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The sharp bound of the third Hankel determinant of the k^{th}-root transformation for bounded turning functions

Hari M. Srivastava[1,2,3,4,5,6], Biswajit Rath[7], K. Sanjay Kumar [8], and D. Vamshee Krishna[9]

Abstract. The objective of this paper is to estimate the sharp bound of the third Hankel determinant for the k^{th}-root transformation to the class of functions whose derivative has a positive real part satisfying the normalized conditions $f(0) = 0$ and $f'(0) = 1$ in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

1. Introduction

Let $\mathcal{A}$ be the family of mappings f of the type given by

$$f(z) = z + \sum_{t=2}^{\infty} a_t z^t \tag{1}$$

satisfying the normalized conditions $f(0) = 0$ and $f'(0) = 1$ in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ be the subfamily of $\mathcal{A}$, possessing univalent (schlicht) mappings.

Let k be a positive integer. A domain $\mathbb{U} \in \mathbb{C}$ is said to be *k-fold symmetric* if a rotation of $\mathbb{U}$ about the origin through an angle $\frac{2\pi}{k}$ carries $\mathbb{U}$ to itself. A function f is said to be *k-fold symmetric* in $\mathbb{D}$ if $f\left(e^{\frac{2\pi i}{k}} z\right) = e^{\frac{2\pi i}{k}} f(z)$ for every $z \in \mathbb{D}$. If f is regular and k-fold symmetric in $\mathbb{D}$, then

$$f(z) = b_1 z + b_{k+1} z^{k+1} + b_{2k+1} z^{2k+1} + \ldots . \tag{2}$$

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Corresponding author: Biswajit Rath

Conversely, if f is given by (2), then f is k-fold symmetric inside the circle of convergence of the series (see [5]). The k^{th}-root transform for the mapping f in (1) is given by

$$G(z) := \left[f(z^k)\right]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}. \tag{3}$$

In this sequel, we have introduced and interpreted the concept of the Hankel determinant for $G(z)$ for f in (1), with $q, t, k \in \mathbb{N} = \{1, 2, 3, ...\}$, as follows:

$$H_{q,t,k}(f) = \begin{vmatrix} b_{k(t-1)+1} & b_{kt+1} & \cdots & b_{k(t+q-2)+1} \\ b_{kt+1} & b_{k(t+1)+1} & \cdots & b_{k(t+q-1)+1} \\ \vdots & \vdots & \vdots & \vdots \\ b_{k(t+q-2)+1} & b_{k(t+q-1)+1} & \cdots & b_{k[t+2(q-1)-1]+1} \end{vmatrix} \quad (b_1 = 1). \tag{4}$$

This determinant has been studied by many authors, for different combinations of q and t in (4); when $k = 1$, it yields various types of Hankel determinants. For $q = 2$ and $t = 1$, we obtain the famous Fekete–Szegö functional mathematically denoted by $|H_{2,1}(f)| := |a_3 - a_2^2|$ (see [25]). Shi et al. [20] estimated an upper bound for Hankel determinant of the inverse of certain analytic functions subordinated to the exponential function. Recently, Srivastava et al. [22] estimated of the fourth Hankel determinant for a class of analytic functions with bounded turnings involving cardioid domains. Very recently, Srivastava et al. [23] calculated an upper bound to the second Hankel determinant for subclasses of bi-univalent functions associated with a nephroid domain and Rath et al. have obtained a sharp bound of certain second Hankel determinants for the class of inverses of starlike functions with respect to symmetric points [16].

Ali et al. [1] derived exact estimates for $|b_{2k+1} - \mu b_{k+1}^2|$, which represents the generalized Fekete–Szegö functional related to the function $G(z)$, when f is a member of specific subfamilies of S.

A small number of papers has been dedicated to $H_{3,1,1}(f) := H_{3,1}(f)$, named as the third-order Hankel determinant obtained for $q = 3,\ t = 1$ and $k = 1$ in (4). Babalola [2] is the first one, who tried to estimate an upper bound for the family of functions, namely bounded turning, starlike and convex, symbolized as $\Re$, S^* and $\mathcal{K}$, respectively, fulfilling $\text{Re}\{f'(z)\} > 0$, $\text{Re}\Big\{\frac{zf'(z)}{f(z)}\Big\} > 0$ and $\text{Re}\Big\{1 + \frac{zf''(z)}{f'(z)}\Big\} > 0$ in the unit disc $\mathbb{D}$. As a consequence of this paper, many articles containing results associated with the third Hankel determinant (see [9, 10, 17, 18, 19, 21, 24, 26]) for specific subsets of holomorphic functions were obtained. For our study in this paper, we choose $H_{3,1,k}(f)$, called as the third order Hankel determinant of the k^{th}-root tranformation for bounded turning functions.

The collection $\mathcal{P}$ includes all functions g, each one called as *Carathéodory function* [4], of the form:

$$g(z) = 1 + \sum_{t=1}^{\infty} c_t z^t, \tag{5}$$

holomorphic in $\mathbb{D}$ and satisfying $\mathrm{Re}\{g(z)\} > 0$, for $z \in \mathbb{D}$. The classes $\Re$ and $\mathcal{P}$ are invariant under the rotations, by Carathéodory Theorem (see [6, Vol. I, p. 80, Theorem 3]).

For the proof of our main result, we need the following lemmas. They contain the well-known formulas: for c_2 (e.g., [14, p. 166]), for c_3 due to Libera and Zlotkiewicz [12, 13] and for c_4 ([11]).

Lemma 1.1 ([14], p. 41). *If $g \in \mathcal{P}$, then $|c_t| \leq 2$ for $t \in \mathbb{N}$; the equality occurs for the function $g(z) = \frac{1+z}{1-z}$, $z \in \mathbb{D}$.*

Lemma 1.2 ([11, 12, 13]). *If $g \in \mathcal{P}$ and $c_1 \geq 0$, then*

$$2c_2 = c_1^2 + t\zeta,$$

$$4c_3 = c_1^3 + 2c_1 t\zeta - c_1 t\zeta^2 + 2t\left(1 - |\zeta|^2\right)\eta,$$

and

$$\begin{aligned} 8c_4 = {} & c_1^4 + 3c_1^2 t\zeta + \left(4 - 3c_1^2\right) t\zeta^2 + c_1^2 t\zeta^3 + 4t\left(1 - |\zeta|^2\right)\left(1 - |\eta|^2\right)\xi \\ & + 4t\left(1 - |\zeta|^2\right)\left(c_1\eta - c_1\zeta\eta - \bar{\zeta}\eta^2\right), \end{aligned}$$

where $t := 4 - c_1^2$, for some ζ, η and ξ such that $|\zeta| \leq 1$, $|\eta| \leq 1$ and $|\xi| \leq 1$.

2. Some important results

Result 2.1. For $k \in \mathbb{N}$,

$$-15 + 15k + 45k^2 - 47k^3 - 26k^4 = -15(k-1)(3k^2-1) - 2k^3 - 26k^4 \leq 0.$$

Result 2.2. It is clear that

$$-4k(-15 + 39k^2 - 12k^3) = 48k^3(k-4) + 36k^3 + 60k > 0, k \in \mathbb{N}, k \geq 4$$

and

$$-4k(-15 + 39k^2 - 12k^3) = 48k^3(k-4) + 36k^3 + 60k < 0, k = 1, 2, 3.$$

Result 2.3. Suppose that $c \in [0, 2]$ and $k \in \mathbb{N}$. Then

$$4k(c^2(-240k + 48k^2 + 432k^3) - c^4(-15k - 24k^2 + 45k^3)) \geq 0.$$

Proof.

$$\begin{aligned} (-240k + 48k^2 + 432k^3) - 4(-15k - 24k^2 + 45k^3) &= -180k + 144k^2 + 252k^3 \\ &= 36k(7k^2 + 4k - 5) > 0. \end{aligned}$$

Using the above fact and the inequality $4 - c^2 \geq 0$, we can easily see that

$$4k[c^2(-240k + 48k^2 + 432k^3) - c^4(-15k - 24k^2 + 45k^3)] \geq 0, k \in \mathbb{N}.$$

$\square$

Result 2.4. Suppose that $S : [0, 2] \to \mathbb{R}$, $3 \leq k \in \mathbb{N}$ and $S(c)$ is defined as follows:

$$S(c) = -4k(-256k^2(-5 + 4k) + c^4(8k^2 - 28k^3) + c^2(80k^2 + 224k^3)).$$

Then $S(c) \geq 0$.

Proof. $S(c)$ can be rewritten as follows:

$$\begin{aligned} S(c) &= 4096k^4 - 5120k^3 - 4(80c^2 + 8c^4)k^3 - 4(224c^2 - 28c^4)k^4 \\ &\geq 4096k^4 - 5120k^3 - 1792k^3 - 1792k^4 \\ &= 2304k^3(k-3) \geq 0 \text{ for } 3 \leq k \in \mathbb{N}. \end{aligned}$$

$\square$

Result 2.5. For $c \in [0, 2]$ and $k \in \mathbb{N}$,

$$-4k(-36c^2k^3 + 9c^4k^3) = 36k^4c^2(4 - c^2) \geq 0.$$

Result 2.6. The following facts hold true.

1. For $4 \leq k \in \mathbb{N}$,
   (a) $1440k^3 - 576k^4 < 0$,
   (b) $360k^2 - 288k^3 - 504k^4 < 0$,
   (c) $1728k^3 - 576k^4 < 0$.
2. For $c \in (0, 2)$,
   (a) $-1440c^3 + 360c^5 - 45c^6 < 0$,
   (b) $-17280c^2 + 2016c^3 + 2160c^4 - 504c^5 + 26c^6 < 0$,
   (c) $-17280c^2 + 3168c^3 + 2160c^4 - 792c^5 + 73c^6 < 0$,
   (d) $-14976c^2 + 2016c^3 + 1584c^4 - 504c^5 + 26c^6 < 0$,
   (e) $-21888c^2 + 3168c^3 + 3312c^4 - 792c^5 + 73c^6 < 0$.

Result 2.7. Suppose that $\Phi_1, \Phi_2 : [0,2] \times [0,1] \to \mathbb{R}$ are defined as follows:

$$\begin{aligned}\Phi_1(c,x) = {} & 2304c^2 + 2016c^3 - 576c^4 - 504c^5 + 26c^6 + 36864x \\ & + 9216cx - 18432c^2x - 2304c^3x + 2496c^4x - 48c^6x \\ & + 2304cx^2 + 4608c^2x^2 - 3168c^3x^2 - 1872c^4x^2 \\ & + 648c^5x^2 + 180c^6x^2 - 20480x^3 - 9216cx^3 \\ & + 10752c^2x^3 + 2304c^3x^3 - 960c^4x^3 - 112c^6x^3 \\ & - 2304cx^4 + 576c^2x^4 + 1152c^3x^4 - 288c^4x^4 - 144c^5x^4 + 36c^6x^4\end{aligned}$$

and

$$\begin{aligned}\Phi_2(c,x) = & - 6912c^2 + 1152c^3 + 1728c^4 - 288c^5 + 47c^6 \\ & - 23040cx + 4608c^3x - 624c^4x + 288c^5x + 156c^6x \\ & + 7680c^2x^2 - 1152c^3x^2 - 1536c^4x^2 + 288c^5x^2 \\ & - 96c^6x^2 - 20480x^3 + 23040cx^3 + 3840c^2x^3 \\ & - 4608c^3x^3 + 192c^4x^3 - 288c^5x^3 + 32c^6x^3.\end{aligned}$$

Then (a) $\Phi_1(c,x) \leq 19755$,

(b) $\Phi_2(c,x) \leq 3008$.

Proof of (a). The only real solution $(c,x) \in [0,2] \times [0,1]$ of the system $\frac{\partial \Phi_1}{\partial c} = 0$ and $\frac{\partial \Phi_1}{\partial x} = 0$ by a numerical computation is $(c,x) \approx (0.40539, 0.75649)$. Hence $\Phi_1(c,x) \leq \Phi_1(0.40539, 0.75649) = 19755$. □

Proof of (b). We can write $\Phi_2(c,x)$ as

$$\begin{aligned}\Phi_2(c,x) = & - 6912c^2 + 1152c^3 + 1728c^4 - 288c^5 + 47c^6 \\ & + \left(-23040c + 4608c^3 - 624c^4 + 288c^5 + 156c^6\right) x \\ & + \left(7680c^2 - 1152c^3 - 1536c^4 + 288c^5 - 96c^6\right) x^2 \\ & + \left(-20480 + 23040c + 3840c^2 - 4608c^3 + 192c^4 - 288c^5 + 32c^6\right) x^3.\end{aligned}$$

Since $-23040c + 4608c^3 - 624c^4 + 288c^5 + 156c^6 \leq 0$, we have

$$\begin{aligned}\Phi_2(c,x) \leq & - 6912c^2 + 1152c^3 + 1728c^4 - 288c^5 + 47c^6 \\ & + \left(-23040c + 4608c^3 - 624c^4 + 288c^5 + 156c^6\right) x^2 \\ & + \left(7680c^2 - 1152c^3 - 1536c^4 + 288c^5 - 96c^6\right) x^2 \\ & + \left(-20480 + 23040c + 3840c^2 - 4608c^3 + 192c^4 - 288c^5 + 32c^6\right) x^3 \\ = & - 6912c^2 + 1152c^3 + 1728c^4 - 288c^5 + 47c^6 \\ & + \left(-23040c + 7680c^2 + 3456c^3 - 2160c^4 + 576c^5 + 60c^6\right) x^2 \\ & + \left(-20480 + 23040c + 3840c^2 - 4608c^3 + 192c^4 - 288c^5 + 32c^6\right) x^3.\end{aligned}$$

Also, $-23040c+7680c^2+3456c^3-2160c^4+576c^5+60c^6 \leq 0$. Therefore

$$\begin{aligned}\Phi_2(c,x) \leq& -6912c^2+1152c^3+1728c^4-288c^5+47c^6\\ &+\left(-23040c+7680c^2+3456c^3-2160c^4+576c^5+60c^6\right)x^3\\ &+\left(-20480+23040c+3840c^2-4608c^3+192c^4-288c^5+32c^6\right)x^3\\ =& -6912c^2+1152c^3+1728c^4-288c^5+47c^6\\ &+\left(-20480+11520c^2-1152c^3-1968c^4+288c^5+92c^6\right)x^3.\end{aligned}$$

We can easily see that $-20480+11520c^2-1152c^3-1968c^4+288c^5+92c^6 \leq 0$.

Hence $\Phi_2(c,x) \leq -6912c^2+1152c^3+1728c^4-288c^5+47c^6 \leq 3008$. $\square$

Result 2.8. Suppose that $\Psi_1 : [0,2]\times[0,1] \to \mathbb{R}$ is defined as follows:

$$\Psi_1(c,x) = -15c^6+60c^4(4-c^2)x.$$

Then $\Psi_1(c,x) \leq -15c^6+240c^4-60c^6 = 240c^4-75c^6 \leq 364.089 < 365$.

Result 2.9. Suppose that $\Psi_2 : [0,2]\times[0,1] \to \mathbb{R}$ is defined as follows:

$$\begin{aligned}\Psi_2(c,x) =& -1440c^3+360c^5-45c^6-3840c^2x^2+1440c^3x^2\\ &+1200c^4x^2-360c^5x^2-60c^6x^2.\end{aligned}$$

Then

$$\Psi_2(c,x) = -15c^2\left(24c(4-c^2)+3c^4+4(-2+c)^2(2+c)(8+c)x^2\right) \leq 0.$$

Result 2.10. Suppose that $\Psi_1, \Psi_2, \Psi_3, \Psi_4 : [0,2]\times[0,1] \to \mathbb{R}$ are defined as follows:

$$\begin{aligned}\Psi_1(c,x) =& -15c^6+240c^4x-60c^6x,\\ \Psi_2(c,x) =& -1440c^3+360c^5-45c^6-3840c^2x^2+1440c^3x^2+1200c^4x^2\\ &-360c^5x^2-60c^6x^2,\\ \Psi_3(c,x) =&\ 1152c^3-288c^5+47c^6-23040cx-6912c^2x+4608c^3x+1104c^4x\\ &+288c^5x+156c^6x+768c^2x^2-1152c^3x^2+192c^4x^2+288c^5x^2\\ &-96c^6x^2-20480x^3+23040cx^3+10752c^2x^3-4608c^3x^3\\ &-1536c^4x^3-288c^5x^3+32c^6x^3,\\ \Psi_4(c,x) =& -17280c^2+2016c^3\\ &+2160c^4-504c^5+26c^6+9216cx+2304c^2x-2304c^3x-384c^4x\\ &-48c^6x-32256x^2+2304cx^2+23040c^2x^2-3168c^3x^2-4464c^4x^2\\ &+648c^5x^2+180c^6x^2+16384x^3-9216cx^3-9984c^2x^3+2304c^3x^3\\ &+1920c^4x^3-112c^6x^3-2304x^4-2304cx^4+1728c^2x^4\end{aligned}$$

$$+ 1152c^3x^4 - 432c^4x^4 - 144c^5x^4 + 36c^6x^4.$$

Then the following inequalities are true:
(a) $\Psi_4(c,x) \leq 0$,
(b) $\Psi_3(c,x) + \Psi_4(c,x) \leq 0$,
(c) $\Psi_2(c,x) + \Psi_3(c,x) + \Psi_4(c,x) \leq 0$,
(d) $15c^6 + \Psi_1(c,x) + \Psi_2(c,x) + \Psi_3(c,x) + \Psi_4(c,x) \leq 0$,
(e) $\Psi_1(c,x) + \Psi_2(c,x) + \Psi_3(c,x) + \Psi_4(c,x) \leq 0$.

Proof of (a). We have

$$\begin{aligned}
\Psi_4(c,x) =& - 17280c^2 + 2016c^3 + 2160c^4 - 504c^5 + 26c^6 \\
&+ \left(9216c + 2304c^2 - 2304c^3 - 384c^4 - 48c^6\right) x \\
&+ \left(-32256 + 2304c + 23040c^2 - 3168c^3 - 4464c^4 + 648c^5 + 180c^6\right) x^2 \\
&+ \left(16384 - 9216c - 9984c^2 + 2304c^3 + 1920c^4 - 112c^6\right) x^3 \\
&+ \left(-2304 - 2304c + 1728c^2 + 1152c^3 - 432c^4 - 144c^5 + 36c^6\right) x^4 \\
\leq& - 17280c^2 + 2016c^3 + 2160c^4 - 504c^5 + 26c^6 \\
&+ 3\left(9216c + 2304c^2 - 2304c^3 - 384c^4 - 48c^6\right) x \\
&+ \left(-32256 + 2304c + 23040c^2 - 3168c^3 - 4464c^4 + 648c^5 + 180c^6\right) x^2 \\
&+ \left(16384 - 9216c - 9984c^2 + 2304c^3 + 1920c^4 - 112c^6\right) x^3 \\
&+ \left(-2304 - 2304c + 1728c^2 + 1152c^3 - 432c^4 - 144c^5 + 36c^6\right) x^4 \\
=:& \ \psi_4(c,x).
\end{aligned}$$

By a numerical computation, all the real solutions $(c,x) \in (0,2) \times (0,1)$ of the system of equations $\frac{\partial \psi_4}{\partial c} = 0$ and $\frac{\partial \psi_4}{\partial x} = 0$ are obtained as follows: $(c_0,x_0) \approx (0,0)$, $(c_1,x_1) \approx (1.0541, 0.896071)$, $(c_2,x_2) \approx (0.683796, 0.54316)$. But $\psi_4(c_0,x_0) \approx 0$, $\psi_4(c_1,x_1) \approx -1447.46$, $\psi_4(c_2,x_2) \approx -1976.54$.
Hence $\Psi_4(c,x) \leq \psi_4(c,x) \leq 0$ □

Proof of (b). We can write $\Psi_3(c,x) + \Psi_4(c,x)$ as follows:

$$\begin{aligned}
\Psi_3 + \Psi_4 =& - 17280c^2 + 3168c^3 + 2160c^4 - 792c^5 + 73c^6 \\
&+ (-13824c - 4608c^2 + 2304c^3 + 720c^4 + 288c^5 + 108c^6)x \\
&+ (-32256 + 2304c + 23808c^2 - 4320c^3 - 4272c^4 + 936c^5 + 84c^6)x^2 \\
&+ (-4096 + 13824c + 768c^2 - 2304c^3 + 384c^4 - 288c^5 - 80c^6)x^3 \\
&+ (-2304 - 2304c + 1728c^2 + 1152c^3 - 432c^4 - 144c^5 + 36c^6)x^4.
\end{aligned}$$

Since

$$\begin{aligned}
&-17280c^2+3168c^3+2160c^4-792c^5+73c^6 \leq 0,\\
&-13824c-4608c^2+2304c^3+720c^4+288c^5+108c^6 \leq 0,\\
&-32256+2304c+23808c^2-4320c^3-4272c^4+936c^5+84c^6 \leq 0,
\end{aligned}$$

we have

$$\begin{aligned}
\Psi_3+\Psi_4 \leq{} & \left(-17280c^2+3168c^3+2160c^4-792c^5+73c^6\right)x^3\\
&+(-13824c-4608c^2+2304c^3+720c^4+288c^5+108c^6)x^3\\
&+(-32256+2304c+23808c^2-4320c^3-4272c^4+936c^5+84c^6)x^3\\
&+(-4096+13824c+768c^2-2304c^3+384c^4-288c^5-80c^6)x^3\\
&+(-2304-2304c+1728c^2+1152c^3-432c^4-144c^5+36c^6)x^4\\
={} & \left(-36352+2304c+2688c^2-1152c^3-1008c^4+144c^5+185c^6\right)x^3\\
&+\left(-2304-2304c+1728c^2+1152c^3-432c^4-144c^5+36c^6\right)x^4.
\end{aligned}$$

Also, since

$$\begin{aligned}
&-36352+2304c+2688c^2-1152c^3-1008c^4+144c^5+185c^6 \leq 0,\\
&-2304-2304c+1728c^2+1152c^3-432c^4-144c^5+36c^6 \leq 0,
\end{aligned}$$

we have $\Psi_3+\Psi_4 \leq 0$. □

Proof of (c). Proof follows from Part (b) and Result 2.9. □

Proof of (d). We can write $15c^6+\Psi_1(c,x)+\Psi_2(c,x)+\Psi_3(c,x)+\Psi_4(c,x)$ as follows:

$$\begin{aligned}
&15c^6+\Psi_1+\Psi_2+\Psi_3+\Psi_4\\
&\quad = -17280c^2+1728c^3+2160c^4-432c^5+28c^6\\
&\qquad +(-13824c-4608c^2+2304c^3+960c^4+288c^5+48c^6)x\\
&\qquad +(-32256+2304c+19968c^2-2880c^3-3072c^4+576c^5+24c^6)x^2\\
&\qquad +(-4096+13824c+768c^2-2304c^3+384c^4-288c^5-80c^6)x^3\\
&\qquad +(-2304-2304c+1728c^2+1152c^3-432c^4-144c^5+36c^6)x^4.
\end{aligned}$$

Since

$$\begin{aligned}
&-17280c^2+1728c^3+2160c^4-432c^5+28c^6 \leq 0,\\
&-13824c-4608c^2+2304c^3+960c^4+288c^5+48c^6 \leq 0,\\
&-32256+2304c+19968c^2-2880c^3-3072c^4+576c^5+24c^6 \leq 0,
\end{aligned}$$

we conclude that

$$
\begin{aligned}
&15c^6 + \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 \\
&\quad \leq \left(-17280c^2 + 1728c^3 + 2160c^4 - 432c^5 + 28c^6\right) x^3 \\
&\qquad + (-13824c - 4608c^2 + 2304c^3 + 960c^4 + 288c^5 + 48c^6)x^3 \\
&\qquad + (-32256 + 2304c + 19968c^2 - 2880c^3 - 3072c^4 + 576c^5 + 24c^6)x^3 \\
&\qquad + (-4096 + 13824c + 768c^2 - 2304c^3 + 384c^4 - 288c^5 - 80c^6)x^3 \\
&\qquad + (-2304 - 2304c + 1728c^2 + 1152c^3 - 432c^4 - 144c^5 + 36c^6)x^4 \\
&\quad = \left(-36352 + 2304c - 1152c^2 - 1152c^3 + 432c^4 + 144c^5 + 20c^6\right) x^3 \\
&\qquad + \left(-2304 - 2304c + 1728c^2 + 1152c^3 - 432c^4 - 144c^5 + 36c^6\right) x^4.
\end{aligned}
$$

Since

$$
\begin{aligned}
&- 36352 + 2304c - 1152c^2 - 1152c^3 + 432c^4 + 144c^5 + 20c^6 \leq 0, \\
&- 2304 - 2304c + 1728c^2 + 1152c^3 - 432c^4 - 144c^5 + 36c^6 \leq 0,
\end{aligned}
$$

we have $15c^6 + \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 \leq 0$. □

Proof of (e). It follows from the above proof of (d). □

Result 2.11. Suppose that $\omega_1, \omega_2, \omega_3, \omega_4 : [0, 2] \times [0, 1] \to \mathbb{R}$ are defined as follows:

$$
\begin{aligned}
\omega_1(c, x) &= -15c^6 - 240c^4x + 60c^6x, \\
\omega_2(c, x) &= - 1440c^3 + 360c^5 - 45c^6 - 3840c^2x^2 + 1440c^3x^2 + 1200c^4x^2 \\
&\quad - 360c^5x^2 - 60c^6x^2 = \Psi_2(c, x), \\
\omega_3(c, x) &=1152c^3 - 288c^5 + 47c^6 + 23040cx + 6912c^2x - 4608c^3x \\
&\quad - 1104c^4x - 288c^5x - 156c^6x + 768c^2x^2 - 1152c^3x^2 + 192c^4x^2 \\
&\quad + 288c^5x^2 - 96c^6x^2 - 12288x^3 - 23040cx^3 - 2560c^2x^3 + 4608c^3x^3 \\
&\quad + 1536c^4x^3 + 288c^5x^3 - 32c^6x^3, \\
\omega_4(c, x) &= - 17280c^2 + 2016c^3 + 2160c^4 - 504c^5 + 26c^6 - 6144cx \\
&\quad - 2304c^2x + 768c^3x + 384c^4x + 192c^5x + 48c^6x - 32256x^2 \\
&\quad + 2304cx^2 + 23040c^2x^2 - 3168c^3x^2 - 4464c^4x^2 \\
&\quad + 648c^5x^2 + 180c^6x^2 + 16384x^3 + 6144cx^3 + 1792c^2x^3 \\
&\quad - 768c^3x^3 - 1920c^4x^3 - 192c^5x^3 + 112c^6x^3 - 2304x^4 \\
&\quad - 2304cx^4 + 1728c^2x^4 + 1152c^3x^4 - 432c^4x^4 - 144c^5x^4 + 36c^6x^4.
\end{aligned}
$$

Then the following inequalities are true:

(a) $\omega_4(c, x) \leq 0$,

(b) $\omega_3(c, x) + \omega_4(c, x) \leq 0$,

(c) $\omega_2(c,x)+\omega_3(c,x)+\omega_4(c,x)\le 0$,
(d) $\omega_1(c,x)+\omega_2(c,x)+\omega_3(c,x)+\omega_4(c,x)\le 0$,
(e) $15c^6+\omega_1(c,x)+\omega_2(c,x)+\omega_3(c,x)+\omega_4(c,x)\le 0$.

Proof of (a). We can express $\omega_4(c,x)$ as follows:

$$\begin{aligned}
\omega_4(c,x) =& -17280c^2+2016c^3+2160c^4-504c^5+26c^6\\
&-\left(48c(4-c^2)(32+12c+4c^2+c^3)\right)x\\
&-\left(36(2-c)(4-c^2)(112+48c-28c^2-5c^3)\right)x^2\\
&+\left(16(4-c^2)(256+96c+92c^2+12c^3-7c^4)\right)x^3\\
&-\left(36(4-c^2)^2(4-c^2+4c)\right)x^4\\
\le&(-17280c^2+2016c^3+2160c^4-504c^5+26c^6)x^3\\
&-\left(48c(4-c^2)(32+12c+4c^2+c^3)\right)x^3\\
&-\left(36(2-c)(4-c^2)(112+48c-28c^2-5c^3)\right)x^3\\
&+\left(16(4-c^2)(256+96c+92c^2+12c^3-7c^4)\right)x^3\\
&-\left(36(4-c^2)^2(4-c^2+4c)\right)x^4\\
=&\left(-15872+2304c+5248c^2-1152c^3-3840c^4+144c^5+366c^6\right)x^3\\
&-\left(36(4-c^2)^2(4-c^2+4c)\right)x^4\le 0,
\end{aligned}$$

because $-15872+2304c+5248c^2-1152c^3-3840c^4+144c^5+366c^6\le 0$. □

Proof (b). We can write $\omega_3(c,x)+\omega_4(c,x)$ as

$$\begin{aligned}
\omega_3+\omega_4 =& -17280c^2+3168c^3+2160c^4-792c^5+73c^6\\
&+(16896c+4608c^2-3840c^3-720c^4-96c^5-108c^6)x\\
&+(-32256+2304c+23808c^2-4320c^3-4272c^4+936c^5+84c^6)x^2\\
&+(4096-16896c-768c^2+3840c^3-384c^4+96c^5+80c^6)x^3\\
&+(-2304-2304c+1728c^2+1152c^3-432c^4-144c^5+36c^6)x^4\\
\le& -17280c^2+3168c^3+2160c^4-792c^5+73c^6\\
&+1.9(16896c+4608c^2-3840c^3-720c^4-96c^5-108c^6)x\\
&+(-32256+2304c+23808c^2-4320c^3-4272c^4+936c^5+84c^6)x^2\\
&+(4096-16896c-768c^2+3840c^3-384c^4+96c^5+80c^6)x^3\\
&+(-2304-2304c+1728c^2+1152c^3-432c^4-144c^5+36c^6)x^4\\
=:& \ \Omega(c,x).
\end{aligned}$$

By a numerical computation, all the real solutions $(c,x)\in(0,2)\times(0,1)$ of the system of equations $\frac{\partial\Omega}{\partial c}=0$ and $\frac{\partial\Omega}{\partial x}=0$ are the following:

$(c_0, x_0) \approx (0,0)$, $(c_1, x_1) \approx (1.10825, 0.697702)$, $(c_2, x_2) \approx (0.682851, 0.430935)$. Corresponding to these values, we obtain $\Omega(c_0, x_0) \approx 0$, $\Omega(c_1, x_1) \approx -827.758$, $\Omega(c_2, x_2) \approx -1298.56$. Hence $\omega_3 + \omega_4 \leq \Omega(c, x) \leq 0$. $\square$

Proof of (c). It follows from the proof of Part (b) and Result 2.9. $\square$

Proof of (d). We can write $\omega_1(c, x)$ as follows:

$$\omega_1(c, x) = -15[c^2 + 4(4 - c^2)x]c^4 \leq 0$$

Now from Part (c), we can easily see that $\omega_1(c, x) + \omega_2(c, x) + \omega_3(c, x) + \omega_4(c, x) \leq 0$. $\square$

Proof of (e). It follows from $15c^6 + \omega_1(c, x) = -60x(4 - c^2)c^4 \leq 0$ and the proof of (c). $\square$

Result 2.12. Suppose that $\gamma_1 : [0, 2] \times [0, 1] \to \mathbb{R}$ is defined as follows:

$$\begin{aligned}\gamma_1(c, x) = &- 2304c^2 + 2016c^3 + 576c^4 - 504c^5 + 26c^6 + 36864x \\ &- 6144cx - 18432c^2x + 768c^3x + 2112c^4x + 192c^5x + 48c^6x \\ &+ 2304cx^2 + 9216c^2x^2 - 3168c^3x^2 - 3024c^4x^2 \\ &+ 648c^5x^2 + 180c^6x^2 - 20480x^3 + 6144cx^3 + 17920c^2x^3 \\ &- 768c^3x^3 - 3648c^4x^3 - 192c^5x^3 + 112c^6x^3 - 2304cx^4 \\ &+ 576c^2x^4 + 1152c^3x^4 - 288c^4x^4 - 144c^5x^4 + 36c^6x^4.\end{aligned}$$

Then $\gamma_1(c, x) \leq 19521$.

Proof. We can rewrite $\gamma_1(c, x)$ as follows:

$$\begin{aligned}\gamma_1(c, x) = &- 2304c^2 + 2016c^3 + 576c^4 - 504c^5 + 26c^6 + 36864x \\ &+ (-6144c - 18432c^2 + 768c^3 + 2112c^4 + 192c^5 + 48c^6)x \\ &+ (2304c + 9216c^2 - 3168c^3 - 3024c^4 + 648c^5 + 180c^6)x^2 \\ &+ (-20480 + 6144c + 17920c^2 - 768c^3 - 3648c^4 - 192c^5 + 112c^6)x^3 \\ &+ (-2304c + 576c^2 + 1152c^3 - 288c^4 - 144c^5 + 36c^6)x^4.\end{aligned}$$

Since $-6144c - 18432c^2 + 768c^3 + 2112c^4 + 192c^5 + 48c^6 \leq 0$, we have

$$\begin{aligned}\gamma_1(c, x) \leq &- 2304c^2 + 2016c^3 + 576c^4 - 504c^5 + 26c^6 + 36864x \\ &+ (-6144c - 18432c^2 + 768c^3 + 2112c^4 + 192c^5 + 48c^6)x^2 \\ &+ (2304c + 9216c^2 - 3168c^3 - 3024c^4 + 648c^5 + 180c^6)x^2 \\ &+ (-20480 + 6144c + 17920c^2 - 768c^3 - 3648c^4 - 192c^5 + 112c^6)x^3 \\ &+ (-2304c + 576c^2 + 1152c^3 - 288c^4 - 144c^5 + 36c^6)x^4 \\ =: &\ \Gamma_1(c, x).\end{aligned}$$

By a numerical computation, all the real solutions $(c,x)\in(0,2)\times(0,1)$ of the system of equations $\frac{\partial\Gamma_1}{\partial c}=0$ and $\frac{\partial\Gamma_1}{\partial x}=0$ are obtained as follows: $(c_0,x_0)\approx(0.929604,0.962107),(c_1,x_1)\approx(0.155788,0.780765)$. Then we have $\Gamma_1(c_0,x_0)\approx 19520.4<19521$ and $\Gamma_1(c_1,x_1)\approx 19013.3<19521$. Hence $\gamma_1(c,x)\le\Gamma_1(c,x)<19521$. $\square$

Result 2.13. Suppose that $\gamma_2:[0,2]\times[0,1]\to\mathbb{R}$ is defined as follows:

$$\begin{aligned}\gamma_2(c,x)=&\,6912c^2+1152c^3-1728c^4-288c^5+47c^6+23040cx-4608c^3x\\&+624c^4x-288c^5x-156c^6x-6144c^2x^2-1152c^3x^2\\&+1920c^4x^2+288c^5x^2-96c^6x^2-12288x^3-23040cx^3\\&+4352c^2x^3+4608c^3x^3-192c^4x^3+288c^5x^3-32c^6x^3\le 14081.\end{aligned}$$

Then $\gamma_2(c,x)\le 14081$.

Proof. By a numerical computation, the only real solution $(c,x)\in(0,2)\times(0,1)$ of the system of equations $\frac{\partial\gamma_2}{\partial c}=0$ and $\frac{\partial\gamma_2}{\partial x}=0$ is $(c_0,x_0)\approx(1.42762,0.439375)$. Then we have $\gamma_2(c_0,x_0)\approx 14080.3<14081$. Hence $\gamma_2(c,x)<14081$. $\square$

3. A bound for $H_{3,1,k}(f)$

Theorem 1. *If $f\in\Re$, then*

$$\left|H_{3,1,k}(f)\right|\le\frac{1}{4k^2},$$

and the result is sharp for $p(z):=(1+z^3)/(1-z^3)$.

Proof. For $f\in\Re$, there exists a holomorphic function $p\in\mathcal{P}$ such that

$$f'(z)=p(z),\quad z\in\mathbb{D}. \tag{6}$$

Substituting the values for f and p in (6), it simplifies as follows:

$$a_{n+1}=\frac{c_n}{n+1},\ n\in\mathbb{N}. \tag{7}$$

For the mapping f in (1), a calculation gives

$$\begin{aligned}\left[f(z^k)\right]^{\frac{1}{k}}=&\left[z^k+\sum_{n=2}^{\infty}a_nz^{nk}\right]^{\frac{1}{k}}=\left[z+\frac{1}{k}a_2z^{k+1}+\left\{\frac{1}{k}a_3+\frac{(1-k)}{2k^2}a_2^2\right\}z^{2k+1}\right.\\&+\left\{\frac{1}{k}a_4+\frac{(1-k)}{k^2}a_2a_3+\frac{(1-k)(1-2k)}{6k^3}a_2^3\right\}z^{3k+1}\\&+\left\{\frac{1}{k}a_5+\frac{(1-k)}{2k^2}(a_3^2+2a_2a_4)+\frac{(1-k)(1-2k)}{2k^3}a_2^2a_3\right.\\&\left.\left.+\frac{(1-k)(1-2k)(1-3k)}{24k^4}a_2^4\right\}z^{4k+1}+\cdots\right].\end{aligned} \tag{8}$$

Comparing the coefficients of z^{k+1}, z^{2k+1}, z^{3k+1} and z^{4k+1} in the expressions (2) and (8), we obtain

$$\begin{aligned}
b_{k+1} =& \frac{1}{k}a_2;\; b_{2k+1} = \frac{1}{k}a_3 + \frac{(1-k)}{2k^2}a_2^2,\\
b_{3k+1} =& \left[\frac{1}{k}a_4 + \frac{(1-k)}{k^2}a_2a_3 + \frac{(1-k)(1-2k)}{6k^3}a_2^3\right],\\
b_{4k+1} =& \left[\frac{1}{k}a_5 + \frac{(1-k)}{2k^2}(a_3^2+2a_2a_4) + \frac{(1-k)(1-2k)}{2k^3}a_2^2a_3\right.\\
&\left. + \frac{(1-k)(1-2k)(1-3k)}{24k^4}a_2^4\right].
\end{aligned} \tag{9}$$

From (7) and (9), we obtain

$$\begin{aligned}
b_{k+1} =& \frac{c_1}{2k},\\
b_{2k+1} =& -\frac{(-1+k)c_1^2}{8k^2} + \frac{c_2}{3k},\\
b_{3k+1} =& \frac{(-1+k)(-1+2k)c_1^3}{48k^3} - \frac{(-1+k)c_1c_2}{6k^2} + \frac{c_3}{4k}\\
&\text{and}\\
b_{4k+1} =& -\frac{(-1+k)(-1+2k)(-1+3k)c_1^4}{384k^4} + \frac{(-1+k)(-1+2k)c_1^2c_2}{24k^3}\\
&- \frac{(-1+k)c_2^2}{18k^2} - \frac{(-1+k)c_1c_3}{8k^2} + \frac{c_4}{5k}.
\end{aligned} \tag{10}$$

Now, for $q = 3$, $t = 1$ in (4), we have

$$H_{3,1,k}(f) = \begin{vmatrix} 1 & b_{k+1} & b_{2k+1} \\ b_{k+1} & b_{2k+1} & b_{3k+1} \\ b_{2k+1} & b_{3k+1} & b_{4k+1} \end{vmatrix}. \tag{11}$$

Using the values of b_j $(j = k+1, 2k+1, 3k+1, 4k+1)$ from (10) in (11), it simplifies to give

$$\begin{aligned}
&H_{3,1,k}(f)\\
=& \frac{1}{138240k^6}\left[15(1+k)^2(-1-k+2k^2)c_1^6 - 120k(-1-2k+k^2+2k^3)c_1^4c_2\right.\\
&+960k^2(1+k)(-1+k)c_1^2c_2^2 - 2560k^3(1+k)c_2^3\\
&+5760k^3(1+k)c_1c_2c_3 - 720k^2(-1+k^2)c_1^3c_3\\
&\left.-8640k^4c_3^2 + 9216k^4c_2c_4 - 3456k^3(1+k)c_1^2c_4\right].
\end{aligned} \tag{12}$$

Using Lemma 1.2, $c_1 := c$ and $t := (4-c^2)$ in (12), upon simplification we obtain

$$H_{3,1,k}(f) = \frac{1}{138240k^6}\Big[\tau_1(c,\zeta) + \tau_2(c,\zeta)\eta + \tau_3(c,\zeta)\eta^2 + \tau_4(c,\zeta)(1-|\eta|^2)\Big], \tag{13}$$

where

$$\begin{aligned}
\tau_1(c,\zeta) = {} & c^6\left(-15+15k+45k^2-47k^3-26k^4\right) \\
& + \left(4-c^2\right)\Big\{4c^4k\left(15-39k^2+12k^3\right)\zeta + 36k^4c^2\left(4-c^2\right)\zeta^4 \\
& + 4k\left[c^2\left(-240k+48k^2+432k^3\right) - c^4\left(-15k-24k^2+45k^3\right)\right]\zeta^2 \\
& - 4k\left[-256k^2\left(-5+4k\right) + c^4\left(8k^2-28k^3\right) + c^2\left(80k^2+224k^3\right)\right]\zeta^3\Big\}, \\
\tau_2(c,\zeta) = {} & \left(1-|\zeta|^2\right)\left(4-c^2\right)\Big\{c\left(4-c^2\right)\zeta\left(1440k^3-576k^4-144k^4\zeta\right) \\
& + c^3(360k^2-288k^3-504k^4+\left(1728k^3-576k^4\right)\zeta)\Big\}, \\
\tau_3(c,\zeta) = {} & \left(1-|\zeta|^2\right)\left(4-c^2\right)\left\{c^2\left(1728k^3-576k^4\right)\bar{\zeta}\right. \\
& \left. - \left(4-c^2\right)\left(144k^4|\zeta|^2+2160k^4\right)\right\} \\
\tau_4(c,\zeta) = {} & \left(1-|\zeta|^2\right)\left(4-c^2\right)\left\{\left(-1728k^3+576k^4\right)c^2 + 2304k^4\left(4-c^2\right)\zeta\right\}\xi.
\end{aligned}$$

Case I: Let $4 \leq k \in \mathbb{N}$.

Taking modulus on both sides of the expression (13), using Results 2.1, 2.3, 2.4, 2.5, 2.6(1), and since $|\xi| \leq 1$, with $|\zeta| := x \in [0,1]$, $|\eta| := y \in [0,1]$ and $c_1 := c \in [0,2]$ in (13), we obtain

$$\left|H_{3,1,k}(f)\right| \leq \frac{F_1(c,x,y)}{138240k^6}, \tag{14}$$

where $F_1 : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$F_1(c,x,y) = f_1(c,x) + f_2(c,x)y + f_3(c,x)y^2 + f_4(c,x)\left(1-y^2\right), \tag{15}$$

where

$$\begin{aligned}
f_1(c,x) = {} & -c^6\left(-15+15k+45k^2-47k^3-26k^4\right) \\
& + \left(4-c^2\right)\Big\{4c^4k\left(15-39k^2+12k^3\right)x + 36k^4c^2\left(4-c^2\right)x^4 \\
& + 4k\left[c^2\left(-240k+48k^2+432k^3\right) - c^4\left(-15k-24k^2+45k^3\right)\right]x^2 \\
& - 4k\left[-256k^2\left(-5+4k\right) + c^4\left(8k^2-28k^3\right) + c^2\left(80k^2+224k^3\right)\right]x^3\Big\}, \\
f_2(c,x) = {} & \left(1-x^2\right)\left(4-c^2\right)\left\{c\left(4-c^2\right)x\left(-1440k^3+576k^4+144k^4x\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + c^3[-360k^2 + 288k^3 + 504k^4 + \left(-1728k^3 + 576k^4\right)x]\},\\
f_3(c,x) &= \left(1-x^2\right)\left(4-c^2\right)\left\{c^2\left(-1728k^3+576k^4\right)x\right.\\
&\quad \left. + \left(4-c^2\right)\left(144k^4x^2+2160k^4\right)\right\},\\
f_4(c,x) &= \left(1-x^2\right)\left(4-c^2\right)\left\{\left(-1728k^3+576k^4\right)c^2 + 2304k^4\left(4-c^2\right)x\right\}.
\end{aligned}
$$

Now, we will maximize the function $F_1(c,x,y)$ in the region of the parallelepiped $[0,2]\times[0,1]\times[0,1]$, with $c\in[0,2], x\in[0,1], y\in[0,1],\ 4\le k\in\mathbb{N}$.
A. On the vertices of the parallelepiped, from (15), we have

$$
\begin{aligned}
F_1(0,0,0) &= 0,\\
F_1(0,1,0) &= F_1(0,1,1) = 16384k^4 - 20480k^3 \le 16384k^4,\\
F_1(0,0,1) &= 34560k^4,\\
F_1(2,0,0) &= F_1(2,0,1) = F_1(2,1,0) = F_1(2,1,1)\\
&= 960 - 960k - 2880k^2 + 3008k^3 + 1664k^4\\
&\le 960 + 3008k^3 + 1664k^4\\
&\le 960k^4 + 3008k^4 + 1664k^4 = 5632k^4.
\end{aligned}
$$

B. On the edges of $F_1(c,x,y)$, from (15), we have

(a) $c=0$ and $x=0$. Then, for $y\in(0,1)$,

$$F_1(0,0,y) = 34560k^4y^2 \le 34560k^4.$$

(b) $c=0$ and $x=1$, $y\in(0,1)$. Then

$$F_1(0,1,y) = 16384k^4 - 20480k^3 \le 16384k^4.$$

(c) $c=0$ and $y=0$. Then

$$
\begin{aligned}
F_1(0,x,0) &= -20480k^3x^3 + k^4(36864x - 20480x^3)\\
&\le -20480x^3k^3 + 19037k^4\\
&\le 19037k^4, \quad x\in(0,1).
\end{aligned}
$$

(d) $c=0$ and $y=1$. Then

$$
\begin{aligned}
F_1(0,x,1) &= 34560k^4 - 20480k^3x^3 - 256k^4x^2(126 - 64x + 9x^2)\\
&\le 34560k^4, \quad x\in(0,1).
\end{aligned}
$$

(e) $x = 0$ and $y = 0$. Then

$$\begin{aligned}
F_1(c,0,0) &= 15c^6 - 15c^6k - 45c^6k^2 + (-6912c^2 + 1728c^4 + 47c^6)k^3 \\
&\quad + (2304c^2 - 576c^4 + 26c^6)k^4 \\
&\leq 15c^6 - 15c^6k - 45c^6k^2 + 3008k^3 + 2572k^4 \\
&\leq 960 + 3008k^3 + 2572k^4 \\
&\leq 960k^4 + 3008k^4 + 2572k^4 = 6540k^4, \quad c \in (0,2).
\end{aligned}$$

(f) $x = 0$ and $y = 1$. Then

$$\begin{aligned}
F_1(c,0,1) &= 34560k^4 + 15c^6 - 15c^6k + (-1440c^3 + 360c^5 - 45c^6)k^2 \\
&\quad + (1152c^3 - 288c^5 + 47c^6)k^3 \\
&\quad + (-17280c^2 + 2016c^3 + 2160c^4 - 504c^5 + 26c^6)k^4 \\
&\quad \text{(Using Result 2.6(2))} \\
&\leq 34560k^4 + 15c^6 - 15c^6k + (-1440c^3 + 360c^5 - 45c^6)k^2 \\
&\quad + (1152c^3 - 288c^5 + 47c^6)k^3 \\
&\quad + (-17280c^2 + 2016c^3 + 2160c^4 - 504c^5 + 26c^6)k^3 \\
&= 34560k^4 + 15c^6(1-k) + (-1440c^3 + 360c^5 - 45c^6)k^2 \\
&\quad + (-17280c^2 + 3168c^3 + 2160c^4 - 792c^5 + 73c^6)k^3 \\
&\leq 34560k^4, \quad c \in (0,2).
\end{aligned}$$

(g) $x = 1$ and $y = 1$; $x = 1$ and $y = 0$. Then

$$\begin{aligned}
F_1(c,1,y) &= 15c^6 + (240c^4 - 75c^6)k + (-3840c^2 + 1200c^4 - 105c^6)k^2 \\
&\quad + (-20480 + 4608c^2 - 240c^4 + 139c^6)k^3 \\
&\quad + (16384 - 192c^2 - 1200c^4 + 82c^6)k^4 \\
&\leq 15 \times 64 + (365)k + (0)k^2 + (3008)k^3 + (16384)k^4 \\
&= 960 + 365k + 3008k^3 + 16384k^4 \\
&\leq 960k^4 + 365k^4 + 3008k^4 + 16384k^4 = 20717k^4.
\end{aligned}$$

(h) $c = 2,\ y = 0$; or $c = 2,\ y = 1$ or $c = 2,\ x = 0$ or $c = 2,\ x = 1$ or $x = 0,\ y = 0$. Then, for $c \in (0,2)$, $x \in (0,1)$ and $y \in (0,1)$,

$$F_1(2,x,0) = F_1(2,x,1) = F_1(2,0,y) = F_1(2,1,y) \leq 5632k^4.$$

C. Considering the faces of the parallelepiped, from (15), we get

(a) $c = 2,\ x \in (0,1),\ y \in (0,1)$. Then

$$F_1(2,x,y) \leq 5632k^4.$$

(b) $c=0,\ x\in(0,1),\ y\in(0,1)$. Then

$$\begin{aligned}
&F_1(0,x,y)\\
&=36864k^4x-20480k^3x^3-20480k^4x^3+2304k^4(15-x)(1-x)^2(1+x)y^2\\
&\le 36864k^4x-20480k^3x^3-20480k^4x^3+2304k^4(15-x)(1-x)^2(1+x)\\
&=34560k^4-20480k^3x^3-256k^4x^2(126-64x+9x^2)\\
&\le 34560k^4,\ \text{for } x\in(0,1) \text{ and } y\in(0,1).
\end{aligned}$$

(c) $x=0,\ c\in(0,2),\ y\in(0,1)$. Then

$$\begin{aligned}
F_1(c,0,y)=&-c^6(-15+15k+45k^2-47k^3-26k^4)\\
&+(4-c^2)(c^3(-360k^2+288k^3+504k^4)y\\
&+2160(4-c^2)k^4y^2+c^2(-1728k^3+576k^4)(1-y^2))\\
\le&-c^6(-15+15k+45k^2-47k^3-26k^4)\\
&+(4-c^2)[c^3(-360k^2+288k^3+504k^4)y\\
&+2160(4-c^2)k^4y^2+c^2(-1728k^3+576k^4)]\\
\le&-c^6(-15+15k+45k^2-47k^3-26k^4)\\
&+(4-c^2)[c^3(-360k^2+288k^3+504k^4)\\
&+2160(4-c^2)k^4+c^2(-1728k^3+576k^4)]\\
=&\ 34560k^4+15c^6-15c^6k+(-1440c^3+360c^5-45c^6)k^2\\
&+(-6912c^2+1152c^3+1728c^4-288c^5+47c^6)k^3\\
&+(-14976c^2+2016c^3+1584c^4-504c^5+26c^6)k^4.\\
&\text{(using Result 2.6(2))}\\
\le&\ 34560k^4+15c^6-15c^6k+(-1440c^3+360c^5-45c^6)k^2\\
&+(-6912c^2+1152c^3+1728c^4-288c^5+47c^6)k^3\\
&+(-14976c^2+2016c^3+1584c^4-504c^5+26c^6)k^3\\
=&\ 34560k^4+15c^6(1-k)+(-1440c^3+360c^5-45c^6)k^2\\
&+(-21888c^2+3168c^3+3312c^4-792c^5+73c^6)k^3\\
\le&\ 34560k^4.
\end{aligned}$$

(d) For the face $x=1$, we observe that $F_1(c,1,y)$ is independent of y, so it is the same as $\mathbf{B}$(g), i.e., $F_1(c,1,y)\le 20717k^4$ for $c\in(0,2)$ and $y\in(0,1)$.

(e) $y=0,\ c\in(0,2),\ x\in(0,1)$. Then

$$F_1(c,x,0)=-c^6(-15+15k+45k^2-47k^3-26k^4)$$

$$
\begin{aligned}
&+(4-c^2)\Big[-5120k^3x^3+k^4(9216x-5120x^3)\\
&+c^4[60kx+60k^2x^2+k^3(-156x+96x^2-32x^3)\\
&+k^4(48x-180x^2+112x^3-36x^4)]\\
&+c^2[-960k^2x^2+k^3(-1728+1920x^2-320x^3)\\
&+k^4(576-2304x+1152x^2+1408x^3+144x^4)]\Big]\\
\leq&-c^6(-15+15k+45k^2-47k^3-26k^4)\\
&+(4-c^2)\Big[-5120k^3x^3+k^4\left(6144\sqrt{\frac{3}{5}}\right)\\
&+c^4[60kx+60k^2x^2+k^3(0)+k^4(4)]\\
&+c^2[-960k^2x^2+k^3(-128)+k^4(976)]\Big]\\
=&-c^6(-15+15k+45k^2-47k^3-26k^4)\\
&+(4-c^2)\Big[-128c^2k^3+\left(6144\sqrt{\frac{3}{5}}\right)k^4\\
&+976c^2k^4+4c^4k^4+60c^4kx+(-960c^2+60c^4)k^2x^2-5120k^3x\Big]\\
\leq&-c^6(-15+15k+45k^2-47k^3-26k^4)\\
&+(4-c^2)\Big[-128c^2k^3+\left(6144\sqrt{\frac{3}{5}}\right)k^4+976c^2k^4+4c^4k^4+60c^4k\Big]\\
=&15c^6+[-15c^6+60c^4(4-c^2)]k-45c^6k^2+[47c^6-128c^2(4-c^2)]k^3\\
&+\left[26c^6+\left(6144\sqrt{\frac{3}{5}}\right)(4-c^2)+976c^2(4-c^2)+4c^4(4-c^2)\right]k^4\\
\leq&15\times 64+(365)k+(3008)k^3+(19037)k^4\\
<&960k^4+365k^4+3008k^4+19037k^4=23370k^4.
\end{aligned}
$$

(f) $y=1, c\in(0,2),\ x\in(0,1)$. Then

$$
\begin{aligned}
&F_1(c,x,1)\\
&=34560k^4+15c^6+k\Psi_1(c,x)+k^2\Psi_2(c,x)+k^3\Psi_3(c,x)+k^4\Psi_4(c,x),
\end{aligned}
$$

where Ψ_1, Ψ_2, Ψ_3 and Ψ_4 are defined in Result 2.10. Applying Result 2.10, we obtain

$$
\begin{aligned}
&F_1(c,x,1)\\
&\quad< 34560k^4+15c^6+k\Psi_1(c,x)+k^2\Psi_2(c,x)+k^3\{\Psi_3(c,x)+\Psi_4(c,x)\}
\end{aligned}
$$

$$
\begin{aligned}
&< 34560k^4 + 15c^6 + k\Psi_1(c,x) + k^2\left\{\Psi_2(c,x) + \Psi_3(c,x) + \Psi_4(c,x)\right\} \\
&< 34560k^4 + 15c^6 + k\left\{\Psi_1(c,x) + \Psi_2(c,x) + \Psi_3(c,x) + \Psi_4(c,x)\right\} \\
&< 34560k^4 + \left\{15c^6 + \Psi_1(c,x) + \Psi_2(c,x) + \Psi_3(c,x) + \Psi_4(c,x)\right\} \\
&< 34560k^4.
\end{aligned}
$$

D. Now, considering the interior region of the parallelepiped, i.e., $(0,2)\times(0,1)\times(0,1)$, we have from (15),

$$F_1(c,x,y) = f_1(c,x) + f_2(c,x)y + \left\{f_3(c,x) - f_4(c,x)\right\}y^2 + f_4(c,x).$$

Case (I). Suppose that $f_3(c,x) - f_4(c,x) > 0$. Then

$$F_1(c,x,y) \leq f_1(c,x) + f_2(c,x) + f_3(c,x) = F(c,x,1) < 34560k^4.$$

Case (II). Suppose that $f_3(c,x) - f_4(c,x) \leq 0$. Then

$$F_1(c,x,y) \leq f_1(c,x) + f_2(c,x)y + f_4(c,x) \leq f_1(c,x) + f_2(c,x) + f_4(c,x).$$

Now

$$f_1 + f_2 + f_4 = 15c^6 + \Psi_1(c,x)k + \Psi_2(c,x)k^2 + \Phi_2(c,x)k^3 + \Phi_1(c,x)k^4,$$

where Φ_1, Φ_2, Ψ_1 and Ψ_2 are defined in Results 2.7, 2.8, 2.9. Applying Results 2.7, 2.8, 2.9, we obtain

$$f_1 + f_2 + f_4 \leq 15\times 64 + k(365) + k^2(0) + k^3(3008) + k^4(19755) < 34560k^4.$$

Hence $F_1(c,x,y) < 34560k^4$.

In review of Cases **A**, **B**, **C** and **D**, we obtain

$$\max\left\{F_1(c,x,y) : c\in[0,2], x\in[0,1] \text{ and } y\in[0,1]\right\} = 34560k^4. \tag{16}$$

From the expressions (14) and (16), we get

$$\left|H_{3,1}(f)\right| \leq \frac{1}{4k^2},\ k\geq 4. \tag{17}$$

It remains to prove the result for $k = 1,2,3$.

Case II: Let $k = 1,2,3$.

Taking modulus on both sides of the expression (13), using Results 2.1, 2.3, 2.5, 2.6, and since $|\xi|\leq 1$, with $|\zeta| := x\in[0,1]$, $|\eta| := y\in[0,1]$ and $c_1 := c\in[0,2]$ in (13), we obtain

$$\left|H_{3,1}(f)\right| \leq \frac{F_2(c,x,y)}{138240k^6}, \tag{18}$$

where $F_2 : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$F_2(c,x,y) = f_5(c,x) + f_6(c,x)y + f_7(c,x)y^2 + f_8(c,x)\left(1-y^2\right), \tag{19}$$

where

$$
\begin{aligned}
f_5(c,x) = & -c^6\left(-15+15k+45k^2-47k^3-26k^4\right) \\
& + \left(4-c^2\right)\Big\{4c^4k\left(-15+39k^2-12k^3\right)x + 36k^4c^2\left(4-c^2\right)x^4 \\
& + 4k\left(c^2\left(-240k+48k^2+432k^3\right)-c^4\left(-15k-24k^2+45k^3\right)\right)x^2 \\
& + 4k\Big(256k^2+256k^2\left(-4+4k\right)+c^4\left(8k^2-28k^3\right) \\
& + c^2\left(80k^2+224k^3\right)\Big)x^3\Big\}, \\
f_6(c,x) = & \left(1-x^2\right)\left(4-c^2\right)\left\{c\left(4-c^2\right)x\left(1440k^3-384k^4+144k^4x\right)\right. \\
& \left. + c^3(-360k^2+288k^3+504k^4+\left(1728k^3-576k^4\right)x)\right\}, \\
f_7(c,x) = & \left(1-x^2\right)\left(4-c^2\right)\left\{c^2\left(1728k^3-576k^4\right)x\right. \\
& \left. + \left(4-c^2\right)\left(144k^4x^2+2160k^4\right)\right\}, \\
f_8(c,x) = & \left(1-x^2\right)\left(4-c^2\right)\left\{\left(1728k^3-576k^4\right)c^2+2304k^4\left(4-c^2\right)x\right\}.
\end{aligned}
$$

Now we will maximize the function $F_2(c,x,y)$ in the region of the parallelepiped $[0,2]\times[0,1]\times[0,1]$ with $c\in[0,2], x\in[0,1], y\in[0,1], k\in\{1,2,3\}$.

A. On the vertices of the parallelepiped, from (19), we have

$$
\begin{aligned}
& F_2(0,0,0)=0, \\
& F_2(0,1,0)=F_2(0,1,1)=16384k^4-12288k^3\leq 16384k^4, \\
& F_2(0,0,1)=34560k^4, \\
& F_2(2,0,0)=F_2(2,0,1)=F_2(2,1,0)=F_2(2,1,1)=F_1(2,x,y)\leq 5632k^4.
\end{aligned}
$$

B. On the edges of $F_2(c,x,y)$, from (19), we have

(a) $c=0$ and $x=0$. Then for $y\in(0,1)$,

$$F_2(0,0,y)=34560k^4y^2\leq 34560k^4.$$

(b) $c=0$ and $x=1$, $y\in(0,1)$. Then

$$F_2(0,1,y)=16384k^4-12288k^3\leq 16384k^4.$$

(c) $c=0$ and $y=0$. Then

$$
\begin{aligned}
F_2(0,x,0) &= -12288k^3x^3+k^4(36864x-20480x^3) \\
&\leq -12288x^3k^3+19037k^4 \\
&\leq 19037k^4, \quad x\in(0,1).
\end{aligned}
$$

(d) $c = 0$ and $y = 1$. Then

$$\begin{aligned} F_2(0,x,1) &= 34560k^4 - 12288k^3x^3 - 256k^4x^2(126 - 64x + 9x^2) \\ &\leq 34560k^4, \quad y \in (0,1). \end{aligned}$$

(e) $x = 0$ and $y = 0$. Then

$$\begin{aligned} F_2(c,0,0) &= 15c^6 - 15c^6k - 45c^6k^2 + (6912c^2 - 1728c^4 + 47c^6)k^3 \\ &\quad + (-2304c^2 + 576c^4 + 26c^6)k^4 \\ &\leq 15c^6 - 15c^6k - 45c^6k^2 + 7344k^3 + 1664k^4 \\ &\leq 960 + 7344k^3 + 1664k^4 \\ &< 960k^4 + 7344k^4 + 1664k^4 = 9968k^4, \quad c \in (0,2). \end{aligned}$$

(f) $x = 0$ and $y = 1$. Then

$$F_2(c,0,1) = F_1(c,0,1) \leq 34560k^4, \quad c \in (0,2).$$

(g) $x = 1$ and $y = 1$; $x = 1$ and $y = 0$. Then

$$\begin{aligned} F_2(c,1,y) &= 15c^6 + (-240c^4 + 45c^6)k + (-3840c^2 + 1200c^4 - 105c^6)k^2 \\ &\quad + (-12288 + 5120c^2 + 624c^4 - 237c^6)k^3 \\ &\quad + (16384 + 6976c^2 - 4272c^4 + 402c^6)k^4 \\ &\leq 15 \times 64 + (0)k + (0)k^2 + (3194)k^3 + (19501)k^4 \\ &= 960 + 3194k^3 + 19501k^4 \\ &< 960k^4 + 3194k^4 + 19501k^4 = 23655k^4. \end{aligned}$$

(h) $c = 2,\ y = 0$; or $c = 2,\ y = 1$ or $c = 2,\ x = 0$ or $c = 2,\ x = 1$ or $x = 0,\ y = 0$. Then for $c \in (0,2)$, $x \in (0,1)$ and $y \in (0,1)$,

$$F_2(2,x,0) = F_2(2,x,1) = F_2(2,0,y) = F_2(2,1,y) \leq 5632k^4.$$

C. Considering the faces of the parallelepiped, from (19), we get
(a) $c = 2,\ x \in (0,1),\ y \in (0,1)$. Then

$$F_2(2,x,y) \leq 5632k^4.$$

(b) $c = 0,\ x \in (0,1),\ y \in (0,1)$. Then

$$\begin{aligned} &F_2(0,x,y) \\ &= 36864k^4x - 12288k^3x^3 - 20480k^4x^3 + 2304k^4(15 - x)(1 - x)^2(1 + x)y^2 \\ &\leq 36864k^4x - 12288k^3x^3 - 20480k^4x^3 + 2304k^4(15 - x)(1 - x)^2(1 + x) \\ &= 34560k^4 - 12288k^3x^3 - 256k^4x^2(126 - 64x + 9x^2) \\ &\leq 34560k^4 \text{ for, } x \in (0,1) \text{ and } y \in (0,1). \end{aligned}$$

(c) $x = 0,\ c \in (0,2),\ y \in (0,1)$. Then

$$\begin{aligned}
F_2\,(c,0,y) =& - c^6(-15+15k+45k^2-47k^3-26k^4)\\
&+ (4-c^2)(c^3(-360k^2+288k^3+504k^4)y\\
&+ 2160(4-c^2)k^4y^2 + c^2(1728k^3-576k^4)(1-y^2))\\
\leq& - c^6(-15+15k+45k^2-47k^3-26k^4)\\
&+ (4-c^2)[c^3(-360k^2+288k^3+504k^4)y\\
&+ 2160(4-c^2)k^4y^2 + c^2(1728k^3-576k^4)).
\end{aligned}$$

We can easily see that the above function is an increasing function of $y \in (0,1)$. Hence

$$\begin{aligned}
F_2\,(c,0,y) \leq& - c^6(-15+15k+45k^2-47k^3-26k^4)\\
&+ (4-c^2)\left\{c^3(-360k^2+288k^3+504k^4)\right.\\
&\left.+2160(4-c^2)k^4 + c^2(1728k^3-576k^4)\right\}\\
=&\ 34560k^4 + 15c^6 - 15c^6k + (-1440c^3+360c^5-45c^6)k^2\\
&+ (1728(4-c^2)c^2 + 288(4-c^2)c^3 + 47c^6)k^3\\
&+ (-19584c^2 + 2016c^3 + 2736c^4 - 504c^5 + 26c^6)k^4\\
\leq&\ 34560k^4 + 15c^6 - 15c^6k + (-1440c^3+360c^5-45c^6)k^2\\
&+ (1728(4-c^2)c^2 + 288(4-c^2)c^3 + 47c^6)k^4\\
&+ (-19584c^2 + 2016c^3 + 2736c^4 - 504c^5 + 26c^6)k^4\\
=&\ 34560k^4 - 15c^6(k-1) - (360(4-c^2)c^3 + 45c^6)k^2\\
&-(1136 + 3168(2-c) + 792c^3 + 1008(4-c^2) + 73(4-c^2)(4+c^2))k^4\\
\leq&\ 34560k^4.
\end{aligned}$$

(d) For the face $x = 1$, we observe that $F_2\,(c,1,y)$ is independent of y, so it is the same as $\mathbf{B}$(g), i.e.,

$$F_2\,(c,1,y) \leq 23655k^4$$

for $c \in (0,2)$ and $y \in (0,1)$.

(e) $y = 0,\ c \in (0,2),\ x \in (0,1)$. Then

$$\begin{aligned}
F_2(c,x,0) =& - c^6\left(-15+15k+45k^2-47k^3-26k^4\right)\\
&+ \left(4-c^2\right)\left[-3072k^3x^3 + k^4\left(9216x-5120x^3\right)\right.\\
&+c^4\big[-60kx + 60k^2x^2 + k^3(156x+96x^2+32x^3)\\
&+k^4\big(-48x-180x^2-112x^3-36x^4\big)\big]\\
&+c^2\big[-960k^2x^2 + k^3(1728-1536x^2+320x^3)
\end{aligned}$$

$$
\begin{aligned}
&+k^4\big(-576-2304x+2304x^2+3200x^3+144x^4\big)\big]\big]\\
\leq &-c^6\left(-15+15k+45k^2-47k^3-26k^4\right)\\
&+\left(4-c^2\right)\left[k^4\left(6144\sqrt{\frac{3}{5}}\right)+c^4\big[60k^2+k^3(284)+k^4(0)\big]\right.\\
&\left.+c^2\big[k^3(1728)+k^4(2768)\big]\right]\\
= &-15c^6\left(k-1\right)+\left[-45c^6+60c^4\left(4-c^2\right)\right]k^2\\
&+\left[47c^6+1728c^2\left(4-c^2\right)+284c^4\left(4-c^2\right)\right]k^3\\
&+\left(26c^6+6144\sqrt{\frac{3}{5}}\left(4-c^2\right)+2768c^2\left(4-c^2\right)\right)k^4\\
< &\ (186)k^2+(9903)k^3+(22676)k^4\\
\leq &\ (186)k^4+(9903)k^4+(22676)k^4=32765k^4.
\end{aligned}
$$

(f) $y=1, c\in(0,2),\ x\in(0,1)$. Then

$$F(c,x,1)=34560k^4+15c^6+k\omega_1(c,x)+k^2\omega_2(c,x)+k^3\omega_3(c,x)+k^4\omega_4(c,x),$$

where $\omega_1,\omega_2,\omega_3$ and ω_4 are defined in Result 2.11, and applying Result 2.11, we obtain

$$
\begin{aligned}
F(c,x,1) &\leq 34560k^4+15c^6+k\omega_1(c,x)+k^2\omega_2(c,x)+k^3\left\{\omega_3(c,x)+\omega_4(c,x)\right\}\\
&\leq 34560k^4+15c^6+k\omega_1(c,x)+k^2\left\{\omega_2(c,x)+\omega_3(c,x)+\omega_4(c,x)\right\}\\
&\leq 34560k^4+15c^6+k\left\{\omega_1(c,x)+\omega_2(c,x)+\omega_3(c,x)+\omega_4(c,x)\right\}\\
&\leq 34560k^4+\left\{15c^6+\omega_1(c,x)+\omega_2(c,x)+\omega_3(c,x)+\omega_4(c,x)\right\}\\
&\leq 34560k^4.
\end{aligned}
$$

D. Now consider the interior region of the parallelepiped, i.e., $(0,2)\times(0,1)\times(0,1)$.

From (3.15), we have

$$F_2(c,x,y)=f_5(c,x)+f_6(c,x)y+\left\{f_7(c,x)-f_8(c,x)\right\}y^2+f_8(c,x).$$

Case (I). Suppose that $f_7(c,x)-f_8(c,x)>0$. Then

$$F_2(c,x,y)\leq f_5(c,x)+f_6(c,x)+f_7(c,x)=F_2(c,x,1)<34560k^4.$$

Case (II). Suppose that $f_7(c,x)-f_8(c,x)\leq 0$. Then

$$F_2(c,x,y)\leq f_5(c,x)+f_6(c,x)y+f_7(c,x)\leq f_5(c,x)+f_6(c,x)+f_8(c,x).$$

Now

$$f_5+f_6+f_8=15c^6+\omega_1(c,x)k+\Psi_2(c,x)k^2+\gamma_2(c,x)k^3+\gamma_1(c,x)k^4.$$

From Results 2.9, 2.11, 2.12, 2.13 we have

$$f_5 + f_6 + f_8 \le k^3(14081) + k^4(19521) < 34560k^4.$$

Hence $F_2(c, x, y) < 34560k^4$.

In review of Cases **A**, **B**, **C** and **D**, we obtain

$$\max \left\{ F_2(c, x, y) : c \in [0, 2], x \in [0, 1] \text{ and } y \in [0, 1] \right\} = 34560k^4. \tag{20}$$

From expressions (18) and (20), we get

$$\left| H_{3,1,k}(f) \right| \le \frac{1}{4k^2}, \ k = 1, 2, 3. \tag{21}$$

From $p(z)$ we obtain $c_1 = c_2 = c_4 = 0, \ c_3 = 2;$ further, $b_{k+1} = b_{2k+1} = b_{4k+1} = 0, b_{3k+1} = 1/2k$, and the result follows. □

Remark 1. For $k = 1$, the k^{th} root transformation of f reduces to the given function f itself. Therefore, the estimate given in the equations (21) coincides with that of Kowalczyk et al. [8].

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[1]Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada

[2]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

[3]Center for Converging Humanities, Kyung Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, Seoul 02447, Republic of Korea

[4]Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan

[5]Section of Mathematics, International Telematic University Uninettuno, I-00186 Rome, Italy

[6]Department of Applied Mathematics, Chung Yuan Christian University, Chung-Li, Taoyuan City 320314, Taiwan

[7,8]Department of Mathematics, GITAM School of Science, GITAM (Deemed to be University), Visakhapatnam 530045, A.P., India

[9]Department of Mathematics, North-Eastern Hill University (NEHU), Umshing Mawkynroh, Shillong, Meghalaya 793022, India

E-mail address: harimsri@math.uvic.ca[1,2,3,4,5,6],brath@gitam.edu[7],skarri9@gitam.in[8], vamsheekrishna1972@gmail.com[9]