ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 27, Number 2, December 2023
Available online at https://ojs.utlib.ee/index.php/ACUTM

Bialgebras, the Yang–Baxter equation and Manin triples for mock-Lie algebras

Karima Benali, Taoufik Chtioui, Atef Hajjaji, and Sami Mabrouk

Abstract. The aim of this paper is to introduce the notion of a mock-Lie bialgebra which is equivalent to a Manin triple of mock-Lie algebras. The study of a special case called coboundary mock-Lie bialgebra leads to introducing the mock-Lie Yang–Baxter equation on a mock-Lie algebra which is an analogue of the classical Yang–Baxter equation on a Lie algebra. Note that a skew-symmetric solution of mock-Lie Yang–Baxter equation gives a mock-Lie bialgebra. Finally, $\mathcal{O}$-operators are studied to construct a skew-symmetric solution of a mock-Lie Yang–Baxter equation.

1. Introduction

A while ago, a new class of algebras emerged in the literature – the so called mock-Lie algebras. These are commutative algebras satisfying the Jacobi identity. They appeared for the first time in [21] and since then a lot of work is done on this subject, note for example [41, 29]. These algebras live a dual life: as members of a very particular class of Jordan algebras and as strange cousins of Lie algebras.

The theory of Lie bialgebras and Poisson Lie groups dates back to the early 80s. Poisson Lie groups are Lie groups equipped with an additional structure, a Poisson bracket, satisfying a compatibility condition with the group multiplication. The infinitesimal object associated with a Poisson Lie group is the tangent vector space at the origin of the group, which is in a natural way a Lie algebra $\mathfrak{g}$, see, for instance, [17, 36]. The Poisson structure on the group induces on the Lie algebra an additional structure which

Received April 8, 2023.
2020 *Mathematics Subject Classification.* 16W10, 16T10, 16T15, 16T25, 17B38.
Key words and phrases. Mock-Lie algebra, mock-Lie bialgebra, matched pair, Manin triple, mock-Lie Yang–Baxter equation, $\mathcal{O}$-operators.
https://doi.org/10.12697/ACUTM.2023.27.16
Corresponding author: Taoufik Chtioui

is nothing but a Lie algebra structure on the dual vector space $\mathfrak{g}^*$ satisfying a compatibility condition with the Lie bracket on $\mathfrak{g}$ itself. Such a Lie algebra together with its additional structure is called a Lie bialgebra. So a bialgebra structure on a given algebra is obtained by a corresponding set of comultiplication together with the set of compatibility conditions between multiplication and comultiplication [12]. For example, take a finite dimensional vector space V with a given algebraic structure, this can be acheived by equipping the dual space V^* with the same algebraic structure and a set of compatibility conditions between the structures on V and those on V^*. Among the well-known bialgebra structures, we have the associative bialgebra and infinitesimal bialgebra introduced in [6, 31]. Note that these two structures have the same associative multiplications on V and V^*. They are distinguished only by the compatibility conditions, with the comultiplication acting as a homomorphism (respectively a derivation) on the multiplication for the associative bialgebra (respectively the infinitesimal bialgebra). In general, it is quite common to have multiple bialgebra structures that differ only by their compatibility conditions. A good compatibility condition is prescribed on one hand by a strong motivation and potential applications, and on the other hand by a rich structure theory and effective constructions. See also [20, 9, 16, 25, 26, 27, 33, 34, 38, 39, 40, 24, 23] for more details.

One reason for the usefulness of the Lie bialgebra is that it has a coboundary theory, which leads to the construction of Lie bialgebras from solutions of the classical Yang–Baxter equations. The origin of the Yang–Baxter-equations is purely physics. They were first introduced by Baxter, McGuire, and Yang in [17, 18, 37]. Later on, this equation attracts the attention of scientists and becomes one of the most basic equations in mathematical physics [11, 13]. Namely it plays a crucial role for introducing the theory of quantum groups. This exceptional importance can be seen in many other domains like: quantum groups, knot theory, braided categories, analysis of integrable systems, quantum mechanics, non-commutative descent theory, quantum computing, non-commutative geometry, etc. Various forms of the Yang–Baxter-equation and some of their uses in physics are summarized in [35]. Many scientists have found solutions for the Yang–Baxter equation, however the full classification of its solutions remains an open problem. In the theory of Lie bialgebras, it is essential to consider the coboundary case, which is related to the theory of the classical Yang–Baxter equation [12, 10, 8, 30]. We aim to have an analogue in the mock-Lie case.

This paper is organized as follows. In Section 2 we recall some basic definitions and constructions about mock-Lie algebras. Section 3 deals with matched pairs, Manin triples and mock-Lie bialgebras. In Section 4, we introduce and develop the notion of coboundary mock-Lie bialgebra and mock-Lie Yang–Baxter equation. In Section 5, we give the $\mathcal{O}$-operators of mock-Lie algebras and construct a solution of a mock-Lie Yang–Baxter equation.

Unless otherwise specified, all the vector spaces and algebras are finite dimensional over a field $\mathbb{K}$ of characteristic zero.

Notations. Let V and W be two vector spaces.

(1) Denote by $\tau : V \otimes W \to W \otimes V$ the switch isomorphism, $\tau(v \otimes w) = w \otimes v$.
(2) For a linear map $\Delta : V \to \otimes^2 V$, we use the Sweedler's notation $\Delta(x) = \sum_{(x)} x_1 \otimes x_2$ for $x \in V$. We will often omit the summation sign $\sum_{(x)}$ to simplify the notations.
(3) Denote by $V^* = Hom(V, \mathbb{K})$ the linear dual of V. For $\varphi \in V^*$ and $u \in V$, we write $\langle \varphi, u \rangle := \varphi(u) \in \mathbb{K}$.
(4) For a linear map $\phi : V \to W$, we define the map $\phi^* : W^* \to V^*$ by
$$\langle \phi^*(\xi), v \rangle = \langle \xi, \phi(v) \rangle, \ \forall v \in V, \xi \in W^*.$$
(5) For an element x in a mock-Lie algebra $(A, \bullet)$ and $n \geq 2$, define the adjoint map $L(x) : \otimes^n A \to \otimes^n A$ by
$$L(x)(y_1 \otimes \cdots \otimes y_n) = \sum_{i=1}^{n} y_1 \otimes \cdots \otimes y_{i-1} \otimes x \bullet y_i \otimes y_{i+1} \otimes \cdots \otimes y_n \tag{1}$$
for all $y_1, \ldots, y_n \in A$. Conversely, given $Y = y_1 \otimes \cdots \otimes y_n$, we define $L(Y) : \mathfrak{g} \to \otimes^n \mathfrak{g}$ by
$$L(Y)(x) = L(x)(Y), \ \text{for } x \in \mathfrak{g}.$$

2. Preliminaries

In this section, we provide some preliminaries about mock-Lie algebras and left mock-pre-Lie algebras. Our main references are [4, 21, 7, 14].

Definition 2.1. A *mock-Lie algebra* is a pair $(A, \bullet)$ consisting of a vector space A together with a multiplication $\bullet : A \otimes A \to A$ satisfying
$$x \bullet y = y \bullet x, \quad \text{(commutativity)},$$
$$x \bullet (y \bullet z) + y \bullet (z \bullet x) + z \bullet (x \bullet y) = 0, \quad \text{(Jacobi identity)}, \tag{2}$$
for any $x, y, z \in A$. The Jacobi identity (2) is equivalent to
$$x \bullet (y \bullet z) = -(x \bullet y) \bullet z - y \bullet (x \bullet z).$$
In other words, the left multiplication $L : A \to End(A)$ defined by $L(x)y = x \bullet y$, is an anti-derivation on A. Recall that a linear map $D : A \to A$ is called an *anti-derivation* if, for all $x, y \in A$,
$$D(x \bullet y) = -D(x) \bullet y - x \bullet D(y).$$

Example 2.1. Recall that an *anti-associative* algebra is a pair $(A, \star)$ consisting of a vector space A together with a product $\star : A \otimes A \to A$ such that the anti-associator vanishes, i.e,

$$Aass(x, y, z) := (x \star y) \star z + x \star (y \star z) = 0, \quad \forall x, y, z \in A.$$

Let $(A, \star)$ be an anti-associative algebra. Then, $(A, \bullet)$ is a mock-Lie algebra, where $x \bullet y := x \star y + y \star x,$ for all $x, y \in A$.

Examples 2.2. In the following we list some examples of finite dimensional mock-Lie algebras given in [21].

(1) Let A be a 3-dimensional vector space with a basis $\mathcal{B} = \{e_1, e_2, e_3\}$. Then $(A, \bullet)$ is a mock-Lie algebra where the product $\bullet$ is defined on the basis $\mathcal{B}$ by
$$e_1 \bullet e_1 = e_2, \quad e_3 \bullet e_3 = e_2.$$
(2) Let A be a 4-dimensional vector space with a basis $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$. Then $(A, \bullet)$ is a mock-Lie algebra where the product $\bullet$ is defined on the basis $\mathcal{B}$ by
$$e_1 \bullet e_1 = e_2, \quad e_1 \bullet e_3 = e_4.$$
(3) The 3-dimensional commutative Heisenberg algebra $\mathcal{H}_3$ over an algebraically closed field of characteristic not 2 or 3 equipped with the product
$$e_1 \bullet e_2 = e_2 \bullet e_1 = e_3$$
is a mock-Lie algebra.

See [22] for more examples of mock-Lie algebras. Now, we recall the definition of representations of a mock-Lie algebra.

Definition 2.2. A *representation* of a mock-Lie algebra $(A, \bullet)$ is a pair (V, ρ) where V is a vector space and $\rho : A \to End(V)$ is a linear map such that for all $x, y \in A$, the following equality holds:

$$\rho(x \bullet y) = -\rho(x)\rho(y) - \rho(y)\rho(x).$$

Example 2.3. Let $(A, \bullet)$ be a mock-Lie algebra. Then (A, L) is a representation of A on itself, called the adjoint representation.

An equivalent characterisation of representations on mock-Lie algebras is given in the following.

Proposition 2.4. *Let $(A, \bullet)$ be a mock-Lie algebra, V be a vector space and $\rho : A \to End(V)$ a linear map. Then (V, ρ) is a representation of A if and only if the direct sum $A \oplus V$ together with the multiplication defined by*

$$(x + u) \bullet_{A \oplus V} (y + v) = x \bullet y + \rho(x)v + \rho(y)u, \ \forall x, y \in A, \ \forall u, v \in V,$$

is a mock-Lie algebra. This mock-Lie algebra is called the semi-direct product of A and V and it is denoted by $A \ltimes_\rho V$.

Definition 2.3. Let $(A, \bullet)$ be a mock-Lie algebra and consider two representations (V_1, ρ_1) and (V_2, ρ_2). A linear map $\phi : V_1 \to V_2$ is said to be a *morphism of representations* if

$$\rho_2(x) \circ \phi = \phi \circ \rho_1(x),\ \forall x \in A.$$

If ϕ is bijective, then (V_1, ρ_1) and (V_2, ρ_2) are *equivalent* (*isomorphic*).

To relate matched pairs of mock-Lie algebras to mock-Lie bialgebras and Manin triples for mock-Lie algebras in the next section, we need the notions of the coadjoint representation, which is the dual representation of the adjoint representation. In the following, we recall these facts.

Let $(A, \bullet)$ be a mock-Lie algebra and (V, ρ) be a representation of A. Let V^* be the dual vector space of V. Define the linear map $\rho^* : A \to End(V^*)$ as

$$\langle \rho^*(x)u^*, v\rangle = \langle u^*, \rho(x)v\rangle, \quad \forall x \in A,\ v \in V,\ u^* \in V^*, \tag{3}$$

where $\langle \cdot, \cdot\rangle$ is the usual pairing between V and the dual space V^*. With the above notations, we have the following result.

Proposition 2.5. *Let (V, ρ) be a representation of a mock-Lie algebra $(A, \bullet)$. Then (V^*, ρ^*) is a representation of A on V^*.*

Consider the case when $V = A$ and define the linear map $L^* : A \to End(A^*)$ by

$$\langle L^*(x)(\xi), y\rangle = \langle \xi, L(x)y\rangle, \quad \forall x, y \in A,\ \xi \in A^*.$$

Then we have the following corollary.

Corollary 2.6. *Let $(A, \bullet)$ be a mock-Lie algebra and (A, L) be the adjoint representation of A. Then (A^*, L^*) is a representation of $(A, \bullet)$ on A^* which is called the coadjoint representation.*

If there is a mock-Lie algebra structure on the dual space A^*, we denote the left multiplication by $\mathcal{L}$.

Definition 2.4. Let $(A, \bullet)$ be a mock-Lie algebra and (V, ρ) be a representation. A linear map $T : V \to A$ is called an $\mathcal{O}$*-operator* associated to (V, ρ) if T satisfies

$$T(u) \bullet T(v) = T\big(\rho(Tu)v + \rho(Tv)u\big), \quad \forall u, v \in V.$$

In the case $(V, \rho) = (A, L)$, the $\mathcal{O}$-operator T is called a *Rota–Baxter operator* (of weight zero).

Definition 2.5. A *mock-pre-Lie* algebra is a vector space A equipped with a linear map $\cdot : A \otimes A \to A$ satisfying the identity

$$Aass(x, y, z) = -Aass(y, x, z), \quad \forall x, y, z \in A. \tag{4}$$

Recall that $Aass(x,y,z) = (x \cdot y) \cdot z + x \cdot (y \cdot z)$. Therefore, equality (4) is equivalent to

$$(x \star y) \cdot y = (x \cdot y) \cdot z - y \cdot (x \cdot z),$$

where $x \star y = x \cdot y + y \cdot x$ for all $x, y \in A$.

Note that if $(A, \cdot)$ is a mock-pre-Lie algebra, then the product given by

$$x \star y = x \cdot y + y \cdot x, \quad \forall x, y \in A,$$

defines a mock-Lie algebra structure, which is called the *sub-adjacent mock-Lie algebra* of $(A, \cdot)$, and is denoted by A^{ac}. Furthermore, $(A, \cdot)$ is called the *compatible mock-pre-Lie algebra structure* on A^{ac}.

On the other hand, let $\Theta : A \to End(A)$ be defined by $\Theta(x)y = x \cdot y$, for all $x, y \in A$. Then (A, Θ) is a representation of the mock-Lie algebra A^{ac}.

Proposition 2.7. *Let $(A, \bullet)$ be a mock-Lie algebra and (V, ρ) be a representation of A. If T is an $\mathcal{O}$-operator associated to (V, ρ), then $(V, \cdot)$ is a mock-pre-Lie algebra, where*

$$u \cdot v = \rho(Tu)v, \quad \forall u, v \in V.$$

Proposition 2.8. *Let $(A, \bullet)$ be a mock-Lie algebra. Then there is a compatible mock-pre-Lie algebra if and only if there exists an invertible $\mathcal{O}$-operator $T : V \to A$ associated to a representation (V, ρ). Furthermore, the compatible mock-pre-Lie structure on A is given by*

$$x \cdot y = T\big(\rho(x)T^{-1}(y)\big), \quad \forall x, y \in A.$$

3. Matched pairs, Manin triples and mock-Lie bialgebras

In this section, we introduce the notions of Manin triple of a mock-Lie algebra and mock-Lie bialgebras. The equivalence between them is interpreted in terms of matched pairs of mock-Lie algebras.

We first recall the notion of matched pairs of mock-Lie algebras (see [5]). Let $(A, \bullet)$ and $(H, \diamond)$ be two mock-Lie algebras. Let $\rho : A \to End(H)$ and $\mu : H \to End(A)$ be two linear maps. On the direct sum $A \oplus H$ of the underlying vector spaces, define a linear map $\circ : \otimes^2(A \oplus H) \to A \oplus H$ by

$$(x + a) \circ (y + b) = x \bullet y + \mu(b)x + \mu(a)y + a \diamond b + \rho(y)a + \rho(x)b \tag{5}$$

for any $x, y \in A$ and $a, b \in H$.

Theorem 3.1. *Let $(A, \bullet)$ and $(H, \diamond)$ be two mock-Lie algebras. Then the pair $(A \oplus H, \circ)$ is a mock-Lie algebra if and only if (H, ρ) and (A, μ) are representations of $(A, \bullet)$ and $(H, \diamond)$ respectively, and for all $x, y \in A$, $a, b \in H$, the following compatibility conditions are satisfied:*

$$\rho(x)(a \diamond b) + \rho(x)a \diamond b + a \diamond \rho(x)b + \rho(\mu(a)x)b + \rho(\mu(b)x)a = 0, \tag{6}$$

$$\mu(a)(x \bullet y) + \mu(a)x \bullet y + x \bullet \mu(a)y + \mu(\rho(x)a)y + \mu(\rho(y)a)x = 0. \tag{7}$$

Definition 3.1. A *matched pair* of mock-Lie algebras is a quadruple $(A, H; \rho, \mu)$ consisting of two mock-Lie algebras $(A, \bullet)$ and $(H, \diamond)$, together with representations $\rho : A \to End(H)$ and $\mu : H \to End(A)$, respectively, such that the compatibility conditions (6) and (7) are satisfied.

Remark 3.1. We denote the mock-Lie algebra defined by equality (5) by $A \bowtie H$. It is straightforward to show that every mock-Lie algebra which is a direct sum of the underlying vector spaces of two mock-Lie subalgebras can be obtained from a matched pair of mock-Lie algebras as above.

Definition 3.2. A bilinear form ω on a mock-Lie algebra $(A, \bullet)$ is called **invariant** if it satisfies

$$\omega(x \bullet y, z) = \omega(x, y \bullet z), \quad \forall x, y, z \in A.$$

Proposition 3.2. *Let $(A, \bullet)$ be a mock-Lie algebra and (A, L) be the adjoint representation of A on itself. Then (A, L) and (A^*, L^*) are equivalent as representations of the mock-Lie algebra $(A, \bullet)$ if and only if there exists a nondegenerate symmetric invariant bilinear form ω on A.*

Proof. Suppose that there exists a nondegenerate symmetric invariant bilinear form ω on A. Since ω is nondegenerate, there exists a linear isomorphism $\phi : A \to A^*$ defined by

$$\langle \phi(x), y \rangle = \omega(x, y), \quad \forall x, y \in A.$$

Hence, for any $x, y, z \in A$, we have

$$\begin{aligned} \langle \phi(L(x)(y)), z \rangle &= \omega(L(x)(y), z) = \omega(x \bullet y, z) = \omega(y, x \bullet z) \\ &= \langle \phi(y), x \bullet z \rangle = \langle L^*(x)\phi(y), z \rangle. \end{aligned}$$

That is, (A, L) and (A^*, L^*) are equivalent. Conversely, in a similar way we can get the conclusion. □

Definition 3.3. A *Manin triple* of mock-Lie algebras is a triple of mock-Lie algebras (A, A^+, A^-) together with a nondegenerate symmetric invariant bilinear form ω on A such that the following conditions are satisfied:
(a) A^+, A^- are mock-Lie subalgebras of A,
(b) $A = A^+ \oplus A^-$ as vector spaces,
(c) A^+ and A^- are isotropic with respect to ω, that is, $\omega(x_+, y_+) = \omega(x_-, y_-) = 0$, for any $x_+, y_+ \in A^+$, $x_-, y_- \in A^-$.

A *homomorphism* between two Manin triples of mock-Lie algebras (A, A^+, A^-) and (B, B^+, B^-) associated to two nondegenerate symmetric invariant bilinear forms ω_1 and ω_2, respectively, is a homomorphism of mock-Lie algebras $f : A \to B$ such that

$$f(A^+) \subset B^+, \quad f(A^-) \subset B^-, \quad (x, y) = \omega_2(f(x), f(y)),\ \forall x, y \in A.$$

If in addition, f is an isomorphism of vector spaces, then the two Manin triples are called *isomorphic*.

Definition 3.4 ([28])**.** Let $(A, \bullet)$ be a mock-Lie algebra. Suppose that there is a mock-Lie algebra structure $(A^*, \diamond)$ on the dual space A^* of A and there is a mock-Lie algebra structure on the direct sum $A \oplus A^*$ of the underlying vector spaces A and A^* such that $(A, \bullet)$ and $(A^*, \diamond)$ are subalgebras and the natural non-degerenate symmetric bilinear form on $A \oplus A^*$ given by

$$\omega_d(x + \xi, y + \eta) := \langle x, \eta\rangle + \langle \xi, y\rangle, \quad \forall x, y \in A,\ \xi, \eta \in A^*, \tag{8}$$

is invariant, then $(A \oplus A^*, A, A^*)$ is called a *standard Manin triple* of mock-Lie algebra associated to standard bilinear form ω_d.

Obviously, a standard Manin triple of mock-Lie algebras is a Manin triple of mock-Lie algebras. Conversely, we have the following result.

Proposition 3.3. *Every Manin triple of mock-Lie algebras is isomorphic to a standard one.*

Proof. Since A^+ and A^- are isotropic under the nondegenerate invariant bilinear form ω on $A^+ \oplus A^-$, then in this case A^- and $(A^+)^*$ are identified by ω and the mock-Lie algebra structure on A^- is transferred to $(A^+)^*$. Hence the mock-Lie algebra structure on $A^+ \oplus A^-$ is transferred to $A^+ \oplus (A^+)^*$. Transferring the nondegenrate bilinear form ω to $A^+ \oplus (A^+)^*$, we obtain the standard bilinear form given by (8). Thus, (A, A^+, A^-) is isomorphic to the stansadrd Manin triple $(A \oplus A^*, A, A^*)$. □

Proposition 3.4 ([28])**.** *Let $(A, \bullet)$ be a mock-Lie algebra. Suppose that there is a mock-Lie algebra structure $(A^*, \diamond)$ on A^*. Then there exists a mock-Lie algebra sructure on the vector space $A \oplus A^*$ such that $(A \oplus A^*, A, A^*)$ is a standard Manin triple of mock-Lie algebras with respect to ω_d defined by* (8) *if and only if $(A, A^*; L^*, \mathcal{L}^*)$ is a matched pair of mock-Lie algebras. Here $\mathcal{L}^*$ is the coadjoint representation of the mock-Lie algebra $(A^*, \diamond)$.*

Proposition 3.5. *Let $(A, \bullet)$ be a mock-Lie algebra. Suppose that there is a mock-Lie algebra structure $(A^*, \diamond)$ on A^*. Then $(A, A^*; L^*, \mathcal{L}^*)$ is a matched pair of mock-Lie algebras if and only if for any $x, y \in A, \xi \in A^*$, we have*

$$\mathcal{L}^*(\xi)(x\bullet y)+(\mathcal{L}^*(\xi)(x))\bullet y+x\bullet(\mathcal{L}^*(\xi)(y))+\mathcal{L}^*(L^*(x)(\xi))(y)+\mathcal{L}^*(L^*(y)(\xi))(x) = 0. \tag{9}$$

Proof. Obiviously, equality (9) is exactly (7) in the case $\rho = L^*$, $\mu = \mathcal{L}^*$. In addition, for any $x, y \in A$, $\xi, \eta \in A^*$, we have

$$\begin{aligned}
&\langle \mathcal{L}^*(\xi)(x \bullet y), \eta\rangle = \langle x \bullet y, \mathcal{L}(\xi)(\eta)\rangle = \langle L(x)(y), \xi \diamond \eta\rangle = \langle y, L^*(x)(\xi \diamond \eta)\rangle;\\
&\langle (\mathcal{L}^*(\xi)(x)) \bullet y, \eta\rangle = \langle L(\mathcal{L}^*(\xi)(x))(y), \eta\rangle = \langle y, L^*(\mathcal{L}^*(\xi)(x))(\eta)\rangle;\\
&\langle x \bullet (\mathcal{L}^*(\xi)(y)), \eta\rangle = \langle L(x)(\mathcal{L}^*(\xi)(y)), \eta\rangle = \langle \mathcal{L}^*(\xi)(y), L^*(x)(\eta)\rangle\\
&\qquad\qquad = \langle y, \mathcal{L}(\xi)(L^*(x)(\eta))\rangle = \langle y, \xi \diamond (L^*(x)(\eta))\rangle;\\
&\langle \mathcal{L}^*(L^*(x)(\xi))(y), \eta\rangle = \langle y, \mathcal{L}(L^*(x)(\xi))(\eta)\rangle = \langle y, (L^*(x)(\xi)) \diamond \eta\rangle;
\end{aligned}$$

$$
\begin{aligned}
\langle \mathcal{L}^*(L^*(y)(\xi))(x), \eta\rangle &= \langle x, \mathcal{L}(L^*(y)(\xi))(\eta)\rangle = \langle x, \eta \diamond (L^*(y)(\xi))\rangle \\
&= \langle x, \mathcal{L}(\eta)((L^*(y)(\xi))\rangle = \langle \mathcal{L}^*(\eta)(x), L^*(y)(\xi)\rangle \\
&= \langle L(\mathcal{L}^*(\eta)(x))(y), \xi\rangle = \langle y, L^*(\mathcal{L}^*(\eta)(x))(\xi)\rangle.
\end{aligned}
$$

Then equality (6) holds if and only if (7) holds. Therefore the conclusion holds. □

Theorem 3.6. *Let $(A, \bullet)$ be a mock-Lie algebra. Suppose that there is a mock-Lie algebra structure " $\diamond$ " on its dual space A^* given by a linear map $\Delta^* : A^* \otimes A^* \to A^*$, that is, $\xi \diamond \eta = \Delta^*(\xi \otimes \eta)$, for any $\xi, \eta \in A^*$. Then $(A, A^*; L^*, \mathcal{L}^*)$ is a matched pair of mock-Lie algebras if and only if $\Delta : A \to A \otimes A$ satisfies the following condition:*

$$
\Delta(x \bullet y) = -\big(L(x) \otimes id + id \otimes L(x)\big)\Delta(y) - \big(L(y) \otimes id + id \otimes L(y)\big)\Delta(x), \quad (10)
$$

for any $x, y \in A$.

Proof. Using Proposition 3.5, we can prove that equality (10) is equivalent to (9). In fact, for any $x, y \in A$, $\xi, \eta \in A^*$, we have

$$
\begin{aligned}
&\langle \mathcal{L}^*(\xi)(x \bullet y), \eta\rangle = \langle x \bullet y, \xi \diamond \eta\rangle = \langle x \bullet y, \Delta^*(\xi \otimes \eta)\rangle = \langle \Delta(x \bullet y), \xi \otimes \eta\rangle;\\
&\langle (\mathcal{L}^*(\xi)(x)) \bullet y, \eta\rangle = \langle L(y)(\mathcal{L}^*(\xi)(x)), \eta\rangle = \langle \mathcal{L}^*(\xi)(x), L^*(y)(\eta)\rangle = \langle x, \mathcal{L}(\xi)(L^*(y)(\eta))\rangle\\
&\qquad = \langle x, \xi \diamond (L^*(y)(\eta))\rangle = \langle (id \otimes L(y))\Delta(x), \xi \otimes \eta\rangle;\\
&\langle x \bullet (\mathcal{L}^*(\xi)(y)), \eta\rangle = \langle L(x)(\mathcal{L}^*(\xi)(y)), \eta\rangle = \langle \mathcal{L}^*(\xi)(y), L^*(x)(\eta)\rangle = \langle y, \mathcal{L}(\xi)(L^*(x)(\eta))\rangle\\
&\qquad = \langle y, \xi \diamond (L^*(x)(\eta))\rangle = \langle (id \otimes L(x))\Delta(y), \xi \otimes \eta\rangle;\\
&\langle \mathcal{L}^*(L^*(x)(\xi))(y), \eta\rangle = \langle y, (L^*(x)(\xi)) \diamond \eta\rangle = \langle (L(x) \otimes id)\Delta(y), \xi \otimes \eta\rangle;\\
&\langle \mathcal{L}^*(L^*(y)(\xi))(x), \eta\rangle = \langle x, (L^*(y)(\xi)) \diamond \eta\rangle = \langle (L(y) \otimes id)\Delta(x), \xi \otimes \eta\rangle.
\end{aligned}
$$

Then equality (9) is equivalent to (10). Hence the conclusion holds. □

Remark 3.2. From the symmetry of the mock-Lie algebras $(A, \bullet)$ and $(A^*, \diamond)$ in the standard Manin triple of mock-Lie algebras with respect to ω_d, we also can consider a linear map $\gamma : A^* \to A^* \otimes A^*$ such that $\gamma^* : A \otimes A \to A$ gives the mock-Lie algebra structure " $\bullet$ " on A. It is straightforward to show that Δ satisfies equality (10) if and only if γ satisfies

$$
\gamma(\xi \diamond \eta) = -\big(\mathcal{L}(\xi) \otimes id + id \otimes \mathcal{L}(\xi)\big)\gamma(\eta) - \big(\mathcal{L}(\eta) \otimes id + id \otimes \mathcal{L}(\eta)\big)\gamma(\xi),
$$

for any $\xi, \eta \in A^*$.

Definition 3.5. Let $(A, \bullet)$ be a mock-Lie algebra. A *mock-Lie bialgebra structure* on A is a symmetric linear map $\Delta : A \to A \otimes A$ such that

(1) $\Delta^* : A^* \otimes A^* \to A^*$ defines a mock-Lie algebra structure on A^*;
(2) Δ satifies Eq. (10), called the compatibility condition.

We denote it by (A, Δ) or (A, A^*).

We can unwrap the compatibility condition equality (10) as

$$
\Delta(x \bullet y) = -(x \bullet y_1) \otimes y_2 - y_1 \otimes (x \bullet y_2) - (y \bullet x_1) \otimes x_2 - x_1 \otimes (y \bullet x_2).
$$

Remark 3.3. The compatibility condition equality (10) is, in fact, a cocycle condition in the zigzag cohomology of mock-Lie algebra introduced in [15]. Indeed, we can regard $A^{\otimes 2}$ as an A-module via the adjoint action (1):

$$x \cdot (y_1 \otimes y_2) = L(x)(y_1 \otimes y_2) = (x \bullet y_1) \otimes y_2 + y_1 \otimes (x \bullet y_2),$$

for $x \in A$ and $y_1 \otimes y_2 \in A^{\otimes 2}$. Then we can think of the linear map $\Delta : A \to A^{\otimes 2}$ as a 1-cochain. Then the differential on Δ is given by

$$\begin{aligned} d^1\Delta(x, y) &= \Delta(x \bullet y) + x \cdot \Delta(y) + y \cdot \Delta(x) \\ &= \Delta(x \bullet y) + L(x)(\Delta(y)) + L(y)(\Delta(x)). \end{aligned}$$

Therefore, equality (10) says exactly that $\Delta \in C^1(A, A^{\otimes 2})$ is a 1-cocycle.

Example 3.7. Let (A, A^*) be a mock-Lie bialgebra on a mock-Lie algebra $(A, \bullet)$. Then $(A^*, \gamma)(or(A^*, A))$ is a mock-Lie bialgebra on the mock-Lie algebra $(A^*, \diamond)$, where γ is given in Remark 3.2.

Definition 3.6. Let (A_1, A_1^*) and (A_2, A_2^*) be two mock-Lie bialgebras. A linear map $\psi : A_1 \to A_2$ is a *homomorphism* of mock-Lie bialgebras if ψ satifies, for any $x, y \in A_1$, the identities

$$\psi(x \bullet_1 y) = \psi(x) \bullet_2 \psi(y), \qquad (\psi \otimes \psi) \circ \Delta_1 = \Delta_2 \circ \psi.$$

Now, combining Proposition 3.4 and Theorem 3.6, we have the following conclusion.

Theorem 3.8. *Let $(A, \bullet)$ be a mock-Lie algebra. Suppose that there is a mock-Lie algebra structure on A^* denoted by " $\diamond$ " which is defined as a linear map $\Delta : A \to A \otimes A$. Then the following conditions are equivalent.*

(1) *$(A \oplus A^*, A, A^*)$ is a standard Manin triple of mock-Lie algebras with respect to ω_d defined by equality (8).*
(2) *$(A, A^*; L^*, \mathcal{L}^*)$ is a matched pair of mock-Lie algebras.*
(3) *(A, A^*) is a mock-Lie bialgebra.*

4. Coboundary mock-Lie bialgebras and the mock-Lie Yang–Baxter equation

In this section, we consider a special class of mock-Lie bialgebras called coboundary mock-Lie bialgebras and introduce the notion of mock-Lie Yang–Baxter equation.

Definition 4.1. A mock-Lie bialgebra (A, A^*) is called *coboundary* if there exists an element $r \in A \otimes A$ such that, for any $x \in A$,

$$\Delta(x) = \big(L(x) \otimes id - id \otimes L(x)\big)r. \tag{11}$$

Lemma 4.1. *Let $(A, \bullet)$ be a mock-Lie algebra and $r \in A \otimes A$. Suppose that the linear map $\Delta : A \to A \otimes A$ is defined by equality (11). Then Δ satisfies the compatibility condition given by equality (10).*

Proof. Let $r = r_1 \otimes r_2 \in A \otimes A$. We use the commutativity and Jacobi identity for mock-Lie algebras. Then, for any $x, y \in A$, we have

$$\begin{aligned}
&-\big(L(x)\otimes id + id\otimes L(x)\big)\Delta(y) - \big(L(y)\otimes id + id\otimes L(y)\big)\Delta(x)\\
=&-\big(L(x)\otimes id + id\otimes L(x)\big)\big(L(y)\otimes id - id\otimes L(y)\big)r\\
&-\big(L(y)\otimes id + id\otimes L(y)\big)\big(L(x)\otimes id - id\otimes L(x)\big)r\\
=&-\big(L(x)\otimes id + id\otimes L(x)\big)\big((y\bullet r_1)\otimes r_2 - r_1\otimes (y\bullet r_2)\big)\\
&-\big(L(y)\otimes id + id\otimes L(y)\big)\big((x\bullet r_1)\otimes r_2 - r_1\otimes (x\bullet r_2)\big)\\
=&-\Big(x\bullet(y\bullet r_1)\otimes r_2 - (x\bullet r_1)\otimes(y\bullet r_2) + (y\bullet r_1)\otimes(x\bullet r_2) - r_1\otimes(x\bullet(y\bullet r_2))\Big)\\
&-\Big(y\bullet(x\bullet r_1)\otimes r_2 - (y\bullet r_1)\otimes(x\bullet r_2) + (x\bullet r_1)\otimes(y\bullet r_2) - r_1\otimes(y\bullet(x\bullet r_2))\Big)\\
=&\big(-x\bullet(y\bullet r_1) - y\bullet(x\bullet r_1)\big)\otimes r_2 + r_1\otimes\big(x\bullet(y\bullet r_2) + y\bullet(x\bullet r_2)\big)\\
=&\big((x\bullet y)\bullet r_1\big)\otimes r_2 - r_1\otimes\big((x\bullet y)\bullet r_2\big) = \big(L(x\bullet y)\otimes id - id\otimes L(x\bullet y)\big)r\\
=&\Delta(x\bullet y).
\end{aligned}$$

Hence the proof. □

Let $\Delta : A \to A\otimes A$ be a linear map and $\sigma : A^{\otimes 3} \to A^{\otimes 3}$ be defined as $\sigma(x\otimes y\otimes z) = y\otimes z\otimes x$, for any $x, y, z \in A$. Let $E_\Delta : A \to A^{\otimes 3}$ be a linear map given by

$$E_\Delta(x) = (id + \sigma + \sigma^2)\big((id\otimes\Delta)\Delta(x)\big).$$

Lemma 4.2. *Let A be a vector space and $\Delta : A \to A\otimes A$ be a linear map. Then the product "$\diamond$" in A^* given by $\Delta^* : A^*\otimes A^* \to A^*$ satisfies the Jacobi identity if and only if $E_\Delta = 0$.*

Proof. For any $\xi, \eta \in A^*$, $x \in A$, we have

$$\langle \xi\diamond\eta, x\rangle = \langle \Delta^*(\xi\otimes\eta), x\rangle = \langle \xi\otimes\eta, \Delta(x)\rangle.$$

Threrefore, for any $\xi, \eta, \nu \in A^*$ and $x \in A$, the Jacobi identity satisfies

$$\begin{aligned}
&\langle J(\xi,\eta,\nu), x\rangle\\
=&\langle \Delta^*(id\otimes\Delta^*)(\xi\otimes\eta\otimes\nu) + \Delta^*(id\otimes\Delta^*)(\eta\otimes\nu\otimes\xi) + \Delta^*(id\otimes\Delta^*)(\nu\otimes\xi\otimes\eta), x\rangle\\
=&\langle \Delta^*(id\otimes\Delta^*)\big(id + \sigma + \sigma^2\big)(\xi\otimes\eta\otimes\nu), x\rangle\\
=&\langle \xi\otimes\eta\otimes\nu, \big(id + \sigma + \sigma^2\big)((id\otimes\Delta)\Delta)(x)\rangle.
\end{aligned}$$

Therefore $J(\xi,\eta,\nu) = 0$, for any $\xi, \eta, \nu \in A^*$, if and only if $E_\Delta = 0$. □

Let $(A, \bullet)$ be a mock-Lie algebra and $r = \sum_i a_i\otimes b_i \in A\otimes A$. Set

$$r_{12} = \sum_i a_i\otimes b_i\otimes 1,\quad r_{13} = \sum_i a_i\otimes 1\otimes b_i,\quad r_{23} = \sum_i 1\otimes a_i\otimes b_i,$$

where 1 is a unit element if $(A, \bullet)$ is unital or a symbol playing a similar role of the unit for the non-unital cases. The operation between two r_{ij} is defined in an obvious way. For example,

$$r_{12} \bullet r_{13} = \sum_{ij} a_i \bullet a_j \otimes b_i \otimes b_j,$$

$$r_{13} \bullet r_{23} = \sum_{ij} a_i \otimes a_j \otimes b_i \bullet b_j,$$

$$r_{23} \bullet r_{12} = \sum_{ij} a_j \otimes a_i \bullet b_j \otimes b_i.$$

Note that the above elements are independent of the existence of the unit. A tensor $r \in A \otimes A$ is called *symmetric* (resp. *skew-symmetric*) if $r = \tau(r)$ (resp. $r = -\tau(r)$). On the other hand, any $r \in A \otimes A$ can be identified as a linear map from the dual space A^* to A in the following way:

$$\langle \xi, r(\eta) \rangle = \langle \xi \otimes \eta, r \rangle, \quad \forall \xi, \eta \in A^*.$$

The tensor $r \in A \otimes A$ is called nondegenerate if the above induced linear map is invertible.

Proposition 4.3. *Let $(A, \bullet)$ be a mock-Lie algebra. Define a linear map $\Delta : A \to A \otimes A$ by equality (11) with some $r \in A \otimes A$ satisfying*

$$\big(L(x) \otimes id - id \otimes L(x)\big)\big(r + \tau(r)\big) = 0, \tag{12}$$

for all $x \in A$. Then

$$E_\Delta(x) + Q(x)[[r, r]] = 0,$$

where

$$[[r, r]] = r_{12} \bullet r_{13} + r_{13} \bullet r_{23} - r_{12} \bullet r_{23},$$

and $Q(x) = \big(L(x) \otimes id \otimes id + id \otimes L(x) \otimes id + id \otimes id \otimes L(x)\big)$ for any $x \in A$.

Proof. Let $r = \sum_i a_i \otimes b_i$, the condition (12) is equivalent to

$$\sum_i (x \bullet a_i) \otimes b_i - a_i \otimes (x \bullet b_i) + (x \bullet b_i) \otimes a_i - b_i \otimes (x \bullet a_i) = 0. \tag{13}$$

Note that $E_\Delta(x)$ is the sum of twelve terms and that $Q(x)[[r, r]]$ is a sum of nine terms, but two terms appear in both sums up to sign and hence are cancelled. Thus $E_\Delta(x) + Q(x)[[r, r]]$ is a sum of seventeen terms. After rearranging the terms suitably, we obtain

$$\begin{aligned}
& E_\Delta(x) + Q(x)[[r, r]] \\
= & \sum_{i,j} \big\{ - (x \bullet b_i) \bullet a_j \otimes b_j \otimes a_i + x \bullet (a_i \bullet a_j) \otimes b_i \otimes b_j + (x \bullet b_i) \bullet b_j \otimes a_i \otimes a_j \\
& - (b_i \bullet b_j) \otimes (x \bullet a_i) \otimes a_j + (a_i \bullet a_j) \otimes (x \bullet b_i) \otimes b_j + (b_i \bullet a_j) \otimes b_j \otimes (x \bullet a_i) \\
& + (a_i \bullet a_j) \otimes b_i \otimes (x \bullet b_j) - a_i \otimes (x \bullet b_i) \bullet a_j \otimes b_j + a_i \otimes a_j \otimes (x \bullet b_i) \bullet b_j \\
& + b_j \otimes (x \bullet a_i) \otimes (b_i \otimes a_j) - b_j \otimes a_i \otimes (x \bullet b_i) \bullet a_j - a_j \otimes (b_i \bullet b_j) \otimes (x \bullet a_i) \\
& + a_j \otimes (x \bullet b_i) \bullet b_j \otimes a_i - a_i \otimes x \bullet (b_i \bullet a_j) \otimes b_j - a_i \otimes (b_i \bullet a_j) \otimes (x \bullet b_j)
\end{aligned}$$

$$+ a_i \otimes (x \bullet a_j) \otimes (b_i \bullet b_j) + a_i \otimes a_j \otimes x \bullet (b_i \bullet b_j)\big\}.$$

Interchanging the indices i and j in the first term and using the Jacobi identity in A, the first term becomes

$$\sum_{i,j} x \bullet (b_j \bullet a_i) \otimes b_i \otimes a_j + b_j \bullet (a_i \bullet x) \otimes b_i \otimes a_j.$$

Using the equality (13), the sum of $b_j \bullet (a_i \bullet x) \otimes b_i \otimes a_j$ and the third and fourth terms is

$$\begin{aligned}
&\sum_{i,j} \big(L(b_j) \otimes id\big)\big((a_i \bullet x) \otimes b_i + (x \bullet b_i) \otimes a_i - b_i \otimes (x \bullet a_i)\big) \otimes a_j \\
&\sum_{j} \big(L(b_j) \otimes id\big) \sum_{i} \big((a_i \bullet x) \otimes b_i + (x \bullet b_i) \otimes a_i - b_i \otimes (x \bullet a_i)\big) \otimes a_j \\
=&\sum_{i,j} \big(L(b_j) \otimes id\big)\big(a_i \otimes (x \bullet b_i)\big) \otimes a_j \\
=&\sum_{i,j} (a_i \bullet b_j) \otimes (x \bullet b_i) \otimes a_j.
\end{aligned}$$

Similarly, the sum of $(a_i \bullet b_j) \otimes (x \bullet b_i) \otimes a_j$ and the fifth term becomes

$$\sum_{i,j} b_j \otimes (x \bullet b_i) \otimes (a_i \bullet a_j) + a_j \otimes (x \bullet b_i) \otimes (a_i \bullet b_j),$$

and the sum of the sixth and seventh term is

$$\sum_{i,j} a_j \otimes b_i \otimes x \bullet (a_i \bullet b_j) + b_j \otimes b_i \otimes x \bullet (a_i \bullet a_j).$$

Finally, the sum of $x \bullet (b_j \bullet a_i) \otimes b_i \otimes a_j$ and the second term in the sum of the expression of $E_\Delta(x) + Q(x)[[r, r]]$ becomes

$$\sum_{i,j} b_j \otimes b_i \otimes a_i \bullet (x \bullet a_j) + a_j \otimes b_i \otimes a_i \bullet (x \bullet b_j).$$

Inserting these results, we find that the expression of $E_\Delta(x) + Q(x)[[r, r]]$ can be written in the form $\sum_i \big(a_i \otimes U_i + b_i \otimes V_i\big)$: In fact,

$$\begin{aligned}
U_i = \sum_j \big\{ &- (b_j \bullet b_i) \otimes (x \bullet a_j) - (b_i \bullet a_j) \otimes (x \bullet b_j) + b_j \otimes x \bullet (a_j \bullet b_i) \\
&+ a_j \otimes x \bullet (b_i \bullet b_j) + (x \bullet b_j) \otimes (a_j \bullet b_i) + (x \bullet a_j) \otimes (b_i \bullet b_j) \\
&- x \bullet (b_i \bullet a_j) \otimes b_j + (x \bullet b_j) \bullet b_i \otimes a_j - (x \bullet b_i) \bullet a_j \otimes b_j \\
&+ a_j \otimes (x \bullet b_i) \bullet b_j + b_j \otimes (x \bullet b_i) \bullet a_j \big\}.
\end{aligned}$$

On the right-hand side, the sum of the first four terms is zero by equality (13), and the sum of the next three terms becomes

$$x \bullet (b_i \bullet b_j) \otimes a_j.$$

By the Jacobi identity in A, the sum of $x \bullet (b_i \bullet b_j) \otimes a_j$ and the eighth term is

$$-b_j \bullet (x \bullet b_i) \otimes a_j.$$

Finally, the sum of $-b_j \bullet (x \bullet b_i) \otimes a_j$ and the last three terms becomes

$$\sum_j -b_j \bullet (x \bullet b_i) \otimes a_j - (x \bullet b_i) \bullet a_j \otimes b_j + a_j \otimes (x \bullet b_i) \bullet b_j + b_j \otimes (x \bullet b_i) \bullet a_j = 0,$$

if we replace x in equality (13) by $x \bullet b_i$. Hence, we get $U_i = 0$. Similarly, we can prove that

$$\begin{aligned} V_i =& \sum_j \big\{(x \bullet b_j) \otimes (a_j \bullet a_i) + b_j \otimes x \bullet (a_j \bullet a_i) + b_j \otimes a_j \bullet (x \bullet a_i) \\ &\quad + (x \bullet a_j) \otimes (b_j \bullet a_i) - a_j \otimes (x \bullet b_j) \bullet a_i\big\} \\ =&0. \end{aligned}$$

Hence the conclusion holds. $\square$

Using the above discussion, we have the following result.

Theorem 4.4. *Let $(A, \bullet)$ be a mock-Lie algebra and $r \in A \otimes A$. Define a bilinear map $\diamond : A^* \otimes A^* \to A^*$ by*

$$\langle \xi \diamond \eta, x\rangle = \langle \Delta^*(\xi \otimes \eta), x\rangle = \langle \xi \otimes \eta, \Delta(x)\rangle,$$

where Δ is defined by equality (11). Then $(A^*, \diamond)$ is a mock-Lie algebra if and only if the following conditions are satisfied:

$$\begin{aligned} &(i) \quad \big(L(x) \otimes id - id \otimes L(x)\big)\big(r + \tau(r)\big) = 0, \\ &(ii) \quad Q(x)[[r, r]] = 0, \end{aligned}$$

for all $x \in A$. Under these conditions, (A, A^*) is a coboundary mock-Lie bialgebra.

Proof. The bracket $\diamond$ is determined by the cobracket $\Delta(x) = \big(L(x) \otimes id - id \otimes L(x)\big)r$. Hence $(A^*, \diamond)$ is a mock-Lie algebra if and only if $\diamond$ is symmetric and satisfies the Jacobi identity.

For any $x \in A,\ \xi, \eta \in A^*$, we have

$$\begin{aligned} \langle \xi \diamond \eta - \eta \diamond \xi, x\rangle &= \langle \Delta^*(\xi \otimes \eta) - \Delta^*(\eta \otimes \xi), x\rangle = \langle \xi \otimes \eta, \Delta(x) - \tau \circ \Delta(x)\rangle \\ &= \langle \xi \otimes \eta, \big(L(x) \otimes id - id \otimes L(x)\big)r - \tau \circ \big(\big(L(x) \otimes id - id \otimes L(x)\big)r\big)\rangle \\ &= \langle \xi \otimes \eta, L(x)r_1 \otimes r_2 - r_1 \otimes L(x)r_2 - r_2 \otimes L(x)r_1 + L(x)r_2 \otimes r_1\rangle \\ &= \langle \xi \otimes \eta, \big(L(x) \otimes id - id \otimes L(x)\big)\big(r + \tau(r)\big)\rangle. \end{aligned}$$

Then r satisfies (i) if and only if $\diamond$ is symmetric. The proof that (ii) holds is equivalent to the condition that $\diamond$ satisfies the Jacobi identity which follows from Lemma 4.2 and Proposition 4.3. Since $\Delta(x) = \big(L(x) \otimes id - id \otimes L(x)\big)r$, the compatibility conditions for a mock-Lie bialgebra in Definition 3.5 hold naturally. Therefore the conclusion follows. □

Remark 4.1. An easy way to satisfy conditions (i) and (ii) in Theorem 4.4 is to assume that r is skew-symmetric and

$$[[r, r]] = 0 \tag{14}$$

respectively. equality (14) is the **mock-Lie Yang–Baxter equation** in the mock-Lie algebra $(A, \bullet)$. A quasitriangular mock-Lie bialgebra is a coboundary mock-Lie bialgebra, in which r is a solution of the *mock-Lie Yang– Baxter equation.* A triangular mock-Lie bialgebra is a coboundary mock-Lie bialgebra, in which r is a skew-symmetric solution of the mock-Lie Yang–Baxter equation.

A direct application of Theorem 4.4 is given as follows.

Theorem 4.5. *Let (A, A^*) be a mock-Lie bialgebra. Then there is a canonical coboundary mock-Lie bialgebra structure on $A \oplus A^*$ such that both $i_1 : A \to A \oplus A^*$ and $i_2 : A^* \to A \oplus A^*$ into the two summands are homomorphisms of mock-Lie bialgebras. Here the mock-Lie bialgebra structure on A is $(A, -\Delta_A)$, where Δ_A is given by equality* (11).

Proof. Let $r \in A \otimes A^* \subset (A \oplus A^*) \otimes (A \oplus A^*)$ correspond to the identity map $id : A \to A$. Let $\{e_1, \cdots, e_n\}$ be a basis of A and $\{f_1, \cdots, f_n\}$ be its dual basis. Then $r = \sum_i e_i \otimes f_i$. Suppose that the mock-Lie algebra structure "$\circ_{D(A)}$" on $A \oplus A^*$ is given by $D(A) = A \bowtie A^*$. Then, by equality (5), we have

$$x \circ_{D(A)} y = x \bullet y, \quad a^* \circ_{D(A)} b^* = a^* \diamond b^*, \quad x \circ_{D(A)} a^* = L^*(x)a^* + \mathcal{L}^*(a^*)x,$$

for any $x, y \in A$, $a^*, b^* \in A^*$. Next we prove that r satisfies the two conditions in Theorem 4.4. If so, then

$$\Delta_{D(A)}(u) = (L_{\circ_{D(A)}}(u) \otimes id_{D(A)} - id_{D(A)} \otimes L_{\circ_{D(A)}}(u))r, \quad \forall u \in D(A)$$

can induce a coboundary mock-Lie bialgebra structure on $D(A)$. Since

$$\langle \sum_i e_i \otimes f_i, f_s \otimes e_t \rangle = \langle e_t, f_s \rangle,$$

we have

$$\begin{aligned}
&\langle [[r, r]]_{D(A)}, (e_s + f_t) \otimes (e_k + f_l) \otimes (e_p + f_q) \rangle \\
=& \sum_{ij} \Big\langle e_i \circ_{D(A)} e_j \otimes f_i \otimes f_j - e_i \otimes f_i \circ_{D(A)} e_j \otimes f_j + e_i \otimes e_j \otimes f_i \circ_{D(A)} f_j, \\
&\qquad (e_s + f_t) \otimes (e_k + f_l) \otimes (e_p + f_q) \Big\rangle \\
=& \sum_{ij} \Big\langle e_i \bullet e_j \otimes f_i \otimes f_j - e_i \otimes (\mathcal{L}^*(f_i)e_j + L^*(e_j)f_i) \otimes f_j + e_i \otimes e_j \otimes f_i \diamond f_j,
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad (e_s + f_t) \otimes (e_k + f_l) \otimes (e_p + f_q)\Big\rangle \\
=&\sum_{ij} \Big(\langle e_i \bullet e_j, f_t\rangle \langle f_i, e_k\rangle \langle f_j, e_p\rangle - \langle e_i, f_t\rangle \langle e_j, f_i \diamond f_l\rangle \langle f_j, e_p\rangle \\
&\qquad - \langle e_i, f_t\rangle \langle f_i, e_j \bullet e_k\rangle \langle f_j, e_p\rangle + \langle e_i, f_t\rangle \langle e_j, f_l\rangle \langle f_i \diamond f_j, e_p\rangle \Big) \\
=&\langle e_k \bullet e_p, f_t\rangle - \langle e_p, f_t \diamond f_l\rangle - \langle f_t, e_p \bullet e_k\rangle + \langle f_t \diamond f_l, e_p\rangle \\
=&0,
\end{aligned}
$$

we get $[[r, r]]_{D(A)} = 0$. Similarly, we prove that

$$\big(L_{\circ_{D(A)}}(u) \otimes id_{D(A)} - id_{D(A)} \otimes L_{\circ_{D(A)}}(u)\big)\big(r + \tau(r)\big) = 0,$$

for all $u \in D(A)$. Hence there is a coboundary mock-Lie bialgebra structure on $D(A)$ by Theorem 4.4. For $e_i \in A$, we have

$$
\begin{aligned}
\Delta_{D(A)}(e_i) &= \sum_j \big\{ e_i \bullet e_j \otimes f_j - e_j \otimes e_i \circ_{D(A)} f_j \big\} \\
&= \sum_j \big\{ e_i \bullet e_j \otimes f_j - e_j \otimes \big(L^*(e_i) f_j + \mathcal{L}^*(f_j) e_i\big) \big\} \\
&= \sum_{j,m} \big\{ e_i \bullet e_j \otimes f_j - \langle f_j, e_i \bullet e_m\rangle e_j \otimes f_m - \langle f_j \diamond f_m, e_i\rangle e_j \otimes e_m \big\} \\
&= -\sum_{j,m} \langle f_j \diamond f_m, e_i\rangle e_j \otimes e_m \\
&= -\Delta_A(e_i).
\end{aligned}
$$

Therefore $i_1 : A \to A \oplus A^*$ is a homomorphism of mock-Lie bialgebras. Similarly, $i_2 : A^* \to A \oplus A^*$ is also a homomorphism of mock-Lie bialgebras since $\Delta_{D(A)}(f_i) = \Delta_{A^*}(f_i)$. □

Remark 4.2. With the above mock-Lie bialgebra structure given in Theorem 4.5, $A \oplus A^*$ is called the double of A. We denote it by $D(A)$.

5. $\mathcal{O}$-operators of mock-Lie algebras and mock-Lie Yang–Baxter equation

In this section, we interpret a solution of the mock-Lie Yang–Baxter equation in terms of $\mathcal{O}$-operators (see [32]). Let V be a vector space. For any $r \in V \otimes V$, r can be regarded as a map from V^* to V in the following way:

$$\langle u^*, r(v^*)\rangle = \langle u^* \otimes v^*, r\rangle, \quad \forall u^*, v^* \in V^*, \tag{15}$$

where $\langle \cdot, \cdot\rangle$ is the ordinary pairing between the vector space V and the dual space V^*. The tensor $r \in V \otimes V$ is called nondegenerate if the above induced linear map is invertible. Moreover, any invertible linear map $T: V^* \to V$ induces a nondegenerate bilinear form $\omega(,)$ on V by

$$\omega(u,v) = \langle T^{-1}(u), v\rangle, \quad \forall u,v \in V.$$

Definition 5.1 ([14])**.** A *symplectic form* on a mock-Lie algebra $(A,\cdot)$ is a skew-symmetric non-degenerate bilinear form ω satisfying

$$\omega(x\bullet y,z) + \omega(y\bullet z,x) + \omega(z\bullet x,y) = 0, \quad \forall x,y,z \in A.$$

A mock-Lie algebra is called *symplectic* if it is endowed with a symplectic form.

Proposition 5.1. *Let $(A,\bullet)$ be a mock-Lie algebra and $r \in A\otimes A$ be skew-symmetric. Then r is a solution of the mock-Lie YBE in A if and only if r satisfies*

$$r(\xi)\bullet r(\eta) = r\big(L^*(r(\xi))\eta + L^*(r(\eta))\xi\big), \quad \forall \xi,\eta \in A^*. \tag{16}$$

Proof. Let $\{e_1,\cdots,e_n\}$ be a basis of A and $\{e_1^*,\cdots,e_n^*\}$ be the dual basis. Since r is skew-symmetric, we can set $r = \sum_{1\le i,j\le n} a_{ij}e_i\otimes e_j,\ a_{ij} = -a_{ji}$. Suppose that $e_i\bullet e_j = \sum_{k=1}^n C_{ij}^k e_k$, where C_{ij}^k's are the structure coefficients the of mock-Lie algebra A on the basis $\{e_1,\cdots,e_n\}$. We get

$$\begin{aligned}
r_{12}\bullet r_{13} &= \big(\sum_{1\le i,j\le n} a_{ij}e_i\otimes e_j\otimes 1\big)\bullet\big(\sum_{1\le p,q\le n} a_{pq}e_p\otimes 1\otimes e_q\big)\\
&= \sum_{1\le i,j,p,q,k\le n} C_{ip}^k a_{ij}a_{pq} e_k\otimes e_j\otimes e_q;\\
r_{13}\bullet r_{23} &= \sum_{1\le i,j,p,q,k\le n} C_{jq}^k a_{ij}a_{pq} e_i\otimes e_p\otimes e_k;\\
r_{12}\bullet r_{23} &= \sum_{1\le i,j,p,q,k\le n} C_{jp}^k a_{ij}a_{pq} e_i\otimes e_k\otimes e_q.
\end{aligned}$$

Then r is a solution of mock-Lie YBE in A if and only if

$$\sum_{1\le i,p\le n}\big(C_{ip}^k a_{ij}a_{pq} + C_{pi}^q a_{kp}a_{ji} - C_{ip}^j a_{ki}a_{pq}\big)e_k\otimes e_j\otimes e_q.$$

On the other hand, by equality (15), we get $r(e_j^*) = \sum_{i=1}^n a_{ij}e_i = -\sum_{i=1}^n a_{ji}e_i$, $1\le j\le n$. If we take $\xi = e_j^*$, $\eta = e_q^*$ and use equality (16), we get

$$\sum_{1\le i,p\le n}\big(C_{ip}^k a_{ij}a_{pq} + C_{pi}^q a_{kp}a_{ji} - C_{ip}^j a_{ki}a_{pq}\big)e_k = 0.$$

Therefore, it is easy to see that r is a solution of the mock-Lie YBE in A if and only if r satisfies equality (16). $\square$

Example 5.2. Let $(A, \bullet)$ be a mock-Lie algebra. Then a Rota-Baxter operator (of weight zero) is an $\mathcal{O}$-operator of A associated to the adjoint representation (A, L) and a skew-symmetric solution of the mock-Lie YBE in A is an $\mathcal{O}$-operator of A associated to the representation (A^*, L^*).

Corollary 5.3. *Let $(A, \bullet)$ be a mock-Lie algebra and $r \in A \otimes A$ be skew-symmetric. Suppose that there is a symmetric nondegenerate invariant bilinear form ω on A. Let $\phi : A \to A^*$ be a linear map given by $\langle \phi(x), y\rangle = \omega(x, y)$ for any $x, y \in A$. Then r is a solution of the mock-Lie YBE if and only if $r\phi$ is a Rota–Baxter operator (of weight zero) on A.*

Proof. For any $x, y \in A$, we have $\phi(L(x)y) = L^*(x)\phi(y)$ since

$$\begin{aligned}\langle \phi(L(x)y), z\rangle &= \omega(x \bullet y, z) = \omega(y \bullet x, z)\\ &= \omega(y, x \bullet z) = \langle L^*(x)\phi(y), z\rangle, \quad \forall x, y, z \in A.\end{aligned}$$

That is, the representations (A, L) and (A^*, L^*) are isomorphic. Let $\xi = \phi(x), \eta = \phi(y)$, then by Proposition 5.1, r is a solution of the mock-Lie YBE in A if and only if

$$\begin{aligned}r\phi(x) \bullet r\phi(y) = r(\xi) \bullet r(\eta) &= r\Big(L^*(r(\xi))\eta + L^*(r(\eta))\xi\Big)\\ &= r\phi\Big(r\phi(x) \bullet y + x \bullet r\phi(y)\Big).\end{aligned}$$

Therefore the conclusion holds. □

Let $(A, \bullet)$ be a mock-Lie algebra. Let (V, ρ) be a representation of A and $\rho^* : A \to gl(V^*)$ be the dual representation. A linear map $T : V \to A$ can be identified as an element in $A \otimes V^* \subset (A \ltimes_{\rho^*} V^*) \otimes (A \ltimes_{\rho^*} V^*)$ as follows. Let $\{e_1, \cdots, e_n\}$ be a basis of A, let $\{v_1, \cdots, v_m\}$ be a basis of V and $\{v_1^*, \cdots, v_m^*\}$ be its dual space of V^*. We set

$$T(v_i) = \sum_{k=1}^{n} a_{ik}e_k,\ i = 1, \cdots, m.$$

Since as a vector space, $Hom(V, A) \cong A \otimes V^*$, we have

$$T = \sum_{i=1}^{m} T(v_i) \otimes v_i^* = \sum_{i=1}^{m}\sum_{k=1}^{n} a_{ik}e_k \otimes v_i^* \in A \otimes V^* \subset (A \ltimes_{\rho^*} V^*) \otimes (A \ltimes_{\rho^*} V^*).$$

Theorem 5.4. *With the above notations, $r = T - \tau(T)$ is a skew-symmetric solution of the mock-Lie YBE in the semi-direct product mock-Lie algebra $(A \ltimes_{\rho^*} V^*)$ if and only if T is an $\mathcal{O}$-operator associated to (V, ρ).*

Proof. We have

$$r = T - \tau(T) = \sum_{i=1}^{m} T(v_i) \otimes v_i^* - \sum_{i=1}^{m} v_i^* \otimes T(v_i),$$

thus we obtain

$$r_{12}\bullet r_{13}=\sum_{i,j=1}^{m}Tv_i\bullet Tv_j\otimes v_i^*\otimes v_j^*-\rho^*(Tv_i)v_j^*\otimes v_i^*\otimes Tv_j-\rho^*(Tv_j)v_i^*\otimes Tv_i\otimes v_j^*;$$

$$r_{12}\bullet r_{23}=\sum_{i,j=1}^{m}-v_i^*\otimes Tv_i\bullet Tv_j\otimes v_j^*+Tv_i\otimes\rho^*(Tv_j)v_i^*\otimes v_j^*+v_i^*\otimes\rho^*(Tv_i)v_j^*\otimes Tv_j;$$

$$r_{13}\bullet r_{23}=\sum_{i,j=1}^{m}v_i^*\otimes v_j^*\otimes Tv_i\bullet Tv_j-Tv_i\otimes v_j^*\otimes\rho^*(Tv_j)v_i^*-v_i^*\otimes Tv_j\otimes\rho^*(Tv_i)v_j^*.$$

By the definition of dual representation, we know

$$\rho^*(Tv_j)v_i^*=\sum_{p=1}^{m}\langle v_i^*,\rho(Tv_j)v_p\rangle v_p^*.$$

Therefore

$$\sum_{1\le i,j\le m}Tv_i\otimes\rho^*(Tv_j)v_i^*\otimes v_j^*=\sum_{1\le i,j,p\le m}\langle v_p^*,\rho(Tv_j)v_i\rangle Tv_p\otimes v_i^*\otimes v_j^*$$

$$=\sum_{1\le i,j\le m}T\Big(\langle v_p^*,\rho(Tv_j)v_i\rangle v_p\Big)\otimes v_i^*\otimes v_j^*=\sum_{1\le i,j\le m}T\Big(\rho(Tv_j)v_i\Big)\otimes v_i^*\otimes v_j^*.$$

Then we get

$$r_{12}\bullet r_{13}=\sum_{i,j=1}^{m}Tv_i\bullet Tv_j\otimes v_i^*\otimes v_j^*-v_i^*\otimes v_j^*\otimes T(\rho(Tv_j)v_i)-v_i^*\otimes T(\rho(Tv_j)v_i)\otimes v_j^*;$$

$$-r_{12}\bullet r_{23}=\sum_{i,j=1}^{m}v_i^*\otimes Tv_i\bullet Tv_j\otimes v_j^*-T(\rho(Tv_j)v_i)\otimes v_i^*\otimes v_j^*-v_i^*\otimes v_j^*\otimes T(\rho(Tv_i)v_j);$$

$$r_{13}\bullet r_{23}=\sum_{i,j=1}^{m}v_i^*\otimes v_j^*\otimes Tv_i\bullet Tv_j-T(\rho(Tv_i)v_j)\otimes v_i^*\otimes v_j^*-v_i^*\otimes T(\rho(Tv_i)v_j)\otimes v_j^*.$$

Hence, we obtain

$$\begin{aligned}&r_{12}\bullet r_{13}+r_{13}\bullet r_{23}-r_{12}\bullet r_{23}\\&=\sum_{1\le i,j\le m}\Bigg\{\Big(Tv_i\bullet Tv_j-T(\rho(Tv_i)v_j)-T(\rho(Tv_j)v_i)\Big)\otimes v_i^*\otimes v_j^*\\&\qquad+v_i^*\otimes v_j^*\otimes\Big(Tv_i\bullet Tv_j-T(\rho(Tv_i)v_j)-T(\rho(Tv_j)v_i)\Big)\\&\qquad+v_i^*\otimes\Big(Tv_i\bullet Tv_j-T(\rho(Tv_i)v_j)-T(\rho(Tv_j)v_i)\Big)\otimes v_j^*\Bigg\}.\end{aligned}$$

So r is a solution of the mock-Lie YBE in the semi-direct product mock-Lie algebra $(A\ltimes_{\rho^*}V^*)$ if and only if T is an $\mathcal{O}$-operator associated to (V,ρ). □

Combining Proposition 5.1 and Theorem 5.4, we have the following conclusion.

Corollary 5.5. *Let $(A,\bullet)$ be a mock-Lie algebra and (V,ρ) be a representation of A. Set $\widehat{A}=A\ltimes_{\rho^*}V^*$. Let $T:V\to A$ be a linear map. Then the following conditions are equivalent.*

(1) *T is an $\mathcal{O}$-operator of A associated to (V,ρ).*
(2) *$T-\tau(T)$ is a skew-symmetric solution of the mock-Lie YBE in the Jordan algebra $\widehat{A}$.*
(3) *$T-\tau(T)$ is an $\mathcal{O}$-operator of the mock-Lie algebra $\widehat{A}$ associated to $(\widehat{A}^*,L^*_{\widehat{A}})$.*

Remark 5.1. The equivalence between the above (1) and (3) can be obtained by a straightforward proof and then Theorem 5.4 follows from this equivalence and Proposition 5.1.

The following conclusion reveals the relationship between mock-pre-Lie algebras and the mock-Lie algebras with a symplectic form:

Proposition 5.6. *Let $(A,\bullet)$ be a mock-Lie algebra with a symplectic form ω. Then there exists a compatible pre-mock-Lie algebra structure "$\cdot$" on A given by*

$$\omega(x\cdot y,z)=\omega(y,x\bullet z),\quad \forall x,y,z\in A.$$

Proof. Define a linear map $T:A\to A^*$ by $\langle T(x),y\rangle=\omega(x,y)$ for any $x,y\in A$. For any $\xi,\eta,\gamma\in A^*$, since T is invertible, there exist $x,y,z\in A$ such that $Tx=\xi, Ty=\eta, Tz=\gamma$. Then $T^{-1}:A^*\to A$ is an $\mathcal{O}$-operator of A associated to (A^*,L^*) since for any $x,y,z\in A$, we have

$$\begin{aligned}\langle T(x\bullet y),z\rangle&=\omega(x\bullet y,z)=\omega(y,x\bullet z)+\omega(x,y\bullet z)\\&=\langle L^*(x)T(y),z\rangle+\langle L^*(y)T(x),z\rangle.\end{aligned}$$

By Proposition 2.8, there is a compatible mock-pre-Lie algebra structure "$\cdot$" on A given by

$$x\cdot y=T^{-1}\big(L^*(x)T(y)\big),\quad \forall x,y\in A,$$

which implies that

$$\begin{aligned}\omega(x\cdot y,z)&=\langle T(x\cdot y),z\rangle=\langle L^*(x)T(y),z\rangle\\&=\langle T(y),x\bullet z\rangle=\omega(y,x\bullet z),\quad \forall x,y,z\in A.\end{aligned}$$

Hence the proof. □

The following conclusion provides a construction of solutions of the mock-Lie YBE in certain mock-Lie algebras from mock-pre-Lie algebras.

Corollary 5.7. *Let $(A,\cdot)$ be a mock-pre-Lie algebra. Let $\{e_1,\cdots,e_n\}$ be a basis of A and $\{e_1^*,\cdots,e_n^*\}$ be the dual basis. Then*

$$r=\sum_{i=1}^{n}(e_i\otimes e_i^*-e_i^*\otimes e_i)$$

is a skew-symmetric solution of the mock-Lie YBE in the mock-Lie algebra $(A^{ac})\ltimes_{\Theta^*}(A^{ac})^*$.

Proof. It follows from Theorem 5.4 and the fact that the identity map id is an $\mathcal{O}$-operator of the sub-adjacent mock-Lie algebra A^{ac} of a mock-pre-Lie algebra associated to the representation (A,Θ). $\square$

Further discussions

In this paper, we consider the D-bialgebra of mock-Lie type and the corresponding Yang–Baxter algebras. The category of connected, simply-connected Poisson–Lie groups is equivalent to the category of Lie bialgebras [26]. Therefore, a basic problem in the theory of Poisson manifolds is the classification of Lie bialgebras. A fundamental contribution to this question is the Theorem of Belavin and Drinfeld [19], which contains the classification of all the simple factorizable complex Lie bialgebras. In particular, the author of [1], uses algebro-geometric methods in order to derive classification results for so-called D-bialgebra structures on the power series algebra $\mathfrak{g}[[z]]$ for certain central simple Lie (resp. associative) algebras $\mathfrak{g}$. These structures are closely related to a version of the classical (resp. associative) Yang–Baxter equation (CYBE) over $\mathfrak{g}$ (for more details see [2, 3]). Therefore, it is interesting to introduce and classify simple mock-Lie (bi)algebras using the corresponding Yang–Baxter equation.

Acknowledgements

The authors would like to thank the referee for valuable comments and suggestions on this article.

References

[1] R. Abedin, *Classification of D-bialgebra structures on power series algebras*, arXiv:2301.13022.v2[math.AG] (2023), 45 pp. DOI

[2] R. Abedin, S. Maximov, and A. Stolin, *Topological Manin pairs and (n,s)-type series*, Lett. Math. Phys. **113** (2023), 1–28.

[3] R. Abedin, S. Maximov, A. Stolin, and E. Zelmanov, *Topological Lie bialgebras structures and their classification over $\mathfrak{g}[[x]]$*, arXiv:2203.01105v3[math.RA] (2022), 45 pp. DOI

[4] A. L. Agore and G. Militaru, *On a type of commutative algebras*, Linear Algebra Appl. **485** (2015), 222–249.

[5] A. L. Agore and G. Militaru, *Algebraic constructions for Jacobi–Jordan algebras*, Linear Algebra Appl. **630** (2021), 158–178.

[6] M. Aguiar, *Infinitesimal Hopf algebras*, Contemp. Math. **267** (2000), 1–29.

[7] S. Attan, *Representations and O-operators of Hom-(pre)-Jacobi–Jordan algebras*, arXiv:2105.14650[math.RA] (2021), 19 pp. DOI

[8] C. Bai, *Left-symmetric bialgebras and an analogue of the classical Yang–Baxter equation*, Commun. Contemp. Math. **10** (2008), 221–260.

[9] C. Bai, *Double constructions of Frobenius algebras, Connes cocycles and their duality*, J. Noncommut. Geom. **4** (2010), 475–530.

[10] C. Bai, *A unified algebraic approach to the classical Yang–Baxter equation*, J. Phys. A. **40** (2007), 11073–11082.

[11] C. Bai, L. Guo, and T. Ma, *Bialgebras, Frobenius algebras and associative Yang–Baxter equations for Rota–Baxter algebras*, arXiv:2112.10928v1[math.QA] (2021), 27 pp. DOI

[12] C. Bai, L. Guo, G. Liu, and T. Ma, *Rota–Baxter Lie bialgebras, classical Yang–Baxter equations and special L-dendriform bialgebras*, arXiv:2207.08703v1[math.QA] (2022), 29 pp. DOI

[13] C. Bai, O. Bellier, L. Guo, and X. Ni, *Splitting of operations, Manin products and Rota–Baxter operators*, Int. Math. Res. Not. **3** (2013), 485–524.

[14] A. Baklouti and S. Benayadi, *Symplectic Jacobi–Jordan algebras*, Linear Multilinear Algebra **69** (2021), 1557–1578.

[15] A. Baklouti, S. Benayadi, A. Makhlouf, and S. Mansour, *Cohomology and deformations of Jacobi–Jordan algebras*, ArXiv: 2109.12364v1[math.RA] (2021), 23 pp. DOI

[16] I. Basdouri, M. Fadous, S. Mabrouk, and A. Makhlouf, *Poisson superbialgebras*, arXiv:2205.06184v1[math.RA] (2022), 24 pp. DOI

[17] R. J. Baxter, *Partition function fo the eight-vertex lattice model*, Ann. Physics **70** (1972), 193–228.

[18] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London, 1982.

[19] A. A. Belavin and V. G. Drinfeld, *Triangle equations and simple Lie algebras*, Math. Phys. Rev. **4** (1984), 93–165.

[20] D. Burde, *Left-symmetric algebras and pre-Lie algebras in geometry and physics*, Cent. Eur. J. Math. **4** (2006), 323–357.

[21] D. Burde and A. Fialowski, *Jacobi–Jordan algebras,* Linear Algebra Appl. **495** (2014), 586–594.

[22] L. M. Camacho, I. Kaygorodov, V. Lopatkin, and Mohamed A. Salim, *The variety of dual mock-Lie algebras*, Commun. Math. **28** (2020) 161–178.

[23] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, Cambridge, 1994.

[24] M. L. Dassoundo, *Pre-anti-flexible bialgebras*, Comm. Algebra, **49** (2021), 4050–4078.

[25] M. L. Dassoundo, C. Bai, and M. N. Hounkonnou, *Anti-flexible bialgebras*, J. Algebra Appl. **21** (2022), 2250212. DOI

[26] V. G. Drinfeld, *Hamiltonian structures of Lie groups, Lie bialgebras and the geometric meaning of the classical Yang–Baxter equations*, Soviet Math. Dokl. **27** (1983), 68–71.

[27] M. Fadous, S. Mabrouk, and A. Makhlouf, *On Hom-Lie superbialgebras*, Comm. Algebra **47** (2019), 114–137.

[28] C. E. Haliya and G. D. Houndedji, *On a pre-Jacobi-Jordan algebra: relevant properties and double construction*, arXiv:2007.06736v1[math.RA] (2020), 20 pp. DOI

[29] C. E. Haliya and G. D. Houndedji, *Hom-Jacobi-Jordan and Hom-antiassociative algebras with symmetric invariant nondegenerate bilinear forms*, Quasigroups Related Systems **29** (2021), 61–88.
[30] D. Hou, X. Ni, and C. Bai, *Pre-Jordan algebras,* Math. Scand. (2013), 19–48.
[31] S. A. Joni and G. C. Rota, *Coalgebras and bialgebras in combinatories*, Stud. Appl. Math. **61** (1979), 93–139.
[32] B. A. Kupershmidt, *What a classical r-matrix really is*, J. Nonlinear Math. Phys. **6** (1999), 448–488.
[33] A. Makhlouf and F. Panaite, *Yetter–Drinfeld modules for Hom-bialgebras*, J. Math. Phys. **55** (2014), 013501. DOI
[34] T. Ma, A. Makhlouf, and S. Silvestrov, *Rota—Baxter cosystems and coquasitriangular mixed bialgebras*, J. Algebra Appl. **20** (2021), 2150064. DOI
[35] J. H. H. Perk and H. Au-Yang, *Yang–Baxter equations,* arXiv:math-ph/0606053v1 (2006), 22 pp. DOI
[36] G. C. Rota, *Baxter Operators, an Introduction.* In: Gian-Carlo Rota on Combinatorics: Introductory Papers and Commentaries, Joseph P. S. Kung editor, Birkhäuser, Boston, 1995.
[37] C. N. Yang, *Some exact results for the many-body problem in one dimension with replusive delta-function interaction.* Phys. Rev. Lett. **19** (1967) 1312–1315.
[38] V. N. Zhelyabin, *Jordan bialgebras and their relation to Lie bialgebras*, Algebra Logic **36** (1997), 1–15.
[39] V. N. Zhelyabin, *Jordan D-bialgebras and symplectic forms on Jordan algebras*, Siberian Adv. Math. **10** (2000), 142–150.
[40] V. N. Zhelyabin, *On a class of Jordan D-bialgebras*, St. Petersburg Math. J. **11** (2000), 589–609.
[41] P. Zusmanovich, *Special and exceptional mock-Lie algebras*, Linear Algebra Appl. **518** (2017), 79–96.

University of Sfax, Faculty of Sciences, BP 1171, 3038 Sfax, Tunisia
E-mail address: karimabenali172@yahoo.fr
E-mail address: atefhajjaji100@gmail.com

University of Gabes, Faculty of Sciences Gabes, City Riadh 6072 Zrig, Gabes, Tunisia
E-mail address: chtioui.taoufik@yahoo.fr

University of Gafsa, Faculty of Sciences, 2112 Gafsa, Tunisia
E-mail address: mabrouksami00@yahoo.fr