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Osculating mate of a Frenet curve in the Euclidean 3-space

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Abstract. A new kind of partner curves called osculating mate of a Frenet curve is introduced. Some characterizations for osculating mate are obtained and using the obtained results some special curves such as slant helix, spherical helix, C-slant helix and rectifying curve are constructed.

1. Introduction

The most important and fascinating topic of curve theory is finding characterizations for a curve or a pair of curves known as special curves or partner curves. Helices, slant helices, rectifying curves, spherical curves, etc. are common examples of such curves. Especially, the helices are seen in many areas such as nature, design of mechanic tools and highways, simulation of kinematic motion or architect, nucleic acids and molecular model of DNA [19, 20, 23, 24, 25]. Helices are also important in physics since they are used in helical gears, shapes of springs and elastic rods [9, 12]. A helix α is defined by that the tangent of α always makes a constant angle with a fixed direction and a necessary and sufficient condition for a curve α to be a helix is that $\frac{\tau}{\kappa}(s)$ is constant, where κ is the first curvature (or curvature) and τ is the second curvature (or torsion) of α [2, 21]. Another type of special curves is the slant helix, which is defined by the property that there is always a constant angle between the principal normal line of the curve and a fixed direction. This special curve was first defined by Izumiya and Takeuchi [11]. Later, Zıplar et al. [26] have introduced a new special curve called Darboux helix and they have obtained that a curve is a Darboux helix iff the curve is a slant helix.

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Furthermore, a special curve can be defined by considering its position vector. A curve α in the Euclidean 3-space E^3 for which the position vector of the curve is always contained in its rectifying plane (respectively, osculating plane or normal plane) is called a rectifying curve or briefly rectifying (respectively, osculating curve or normal curve) [5]. Rectifying curves, normal curves and osculating curves satisfy Cesaro's fixed point condition [18]. Namely, rectifying, normal and osculating planes of such curves always contain a particular point. Moreover, Darboux vectors (centrodes) and rectifying curves are related and used in different branches of science such as kinematics, mechanics and differential geometry of curves of constant precession [6].

Kızıltuğ et al. [13] have defined a new kind of special curves called normal direction curves. Later, Çakmak [7] has studied the same subject in a 3-dimensional compact Lie group and he has also given two similar new curves.

Recently, Deshmukh et al. [8] have studied the natural mate and the conjugate mate of a curve. They have given some new characterizations for a spherical curve, a helix, a rectifying curve and a slant helix. Alghanemi and Khan [1] have given the position vectors of the natural mate and the conjugate mate. Mak [15] has studied these mates in three-dimensional Lie groups. Later, Camcı et al. [4] have studied sequential natural mates of Frenet curves in E^3.

In the present paper, we define the osculating mate of a Frenet curve α in E^3. We give some relations between a Frenet curve and its osculating mate and introduce some applications of osculating mates to a slant helix, a spherical helix, a rectifying curve and a C-slant helix in E^3.

2. Preliminaries

Let $\alpha : I \to E^3$ be a unit speed curve with arclength parameter s. The vector $T(s) = \alpha'(s)$ is called the *unit tangent vector* of α and the function $\kappa(s) = \|\alpha''(s)\|$ is called the *curvature* of α. The unit principal normal vector $N(s)$ of the curve α is defined by $\alpha''(s) = \kappa(s)N(s)$. The unit binormal vector of α is $B(s) = T(s) \times N(s)$. Then, the Frenet frame $\{T, N, B\}$ has the following formula

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

where $\tau = \tau(s)$ is the torsion of the curve α defined by $\tau = -\langle B', N \rangle$ [21]. If $\kappa(s) \neq 0$, the curve α is called a *Frenet curve*. The curve α is a general helix iff $\frac{\tau}{\kappa}(s)$ is constant. Similarly, in [11], the characterization of a slant helix is given by the necessary and sufficient condition

$$\sigma(s) = \frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)' = const. \tag{1}$$

A Frenet curve α is said to be a *Salkowski* (respectively, *anti-Salkowski*) *curve* if its curvature κ is constant but the torsion τ is non-constant (respectively, the torsion τ is constant but the curvature κ is non-constant) [17].

A Frenet curve α is said to be a *spherical curve* if all points of α lie on the same sphere and such a curve is characterized as follows: a Frenet curve α is a spheciral curve iff $(p'q)' + \frac{p}{q} = 0$ holds, where $p = 1/\kappa,\ q = 1/\tau$. Moreover, another characterization for a spherical curve is that a Frenet curve α is a sphecial curve iff $p^2 + (p'q)^2 = a^2$ holds, where $a > 0$ is the radius of the sphere on which α lies [16].

A Frenet curve α is called a *rectifying curve* if the position vector of α always lies on the rectifying plane of the curve [5]. A rectifying curve is characterized by the necessary and sufficient condition that $\frac{\tau}{\kappa}(s) = \frac{1}{c}(s+b)$ holds, where $c \neq 0,\ b$ are real constants and such a curve has the parametrization $\alpha(s) = (s + b)T(s) + cB(s)$ [5].

The vector W defined by $W = \frac{\tau T + \kappa B}{\sqrt{\kappa^2 + \tau^2}}$ is called the *unit Darboux vector* of α. Then, the frame $\{N, C = W \times N, W\}$ is called the *alternative frame* of α. A curve α is called a *Darboux helix* if the unit Darboux vector W makes a constant angle with a fixed direction and the curve α is a Darboux helix iff α is a slant helix [26]. A curve α is said to be a *C-slant helix* if the unit vector C always makes a constant angle with a fixed direction. A necessary and sufficient condition for a curve α to be a C-slant helix is that the function

$$\mu(s) = \frac{(f^2 + g^2)^{3/2}}{f^2(\frac{g}{f})'} \tag{2}$$

is constant [22].

3. Osculating mates of a Frenet curve in E^3

In this section, we define the osculating mate of a Frenet curve in E^3 and give some characterizations for this curve.

Definition 1. Let $\alpha : I \subset \mathbb{R} \to E^3$ be a unit speed Frenet curve. The curve β defined by

$$\beta(s) = \int (x_1(s)T(s) + x_2(s)N(s))\, ds \tag{3}$$

and satisfying the conditions $x_1^2(s)+x_2^2(s)=1$ and $\beta'' \perp sp\{T,N\}$ is called the *osculating mate* of the curve α where s is the arclength parameter of α and $x_1(s), x_2(s)$ are differentiable functions of s.

Unless otherwise stated, hereafter when we talk about the concept of curves we will mean Frenet curves.

Theorem 1. *The Frenet apparatus of the osculating mate β is computed as follows*

$$\begin{cases} \bar{T} = \sin\left(\int \kappa(s)ds\right)T + \cos\left(\int \kappa(s)ds\right)N, \ \bar{N} = B, \\ \bar{B} = \cos\left(\int \kappa(s)ds\right)T - \sin\left(\int \kappa(s)ds\right)N, \end{cases} \tag{4}$$

$$\bar{\kappa} = \varepsilon_1 \tau \cos\left(\int \kappa(s)ds\right), \ \bar{\tau} = \tau \sin\left(\int \kappa(s)ds\right), \tag{5}$$

where $\varepsilon_1 = \pm 1$ is chosen such as $\bar{\kappa} > 0$.

Proof. Let $\left\{\bar{T}, \bar{N}, \bar{B}; \bar{\kappa}, \bar{\tau}\right\}$ be the Frenet apparatus of the osculating mate β. From Definition 1, it follows that $\beta' = \bar{T} = x_1 T + x_2 N$. Differentiating the last equality we have

$$\bar{T}' = \left(x_1' - x_2\kappa\right)T + \left(x_2' + x_1\kappa\right)N + x_2\tau B, \tag{6}$$

which gives the system

$$x_1' - x_2\kappa = 0, \ x_2' + x_1\kappa = 0, \ x_2\tau \neq 0. \tag{7}$$

The solution of the system (7) is

$$x_1(s) = \sin\left(\int \kappa(s)ds\right), \ x_2(s) = \cos\left(\int \kappa(s)ds\right). \tag{8}$$

Then, it follows that $\bar{T} = \sin\left(\int \kappa ds\right)T + \cos\left(\int \kappa ds\right)N$ and from (6), we have $\bar{\kappa}\bar{N} = \tau\cos\left(\int \kappa ds\right)B$. Hence we obtain

$$\bar{\kappa} = \varepsilon_1 \tau \cos\left(\int \kappa ds\right), \ \bar{N} = B, \tag{9}$$

where $\varepsilon_1 = \pm 1$ is chosen such that $\bar{\kappa} > 0$. Furthermore,

$$\bar{B} = \bar{T} \times \bar{N} = \cos\left(\int \kappa ds\right)T - \sin\left(\int \kappa ds\right)N. \tag{10}$$

Differentiating (10) and using the equality $\bar{\tau}' = -\left\langle \bar{B}', \bar{N}\right\rangle$, we have $\bar{\tau} = \tau\sin\left(\int \kappa ds\right)$. □

Theorem 2. *The curvatures κ and τ of α are computed as*

$$\kappa = \frac{\varepsilon_1 \bar{\kappa}^2}{\bar{\kappa}^2 + \bar{\tau}^2}\left(\frac{\bar{\tau}}{\bar{\kappa}}\right)', \ \tau = \pm\sqrt{\bar{\kappa}^2 + \bar{\tau}^2}. \tag{11}$$

Proof. From (5), we easily get

$$\tau = \pm\sqrt{\bar{\kappa}^2 + \bar{\tau}^2}. \tag{12}$$

By writing the equality (12) in equalities (5), it follows

$$\cos\left(\int \kappa ds\right) = \frac{\pm\varepsilon_1\bar{\kappa}}{\sqrt{\bar{\kappa}^2 + \bar{\tau}^2}},\ \sin\left(\int \kappa ds\right) = \frac{\pm\bar{\tau}}{\sqrt{\bar{\kappa}^2 + \bar{\tau}^2}}, \tag{13}$$

respectively. By taking the derivative of the second equality in (13), we get

$$\kappa\cos\left(\int \kappa ds\right) = \pm\frac{\bar{\kappa}(\bar{\kappa}\bar{\tau}' - \bar{\kappa}'\bar{\tau})}{(\bar{\kappa}^2 + \bar{\tau}^2)^{3/2}}. \tag{14}$$

Putting first equality in (13) into (14) gives $\kappa = \frac{\varepsilon_1\bar{\kappa}^2}{\bar{\kappa}^2+\bar{\tau}^2}\left(\frac{\bar{\tau}}{\bar{\kappa}}\right)'$, which completes the proof. □

From Theorem 1, Theorem 2 and equation (1), we have

$$\bar{\tau} = \tau\sin\left(\int \kappa ds\right),\ \frac{\kappa}{\tau} = \varepsilon_1\bar{\sigma}, \tag{15}$$

which gives the following corollary.

Corollary 1. *i) α is a plane curve iff the osculating mate β is a plane curve.*

ii) α is a helix iff the osculating mate β is a slant helix.

Theorem 3. *The osculating mate β is spherical curve iff the curvatures κ, τ of α satisfy the equality*

$$(\tau\cos x)' = \pm\tau^2\sin x\cos x\sqrt{a^2\tau^2\cos^2 x - 1}, \tag{16}$$

where $a > 0$ is the radius of the sphere and $x(s) = \int \kappa(s)ds$.

Proof. First assume that β lies on a sphere with the radius $a > 0$. Hence, $\bar{p}^2 + (\bar{p}'\bar{q})^2 = a^2$ holds, where $\bar{p} = 1/\bar{\kappa}$, $\bar{q} = 1/\bar{\tau}$. From (5), it follows that $\bar{p}' = \frac{-\varepsilon_1(\tau\cos x)'}{\tau^2\cos^2 x}$. Then, we have

$$\frac{1}{\tau^2\cos^2 x}\left[1 + \frac{((\tau\cos x)')^2}{\tau^4\sin^2 x\cos^2 x}\right] = a^2, \tag{17}$$

which gives (16).

Conversely, assume that (16) holds. By differentiating the first equality in (5), we have

$$-\bar{p}' = \frac{\bar{\kappa}'}{\bar{\kappa}^2} = \frac{-\varepsilon_1(\tau\cos x)'}{\tau^2\cos^2 x}. \tag{18}$$

Putting (16) into (18) gives

$$\bar{p}' = \frac{\mp\varepsilon_1\sin x\sqrt{a^2\tau^2\cos^2 x - 1}}{\cos x}. \tag{19}$$

By the second equality in (5), we obtain $\bar{p}'\bar{q} = \frac{\mp\varepsilon_1\sqrt{a^2\tau^2\cos^2 x-1}}{\tau\cos x}$, and so $\bar{p}^2 + (\bar{p}'\bar{q})^2 = a^2$, i.e., β lies on a sphere with the radius $a > 0$. □

Theorem 4. *The osculating mate β is rectifying iff the function* $\tan\int\kappa ds$ *is a linear function of s.*

Proof. Suppose that β is rectifying. So we have $\frac{\bar{\tau}}{\bar{\kappa}} = \frac{1}{c}(s+b)$, where $c \neq 0$, b are real constants. Considering (5), it follows that $\tan\int\kappa ds = \frac{\varepsilon_1}{c}(s+b)$.

Conversely, let us write $\varepsilon_1\tan\int\kappa ds = (a_1 s + a_2)$, where $a_1 \neq 0$, a_2 are real constants. Let us define $a_1 = \frac{1}{c}$ and $a_2 = \frac{b}{c}$, where $c \neq 0$ is a real constant. Then, we get $\varepsilon_1\tan\int\kappa ds = \frac{1}{c}(s+b)$ and it follows that $c\tau\sin\left(\int\kappa ds\right) = \varepsilon_1(s+b)\tau\cos\left(\int k ds\right)$. Taking into account (5), we obtain $(s+b)\bar{\kappa} - c\bar{\tau} = 0$, which gives that β is rectifying. □

Theorem 5. *The position vector of the osculating mate β is given by*

$$\beta = \left[\int\left(-\frac{\kappa}{\tau}h' + \sin\left(\int\kappa ds\right)\right)ds\right]T - \frac{h'}{\tau}N + hB, \tag{20}$$

where $h(s) = \frac{(dd')'-1}{\tau\cos\int\kappa ds}$ *and* $d = d(s) = \|\beta(s)\|$ *is the distance function of* β.

Proof. For the position vector β, we can write

$$\beta = a_1T + a_2N + a_3B, \tag{21}$$

where $a_i = a_i(s)$, $(i = 1,2,3)$ are smooth functions of s. Differentiating (21) and using (4), we have

$$\left\{\begin{array}{l} \sin\left(\int\kappa ds\right)T + \cos\left(\int\kappa ds\right)N = (a_1' - a_2\kappa)T + (a_1\kappa + a_2' - a_3\tau)N \\ \qquad\qquad + (a_2\tau + a_3')B. \end{array}\right. \tag{22}$$

From (22), we have the following system

$$\left\{\begin{array}{c} a_1' - a_2\kappa = \sin\left(\int\kappa ds\right), \\ a_1\kappa + a_2' - a_3\tau = \cos\left(\int\kappa ds\right), \\ a_2\tau + a_3' = 0. \end{array}\right. \tag{23}$$

From (21), it follows that $d^2 = a_1^2 + a_2^2 + a_3^2$. Differentiating the last equality gives $dd' = a_1a_1' + a_2a_2' + a_3a_3'$. Then, from the system (23), we get

$$dd' = a_1\sin\left(\int\kappa ds\right) + a_2\cos\left(\int\kappa ds\right). \tag{24}$$

Differentiating (24) and taking into account the system (23), we obtain $a_3 = \frac{(dd')'-1}{\tau\cos\left(\int\kappa ds\right)}$. By writing $h(s) = a_3(s)$, from the system (23), we get

$$a_2 = -\frac{h'}{\tau}(dd')', a_1 = \int\left[-\frac{\kappa}{\tau}(h)' + \sin\left(\int\kappa ds\right)\right]ds. \tag{25}$$

Considering (21), we have (20). □

Corollary 2. *Let β be an osculating mate of α.*
i) β is a spherical curve iff $h(s) = \frac{-1}{\tau \cos \int \kappa ds}$.
ii) If β is a rectifying curve, then $h = 0$.

Proof. *i)* β is a spherical curve iff d is a non-zero constant iff $h(s) = \frac{-1}{\tau \cos \int \kappa ds}$.

ii) Since β is a rectifying curve, its distance function d satisfies $d^2(s) = s^2 + c_1 s + c_2$, where c_i $(i = 1, 2)$ are constants [6]. Then we have $h = 0$. □

Theorem 6. *Let β be an osculating mate of α.*
i) β is a Bertrand curve iff the function $(pq')^2 + q^2$ is a non-zero constant.
ii) α is a Bertrand curve iff $\varepsilon_1 \varsigma_1 \bar{\sigma} \mp \varsigma_2 = \frac{1}{\sqrt{\bar{\kappa}^2 + \bar{\tau}^2}}$, where $\varsigma_1 \neq 0$, ς_2 are constants.

Proof. *i)* Since β is a Bertrand curve, we can write $a\bar{\kappa} + b\bar{\tau} = 1$, where $a \neq 0$ and b are constants [3]. Writing (5) in the last equality gives

$$a\varepsilon_1 \cos\left(\int \kappa ds\right) + b \sin\left(\int \kappa ds\right) = \frac{1}{\tau} = q. \tag{26}$$

By differentiating (26), we have

$$-a\varepsilon_1 \sin\left(\int \kappa ds\right) + b \cos\left(\int \kappa ds\right) = \left(\frac{1}{\tau}\right)' \frac{1}{\kappa} = q'p. \tag{27}$$

From (26) and (27), it follows that $(pq')^2 + q^2 = a^2 + b^2$.

Conversely, let $(pq')^2 + q^2$ be a non-zero constant. Define $q = (a^2 + b^2) \cos\theta$ and $pq' = (a^2 + b^2) \sin\theta$, where $a \neq 0$, b are real constants. Differentiating the first equality and writing the result in the second one gives $\theta' = -\kappa$. Then the equality $q = (a^2 + b^2) \cos\theta$ becomes $\tau = \frac{1}{(a^2+b^2)\cos\left(\int \kappa ds + m\right)}$, where m is an integration constant. By taking into account (5), we have

$$\bar{\kappa} = \frac{\varepsilon_1 \cos\left(\int \kappa ds\right)}{(a^2 + b^2)\cos\left(\int \kappa ds + m\right)}, \quad \bar{\tau} = -\frac{\sin\left(\int \kappa ds\right)}{(a^2 + b^2)\cos\left(\int \kappa ds + m\right)}. \tag{28}$$

By writing $A = \left(a^2 + b^2\right)\cos(m)$, $B = \left(a^2 + b^2\right)\sin(m)$ and taking into account (28) it follows that $A\bar{\kappa} + B\bar{\tau} = 1$, i.e., β is a Bertrand curve.

ii) If α is a Bertrand curve, then $\varsigma_1 \kappa + \varsigma_2 \tau = 1$, where $\varsigma_1 \neq 0$, ς_2 are constants. Writing (11) in the last equality, it follows that $\frac{\varepsilon_1 \varsigma_1 \bar{\kappa}^2}{\bar{\kappa}^2 + \bar{\tau}^2}\left(\frac{\bar{\tau}}{\bar{\kappa}}\right)' \pm \varsigma_2 \sqrt{\bar{\kappa}^2 + \bar{\tau}^2} = 1$ or, equivalently, $\varepsilon_1 \varsigma_1 \bar{\sigma} \pm \varsigma_2 = \frac{1}{\sqrt{\bar{\kappa}^2 + \bar{\tau}^2}}$.

Conversely, if $\varepsilon_1 \varsigma_1 \bar{\sigma} \pm \varsigma_2 = \frac{1}{\sqrt{\bar{\kappa}^2 + \bar{\tau}^2}}$ holds, then taking into account (11) we have $\varsigma_1 \kappa + \varsigma_2 \tau = 1$, i.e., α is a Bertrand curve. □

Theorem 7. *Let β be an osculating mate of α.*

i) β is a Mannheim curve iff $\frac{1}{\tau}\cos\int\kappa ds=\varepsilon_1\lambda_1$, where λ_1 is non-zero constant.

ii) α is a Mannheim curve iff $\varepsilon_1\sqrt{\bar{\kappa}^2+\bar{\tau}^2}\bar{\sigma}^3=\lambda_2\left(1+\bar{\sigma}^2\right)$, where λ_2 is non-zero constant.

Proof. i) If β is a Mannheim curve, there is a non-zero constant λ_1 such that $\bar{\kappa}=\lambda_1(\bar{\kappa}^2+\bar{\tau}^2)$ holds [14, 10]. Writing (5) in the last equality gives $\frac{1}{\tau}\cos\int\kappa ds=\varepsilon_1\lambda_1$.

Conversely, if $\frac{1}{\tau}\cos\int\kappa ds=\varepsilon_1\lambda_1$ holds for a non-zero constant λ_1, from (4) and (5), we have that $\bar{\kappa}=\lambda_1(\bar{\kappa}^2+\bar{\tau}^2)$ holds, i.e., β is a Mannheim curve.

ii) If α is a Mannheim curve, the curvatures of α satisfy $\kappa=\lambda_2(\kappa^2+\tau^2)$, where λ_2 is a non-zero constant. Hence we get $\frac{1}{\kappa}=\lambda_2(1+\frac{\tau^2}{\kappa^2})$. By writing (11) in the last equality and considering (1), we obtain $\varepsilon_1\sqrt{\bar{\kappa}^2+\bar{\tau}^2}\bar{\sigma}^3=\lambda_2\left(1+\bar{\sigma}^2\right)$.

The converse is clear. □

Corollary 3. *Let β be an osculating mate of α. Then, α is a Mannheim curve iff the cuvatures of α and β satisfy $\bar{\kappa}=\pm\lambda\tau$, where λ is a non-zero constant.*

Theorem 8. *The curve β be an osculating mate of α.*

i) Let α be a Salkowski curve. Then β is a Salkowski curve iff $\tau=\varepsilon_1 e_3\sec(e_1 s+e_2)$, where e_i $(i=1,2,3)$ are real constants.

ii) Let β be a Salkowski curve with constant curvature $\bar{\kappa}=e_4$. Then α is a Salkowski curve with $\kappa=c>0$ iff $\varepsilon_1 e_4\bar{\tau}''-2c\bar{\tau}\bar{\tau}'=0$ holds.

iii) Let β be an anti-Salkowski curve with constant torsion $\bar{\tau}=e_5$. Then α is a Salkowski curve with $\kappa=c>0$ iff $\varepsilon_1 e_5\bar{\kappa}''+2c\bar{\kappa}\bar{\kappa}'=0$ holds.

Proof. i) Since α is a Salkowski curve, we have $\kappa=e_1>0$ is constant but τ is non-constant. Then, from (4) it follows that $\bar{\kappa}=\varepsilon_1\tau\cos\left(e_1 s+e_2\right),\ \bar{\tau}=\tau\sin\left(e_1 s+e_2\right)$, where e_2 is an integration contant. We get $\frac{\bar{\tau}}{\bar{\kappa}}=\varepsilon_1\tan(e_1 s+e_2)$. Hence β is a Salkowski curve with constant curvature $\bar{\kappa}=e_3>0$ iff $\tau=\varepsilon_1 e_3\sec(e_1 s+e_2)$.

The proofs of (ii) and (iii) are similar to the proof of (i). □

Let now $(\bar{T})$, $(\bar{N})$, $(\bar{B})$ denote the tangent indicatrix, the principal normal indicatrix and the binormal indicatrix of the osculating mate β, respectively. The curvatures of these spherical curves are computed as

$$\kappa_{\bar{T}}=\frac{\sqrt{\bar{\kappa}^2+\bar{\tau}^2}}{\bar{\kappa}},\ \tau_{\bar{T}}=\frac{\bar{\kappa}}{\bar{\kappa}^2+\bar{\tau}^2}\left(\frac{\bar{\tau}}{\bar{\kappa}}\right)', \tag{29}$$

$$\kappa_{\bar{N}} = \frac{\sqrt{\bar{\kappa}^2 + \bar{\tau}^2}}{\bar{\tau}}, \ \tau_{\bar{N}} = \frac{\bar{\kappa}^2}{\bar{\tau}(\bar{\kappa}^2 + \bar{\tau}^2)}(\frac{\bar{\tau}}{\bar{\kappa}})', \tag{30}$$

$$\kappa_{\bar{B}} = \frac{\sqrt{\bar{\kappa}^2 + \bar{\tau}^2}}{\bar{\tau}}, \ \tau_{\bar{B}} = \frac{\bar{\kappa}^2}{\bar{\tau}(\bar{\kappa}^2 + \bar{\tau}^2)}(\frac{\bar{\tau}}{\bar{\kappa}})', \tag{31}$$

respectively [22]. We obtain the following result.

Corollary 4. *The statements given below are equivalent.*
i) The tangent inticatrix $(\bar{T})$ *of* β *is a general helix.*
ii) The osculating mate β *is a slant helix.*
iii) α *is a general helix.*

Proof. From (5), (29) and (1), it follows that $\frac{\tau_{\bar{T}}}{\kappa_{\bar{T}}} = \bar{\sigma} = \varepsilon_1 \frac{\kappa}{\tau}$, which finishes the proof. □

Corollary 5. *The statements given below are equivalent.*
i) The principal normal inticatrix $(\bar{N})$ *of* β *is a general helix.*
ii) The osculating mate β *is a* $C-$*slant helix.*
iii) α *is a slant helix.*

Proof. If we write (5) in (30) and consider (1) and (2), we obtain $\frac{\tau_{\bar{N}}}{\kappa_{\bar{N}}} = \frac{1}{\bar{\mu}} = -\sigma$, which gives the desired results. □

Corollary 6. *The statements given below are equivalent.*
i) The binormal inticatrix $(\bar{B})$ *of* β *is a general helix.*
ii) The osculating mate β *is a slant helix.*
iii) α *is a general helix.*

Proof. Putting (5) into (31) and considering (1), we have $\frac{\tau_{\bar{B}}}{\kappa_{\bar{B}}} = -\bar{\sigma} = -\varepsilon_1 \frac{\kappa}{\tau}$, which gives the desired statements. □

3.1. Osculating type (OT) osculating mates. In this subsection we define osculating type osculating mates (or OT-osculating mates) in E^3 and give the relationships between osculating mates and OT-osculating mates. This section also gives a method to obtain a rectifying curve.

Consider a space curve $\alpha : I \to E^3$ with Frenet triangle $\{T, N, B\}$ and curvatures κ, τ. The vector $\widetilde{D} = \frac{\tau}{\kappa}(s)T(s) + B(s)$ is said to be the *modified Darboux vector* of α [11]. Let now the curve α be a Frenet curve and the curve β be an osculating mate of α. The curve β is called an *osculating-type osculating mate* (or an *OT-osculating mate*) of α, if the position vector of β is always contained in the osculating plane of α.

Considering the definition of OT-osculating mate, we can write

$$\beta(s) = m(s)T(s) + n(s)N(s), \tag{32}$$

where $m(s),\ n(s)$ are non-zero smooth functions of s. From (4),

$$\begin{cases} T = \sin\left(\int \kappa ds\right)\bar{T} + \cos\left(\int \kappa ds\right)\bar{B}, \\ N = \cos\left(\int \kappa ds\right)\bar{T} - \sin\left(\int \kappa ds\right)\bar{B}. \end{cases} \tag{33}$$

Writing (33) in (32) gives

$$\begin{cases} \beta(s) = \left[m\sin\left(\int \kappa ds\right) + n\cos\left(\int \kappa ds\right)\right]\bar{T} \\ \qquad\quad + \left[m\cos\left(\int \kappa ds\right) - n\sin\left(\int \kappa ds\right)\right]\bar{B}. \end{cases} \tag{34}$$

Defining

$$\begin{cases} \zeta(s) = m\sin\left(\int \kappa ds\right) + n\cos\left(\int \kappa ds\right), \\ \eta(s) = m\cos\left(\int \kappa ds\right) - n\sin\left(\int \kappa ds\right), \end{cases} \tag{35}$$

in (34) and differentiating the obtained equality gives

$$\bar{T} = \zeta'\bar{T} + (\zeta\bar{\kappa} - \eta\bar{\tau})\bar{N} + \eta'\bar{B}. \tag{36}$$

Hence we get

$$\eta = a = const,\ \zeta = s + b = \frac{\bar{\tau}}{\bar{\kappa}}a, \tag{37}$$

where $a,\ b$ are non-zero constants. Considering (37), we obtain

$$\beta(s) = a\left(\frac{\bar{\tau}}{\bar{\kappa}}\bar{T} + \bar{B}\right)(s) = a\tilde{\bar{D}}(s), \tag{38}$$

where $\tilde{\bar{D}}$ is the modified Darboux vector of β. Then the following theorem is obtained.

Theorem 9. *Let β be an OT-osculating mate of α. Then*
i) β is a rectifiyng curve.

ii) The position vector β and the modified Darboux vector $\tilde{\bar{D}}$ of an osculating mate β are linearly dependent.

Considering (35), (37) and (33), the last theorem gives a method to construct a rectifying curve by using an osculating mate as follows.

Corollary 7. *The curve β given by the parametrization*

$$\begin{cases} \beta(s) = \left[(s+b)\sin\left(\int \kappa ds\right) + a\cos\left(\int \kappa ds\right)\right]T(s) \\ \qquad\quad + \left[(s+b)\cos\left(\int \kappa ds\right) - a\sin\left(\int \kappa ds\right)\right]N(s) \end{cases} \tag{39}$$

is a rectifying curve and also an osculating mate of α, where $a,\ b$ are non-zero constants.

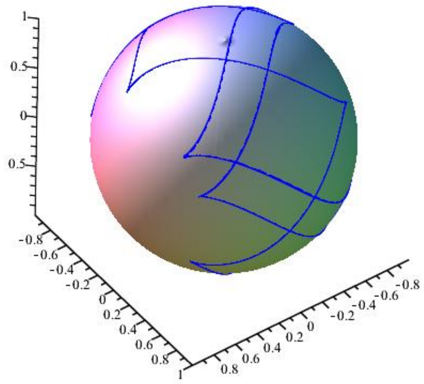

FIGURE 1. (a) Spherical helix α (left). (b) Osculating mate β (right).

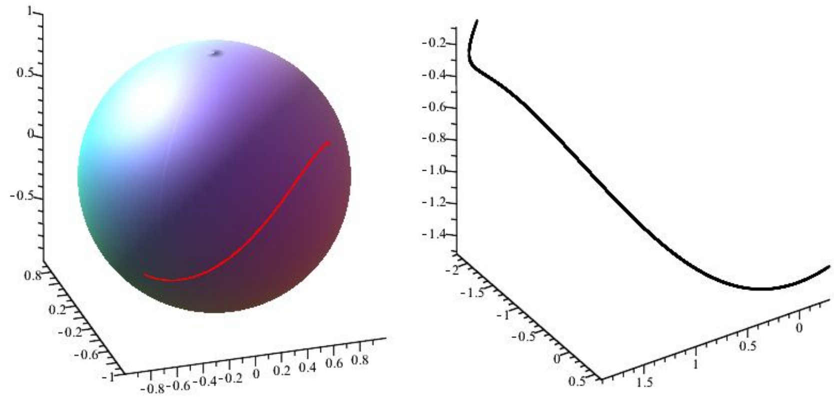

FIGURE 2. (a) Tangent indicatrix $\bar{T}$ (left). (b) OT-osculating mate of α (right).

Example 1. Let us consider the spherical helix α in E^3 defined by $\alpha(t) = \left(\frac{1}{\sqrt{2}}\sin t, \cos t\cos(\sqrt{2}t)+\frac{1}{\sqrt{2}}\sin t\sin(\sqrt{2}t), -\cos t\sin(\sqrt{2}t)+\frac{1}{\sqrt{2}}\sin t\cos(\sqrt{2}t)\right)$, (see Fig. 1(a)). The arc parameter of α is $s = \sin t$. The Frenet apparatus of α is computed as follows,

$$\begin{aligned} T(s) &= \frac{\sqrt{2}}{2}\left(1, -\sin(\sqrt{2}\arcsin s),\ -\cos(\sqrt{2}\arcsin s)\right), \\ N(s) &= \left(0, -\cos(\sqrt{2}\arcsin s),\ \sin(\sqrt{2}\arcsin s)\right), \\ B(s) &= -\frac{\sqrt{2}}{2}\left(1, \sin(\sqrt{2}\arcsin s),\ \cos(\sqrt{2}\arcsin s)\right), \end{aligned}$$

$$\kappa = \frac{1}{\sqrt{1-s^2}}, \quad \tau = -\frac{1}{\sqrt{1-s^2}}.$$

From (4) and (9), the osculating mate β of α is obtained as

$$\beta(s) = \int \left(sT(s) + \cos(\arcsin s)N(s)\right) ds = \left(\beta_1(s), \beta_2(s), \beta_3(s)\right),$$

where

$$\begin{aligned}
\beta_1(s) &= \frac{\sqrt{2}}{4}s^2 + c_1, \\
\beta_2(s) &= \int \left(-\frac{\sqrt{2}}{2}s\sin(\sqrt{2}\arcsin s) - \cos(\sqrt{2}\arcsin s)\cos(\arcsin s)\right) ds, \\
\beta_3(s) &= \int \left(-\frac{\sqrt{2}}{2}s\cos(\sqrt{2}\arcsin s) + \cos(\arcsin s)\sin(\sqrt{2}\arcsin s)\right) ds,
\end{aligned}$$

where c_1 is an integration constant (see Fig. 1(b)). From Corollary 5, the osculating mate β is a slant helix and its tangent indicatrix $\bar{T}$ is a general helix which is plotted in Fig. 2(a). Furthermore, by choosing $a = b = \sqrt{2}$, from (39) an OT-osculating mate of α, which is also a rectifying curve and plotted in Fig. 2(b), is obtained.

4. Conclusions

A new type of associated curves is introduced and called an osculating mate. The relations between a Frenet curve and its osculating mate are obtained. The obtained results allow to construct a slant helix, a C-slant helix, a spherical helix and a rectifying curve by considering an osculating mate of a Frenet curve.

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