *if F is m-closed and $F \subset \mathrm{mCl}(A) \setminus A$, then $F \in \mathcal{H}$;*
(3) *$\mathrm{mCl}(A) \setminus A$ is $mg\mathcal{H}$-open.*

Proof. (1) $\Rightarrow$ (2): Suppose that F is m-closed and $F \subset \mathrm{mCl}(A) \setminus A$. Then $X \setminus F \in m$ and $A \subset X \setminus F$. Since A is $mg\mathcal{H}$-closed, $\mathrm{mCl}(A) \setminus (X \setminus F) \in \mathcal{H}$. We have $F = F \cap \mathrm{mCl}(A) = (X \setminus (X \setminus F)) \cap \mathrm{mCl}(A) = \mathrm{mCl}(A) \setminus (X \setminus F) \in \mathcal{H}$. Hence $F \in \mathcal{H}$.
(2) $\Rightarrow$ (3): Suppose that F is m-closed and $F \subset \mathrm{mCl}(A) \setminus A$. By (2), we have $F \setminus \mathrm{mInt}(\mathrm{mCl}(A) \setminus A) \subset F \in \mathcal{H}$. Therefore, by Theorem 1, $\mathrm{mCl}(A) \setminus A$ is *$mg\mathcal{H}$-open.*
(3) $\Rightarrow$ (1): Suppose that $\mathrm{mCl}(A) \setminus A$ is *$mg\mathcal{H}$-open.* Let $A \subset U$ and U be any m-open set. Then $X \setminus U \subset X \setminus A$ and $X \setminus U$ is m-closed. We have $\mathrm{mCl}(A) \setminus A = \mathrm{mCl}(A) \cap (X \setminus A) \supset \mathrm{mCl}(A) \cap (X \setminus U)$. Since m has property $\mathcal{B}$, $\mathrm{mCl}(A) \cap (X \setminus U)$ is m-closed. By Theorem 1, $(\mathrm{mCl}(A) \cap (X \setminus U)) \setminus \mathrm{mInt}(\mathrm{mCl}(A) \setminus A) = (\mathrm{mCl}(A) \cap (X \setminus U)) \cap [X \setminus \mathrm{mInt}(\mathrm{mCl}(A) \setminus A)] \in \mathcal{H}$. By a simple calculation, we obtain $\mathrm{mInt}(\mathrm{mCl}(A) \setminus A) = \emptyset$ and hence $\mathrm{mCl}(A) \cap (X \setminus U) = \mathrm{mCl}(A) \setminus U \in \mathcal{H}$. Therefore A is $mg\mathcal{H}$-closed. □

Corollary 2. *Let $(X, m, \mathcal{H})$ be a hereditary m-space, m have property $(\mathcal{B})$ and A be a subset of X. Then A is $mg\mathcal{H}$-open if and only if $A \setminus \mathrm{mInt}(A)$ is $mg\mathcal{H}$-open.*

Proof. By Theorem 4, A is $mg\mathcal{H}$-open if and only if $X \setminus A$ is $mg\mathcal{H}$-closed if and only if $\mathrm{mCl}(X \setminus A) \setminus (X \setminus A)$ is $mg\mathcal{H}$-open if and only if $(X \setminus \mathrm{mInt}(A)) \cap A$ is $mg\mathcal{H}$-open if and only if $(A \setminus \mathrm{mInt}(A))$ is $mg\mathcal{H}$-open. □

Corollary 3. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and A be a subset of X. Then, the following properties are equivalent:*
(1) *If F is m-closed and $F \subset \mathrm{mCl}(A) \setminus A$, then $F \in \mathcal{H}$;*
(2) *$\mathrm{mCl}(A) \setminus A$ is $mg\mathcal{H}$-open.*

Proof. (1) $\Rightarrow$ (2): We obtained this in Theorem 4.
(2) $\Rightarrow$ (1): Suppose that $\mathrm{mCl}(A) \setminus A$ is *$mg\mathcal{H}$-open.* Let F be m-closed and $F \subset \mathrm{mCl}(A) \setminus A$. By Theorem 1, $F \setminus \mathrm{mInt}(\mathrm{mCl}(A) \setminus A) \in \mathcal{H}$. Since $\mathrm{mInt}(\mathrm{mCl}(A) \setminus A) = \emptyset$, $F \in \mathcal{H}$. □

Theorem 5. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and A be a subset of X. Then A is $mg\mathcal{H}$-closed if and only if $\mathrm{mCl}(A) \cap F \in \mathcal{H}$ whenever $A \cap F = \emptyset$ and F is m-closed.*

Proof. ($\Rightarrow$) Suppose that A is $mg\mathcal{H}$-closed. Let $A \cap F = \emptyset$ and F be m-closed. Then $A \subset X \setminus F$ and $X \setminus F \in m$. Since A is $mg\mathcal{H}$-closed,

$\mathrm{mCl}(A) \setminus (X \setminus F) = \mathrm{mCl}(A) \cap F \in \mathcal{H}$.

($\Leftarrow$) Let $A \subset U$ and U be an m-open set. Then $X \setminus U$ is m-closed and $A \cap (X \setminus U) = \emptyset$. By assumption, $\mathrm{mCl}(A) \cap (X \setminus U) = \mathrm{mCl}(A) \setminus U \in \mathcal{H}$. Therefore A is $mg\mathcal{H}$-closed. □

Theorem 6. *Let $(X, m, \mathcal{H})$ be a hereditary m-space, m have property $(\mathcal{B})$ and A be a subset of X. Then A is $mg\mathcal{H}$-open if and only if $X \setminus G \in \mathcal{H}$ whenever G is m-open and $\mathrm{mInt}(A) \cup (X \setminus A) \subset G$.*

Proof. ($\Rightarrow$) Suppose that A is $mg\mathcal{H}$-open. Let G be m-open and $\mathrm{mInt}(A) \cup (X \setminus A) \subset G$. Then $X \setminus G \subset X \setminus [\mathrm{mInt}(A) \cup (X \setminus A)] = [(X \setminus \mathrm{mInt}(A)) \cap A)] = \mathrm{mCl}(X \setminus A) \setminus (X \setminus A)$. Since $X \setminus A$ is $mg\mathcal{H}$-closed and $X \setminus G$ is m-closed, by Theorem 4 (2), $X \setminus G \in \mathcal{H}$.

($\Leftarrow$) Let $F \subset A$ and F be an m-closed set. Since m has property $\mathcal{B}$, $\mathrm{mInt}(A) \cup (X \setminus A) \subset \mathrm{mInt}(A) \cup (X \setminus F) \in m$. By assumption, $X \setminus [\mathrm{mInt}(A) \cup (X \setminus F)] \in \mathcal{H}$. But $X \setminus [\mathrm{mInt}(A) \cup (X \setminus F)] = (X \setminus \mathrm{mInt}(A)) \cap F = F \setminus \mathrm{mInt}(A)$. Hence $F \setminus \mathrm{mInt}(A) \in \mathcal{H}$. Therefore A is $mg\mathcal{H}$-open. □

5. Preservation theorems

Now we give a simple proof of the following lemma.

Lemma 3 ([10])**.** *Let $f : (X, m) \to (Y, n)$ be a function. If $\mathcal{H}$ is a hereditary class on X, then $f(\mathcal{H}) = \{f(H) : H \in \mathcal{H}\}$ is a hereditary class on Y.*

Proof. Let $H \in \mathcal{H}$ and $B \subset f(H)$. Let $A = H \cap f^{-1}(B)$. Then $A \subset H$ and $f(A) = f(H \cap f^{-1}(B)) = f(H) \cap B = B$. Hence $B = f(A) \in f(\mathcal{H})$. □

Lemma 4. *Let $f : (X, m) \to (Y, n)$ be a function. If $\mathcal{H}$ is a hereditary class on Y, then $J_H = \{A \subset X : f(A) \in \mathcal{H}\}$ is a hereditary class on X.*

Proof. Let $B \subset A$ and $A \in J_H$. Then $f(B) \subset f(A) \in \mathcal{H}$ and $f(B) \in \mathcal{H}$. Hence $B \in J_H$ and J_H is a hereditary class on X. □

Definition 8. A function $f : (X, m) \to (Y, n)$ is said to be

(1) *M-continuous* [21] if for each $x \in X$ and each $V \in n$ containing $f(x)$, there exists $U \in m$ containing x such that $f(U) \subset V$,

(2) *M-closed* [20] if for each m-closed set F of (X, m), $f(F)$ is n-closed in (Y, n).

Lemma 5 ([21])**.** *Let m be an m-structure with property $(\mathcal{B})$. Then a function $f : (X, m) \to (Y, n)$ is M-continuous if and only if for each $V \in n$, $f^{-1}(V) \in m$.*

Lemma 6 ([20])**.** *A function $f : (X, m) \to (Y, n)$ is M-closed if and only if for each subset B of Y and each $U \in m$ containing $f^{-1}(B)$, there exists $V \in n$ such that $B \subset V$ and $f^{-1}(V) \subset U$.*

Theorem 7. *Let $f : (X, m, \mathcal{H}) \to (Y, n, f(\mathcal{H}))$ be an M-closed and M-continuous function, where m has the property $(\mathcal{B})$. If A is $mg\mathcal{H}$-closed in $(X, m, \mathcal{H})$, then $f(A)$ is $ngf(\mathcal{H})$-closed in $(Y, n, f(\mathcal{H}))$.*

Proof. Let A be $mg\mathcal{H}$-closed and $f(A) \subset V \in n$. Since m has the property $(\mathcal{B})$ and f is M-continuous, by Lemma 6 $A \subset f^{-1}(V) \in m$. Since A is $mg\mathcal{H}$-closed, $\mathrm{mCl}(A) \setminus f^{-1}(V) \in \mathcal{H}$ and $\mathrm{mCl}(A) \subset f^{-1}(V) \cup H$, where $H \in \mathcal{H}$. Hence $f(\mathrm{mCl}(A)) \subset f(f^{-1}(V)) \cup f(H) \subset V \cup f(H)$. Since $\mathrm{mCl}(A)$ is m-closed and f is M-closed, $f(\mathrm{mCl}(A))$ is n-closed. Hence $\mathrm{nCl}(f(A)) \subset f(\mathrm{mCl}(A)) \subset V \cup f(H)$. Therefore we obtain $\mathrm{nCl}(f(A)) \setminus V \subset f(H) \in f(\mathcal{H})$. This shows that $f(A)$ is $ngf(\mathcal{H})$-closed in $(Y, n, f(\mathcal{H}))$. □

Theorem 8. *Let $f : (X, m, J_H) \to (Y, n, \mathcal{H})$ be an M-closed and M-continuous function, where m and n have the property $(\mathcal{B})$. If B is $ng\mathcal{H}$-closed in $(Y, n, \mathcal{H})$, then $f^{-1}(B)$ is mgJ_H-closed in (X, m, J_H).*

Proof. Let $f^{-1}(B) \subset U$ and U be any m-open set of X. Since f is M-closed, by Lemma 6 there exists $V \in n$ such that $B \subset V$ and $f^{-1}(V) \subset U$. Since B is $ng\mathcal{H}$-closed, $\mathrm{nCl}(B) \setminus V \in \mathcal{H}$. Hence $\mathrm{nCl}(B) \subset V \cup H$ for some $H \in \mathcal{H}$. Since n has the property $(\mathcal{B})$, $\mathrm{nCl}(B)$ is n-closed. Since f is M-continuous, we have $\mathrm{mCl}(f^{-1}(B)) \subset f^{-1}(\mathrm{nCl}(B)) \subset f^{-1}(V) \cup f^{-1}(H) \subset U \cup f^{-1}(H)$. Therefore $\mathrm{mCl}(f^{-1}(B)) \setminus U \subset f^{-1}(H) \in J_H$. This shows that $f^{-1}(B)$ is mgJ_H-closed in (X, m, J_H). □

6. Conclusions

The results obtained in this paper are important, and future research could give more insights by exploring further properties of the minimal spaces with hereditary classes such as a fuzzy minimal structure which is a generalization of the concept of fuzzy topology, fuzzy minimal vector spaces and compatible with the concept of fuzzy minimal spaces via hereditary classes.

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Al al-Bayt University, Faculty of Sciences, Department of Mathematics, P.O. Box 130095, Mafraq 25113, Jordan
E-mail address: omarimutah1@yahoo.com
URL: https://orcid.org/0000-0002-6696-1301

2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi, Kumamoto-ken, 869-5142, Japan
E-mail address: t.noiri@nifty.com
URL: https://orcid.org/0000-0002-0862-5297