ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 27, Number 2, December 2023
Available online at https://ojs.utlib.ee/index.php/ACUTM

Examination of generalized Tribonacci dual quaternions

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Abstract. This manuscript deals with introducing and discussing of a new type dual quaternions which are named *generalized Tribonacci dual quaternions* ($\mathcal{GTDQ}$, for short). For this purpose, several new properties, such as Binet formula, generating function, exponential generating function, matrix formula, and determinant equations, are established. In addition to these, some numerical algorithms are constructed. In the last part, some special cases of the family of the $\mathcal{GTDQ}$ are examined regarding r, s, t values and initial values considering concluded results.

1. Introduction

Hamilton's quaternions have various applications and importance over the number theory and they extend complex numbers with a remarkable way [17, 18]. Quaternions are widely used in many areas such as pure and applied mathematics, motion geometry, differential geometry, graph theory, computer animation, robotics, etc. The algebra of quaternions is associative, non-commutative, and a 4-dimensional Clifford algebra. A quaternion is represented by $q = q_0 + q_1 i + q_2 j + q_3 k$, where $q_0, q_1, q_2, q_3 \in \mathbb{R}$ and i, j, k are quaternionic units that satisfy the rules ([17, 18]): $i^2 = -1$, $j^2 = -1$, $k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. Many researchers have studied and examined various types of quaternions in the quaternion family, such as split quaternions, generalized quaternions, etc. [9, 25, 31, 34]. Additionally, the concept of quaternion-valued short-time special affine Fourier transform is introduced and examined in [57].

The most attractive ones in the quaternion family are dual quaternions which have been studied by Majernik [30]. Dual quaternions are determined

Received April 26, 2023.
2020 *Mathematics Subject Classification.* 11B37, 11K31, 11R52.
Key words and phrases. Generalized Tribonacci numbers, dual quaternions, generalized Tribonacci dual quaternions.
https://doi.org/10.12697/ACUTM.2023.27.17
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by the set $\mathbb{H}_D = \{q | q = q_0 + q_1 i + q_2 j + q_3 k,\ q_0, q_1, q_2, q_3 \in \mathbb{R}\}$ where i, j, k satisfy the following rules ([11, 30]):

$$i^2 = j^2 = k^2 = ijk = 0,\ ij = ji = jk = kj = ki = ik = 0. \tag{1}$$

On the other hand, special sequences (or numbers) are popular, interesting, and essential topics of mathematics (especially in number theory). A growing number of ongoing studies has been done on special numbers with different orders. Fibonacci and Lucas numbers ([10, 29]) are the most popular members of the general second-order linear recurrence sequence named Horadam [20]. Horadam polynomials are examined in geometric function theory [56, 58, 59] as well. Besides, the members of third-order linear recurrence sequences, such as Tribonacci, Tribonacci Lucas, Padovan, Pell–Padovan sequences, have also attracted a great attention as members of general third-order linear recurrence sequence named as generalized Tribonacci sequence (see the studies [1, 4–8, 12, 14, 26, 27, 33, 35–55, 63–65]). Generalized Tribonacci sequence $\{T_n(T_0, T_1, T_2; r, s, t)\}_{n \geq 0}$ ($\{T_n\}_{n \geq 0}$, in short) satisfies the recurrence relation

$$T_n = rT_{n-1} + sT_{n-2} + tT_{n-3}, \quad n \geq 3, \tag{2}$$

where the initial values $T_0 = a, T_1 = b, T_2 = c$ are arbitrary integers and $r, s, t \in \mathbb{R}$ [2]. It can be extended to negative subscripts as follows [41]:

$$T_{-n} = -\tfrac{s}{t} T_{-(n-1)} - \tfrac{r}{t} T_{-(n-2)} + \tfrac{1}{t} T_{-(n-3)}, \quad n \in \mathbb{Z}^+, \quad t \neq 0. \tag{3}$$

Hence (2) is valid for every $n \in \mathbb{Z}$.

Additionally, one can observe that the study of various types of quaternions with special numbers components is an intensively discussed topic in the literature. Fibonacci and Lucas real quaternions are examined in [13, 16, 19, 22]. Narayana (or Fibonacci–Narayana) generalized quaternions are discussed in [13]. Padovan and Pell–Padovan quaternions are introduced in [60]. Some properties of Padovan quaternions are studied in [15]. Padovan and Perrin generalized quaternions are determined in [24]. Real quaternions with generalized Tribonacci numbers are examined in [2]. Bicomplex generalized Tribonacci quaternions are observed in [28]. Dual third-order Jacobsthal quaternions are discussed in [3]. Padovan, Perrin and Pell–Padovan dual quaternions are examined in [23]. Dual Fibonacci quaternions are studied in [62] and generalized dual Fibonacci quaternions are examined in [61].

The fundamental aim of this present work is to determine and examine a new type of dual quaternions which are called the *generalized Tribonacci dual quaternions* ($\mathcal{GTDQ}$). These types of quaternions are discussed regarding special initial values and r, s, t values. Additionally, a recurrence relation, a Binet formula, a generating function, an exponential generating function, matrix equalities, determinant equations, and several special properties are studied. Some numerical algorithms are given to find the nth term of $\mathcal{GTDQ}$. At a final glance, special cases are examined in detail using concluded results.

2. Basic notions

This section gives the general notions about dual quaternions ([11, 30]) and basic discussions regarding generalized Tribonacci numbers ([1, 4–8, 12, 14, 26, 27, 32, 33, 35–55, 63–66]).

The dual quaternion $q \in \mathbb{H}_D$ can be written as $q = S_q + \overrightarrow{V}_q$, where $S_q = q_0$ is the scalar part and $\overrightarrow{V}_q = q_1 i + q_2 j + q_3 k$ is the vector part. For $q, p \in \mathbb{H}_D$, regarding (1), the following operations are given:

- $q = p \Leftrightarrow q_0 = p_0, q_1 = p_1, q_2 = p_2, q_3 = p_3$,
- $q \pm p = q_0 \pm p_0 + (q_1 \pm p_1)i + (q_2 \pm p_2)j + (q_3 \pm p_3)k$,
- $\lambda q = \lambda q_0 + \lambda q_1 i + \lambda q_2 j + \lambda q_3 k, \lambda \in \mathbb{R}$,
- $qp = (q_0 + q_1 i + q_2 j + q_3 k)(p_0 + p_1 i + p_2 j + p_3 k)$.

Also, this implies that $S_{q\pm p} = q_0 \pm p_0 = S_q \pm S_p$ and $\overrightarrow{V}_{q\pm p} = \overrightarrow{V}_q \pm \overrightarrow{V}_p$. The conjugate of a dual quaternion q is $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$. Then, $\overline{q} = S_q - \overrightarrow{V}_q$ and the norm of q is $N_q = q\overline{q} = \overline{q}q = q_0^2$ ([30]).

On the other hand, the characteristic equation of generalized Tribonacci numbers (see (2)) is

$$x^3 - rx^2 - sx - t = 0. \tag{4}$$

The roots of (4) are given as

$$x_1 = \tfrac{r}{3} + \alpha + \beta, \quad x_2 = \tfrac{r}{3} + \varepsilon\alpha + \varepsilon^2\beta, \quad x_3 = \tfrac{r}{3} + \varepsilon^2\alpha + \varepsilon\beta,$$

where

$$\begin{aligned} &\alpha = \sqrt[3]{\tfrac{r^3}{27} + \tfrac{rs}{6} + \tfrac{t}{2} + \sqrt{\mu}}, && \beta = \sqrt[3]{\tfrac{r^3}{27} + \tfrac{rs}{6} + \tfrac{t}{2} - \sqrt{\mu}}, \\ &\varepsilon = \tfrac{-1+i\sqrt{3}}{2}, && \mu = \tfrac{r^3 t}{27} - \tfrac{r^2 s^2}{108} + \tfrac{rst}{6} - \tfrac{s^3}{27} + \tfrac{t^2}{4}, \end{aligned}$$

and $x_1 + x_2 + x_3 = r$, $x_1x_2 + x_1x_3 + x_2x_3 = -s$, $x_1x_2x_3 = t$. The following equality is called the Binet formula for generalized Tribonacci numbers ([2]):

$$T_n = \frac{\widetilde{P}x_1^n}{(x_1-x_2)(x_1-x_3)} + \frac{\widetilde{R}x_2^n}{(x_2-x_1)(x_2-x_3)} + \frac{\widetilde{S}x_3^n}{(x_3-x_1)(x_3-x_2)}, \tag{5}$$

where

$$\begin{cases} \widetilde{P} = c - (x_2 + x_3)b + x_2x_3a, \\ \widetilde{R} = c - (x_1 + x_3)b + x_1x_3a, \\ \widetilde{S} = c - (x_1 + x_2)b + x_1x_2a. \end{cases} \tag{6}$$

Howard and Saidak [21] show that the Binet formula of numbers satisfying (4) is valid for every $n \in \mathbb{Z}$ [41]. Besides, a useful way to generate T_n is using the S-matrix ([36, 63]) $S = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, which is studied as a generalization of the R-matrix (see [27, 64]). Then, the matrix representations of some third-order special recurrence sequences with negative subscripts are studied in [32] and [66]. Also, considering these basic facts, Table 1 represents several members of this family according to both initial values and

r, s, t values. Also, some special subfamilies of this quite large sequence family according to r, s, t values can be examined in Table 2. For getting more detailed information see studies [1, 4–8, 12, 14, 26, 27, 32, 33, 35–55, 60, 63–66].

TABLE 1. Several members of generalized Tribonacci numbers.

Name	$\{T_n\} = \{T_n(T_0, T_1, T_2; r, s, t)\}$	Recurrence Relation
Tribonacci	$\{A_n\} = \{T_n(0, 1, 1; 1, 1, 1)\}$	$A_n = A_{n-1} + A_{n-2} + A_{n-3}$
Tribonacci–Lucas	$\{B_n\} = \{T_n(3, 1, 3; 1, 1, 1)\}$	$B_n = B_{n-1} + B_{n-2} + B_{n-3}$
Tribonacci–Perrin	$\{C_n\} = \{T_n(3, 0, 2; 1, 1, 1)\}$	$C_n = C_{n-1} + C_{n-2} + C_{n-3}$
M. Tribonacci	$\{D_n\} = \{T_n(1, 1, 1; 1, 1, 1)\}$	$D_n = D_{n-1} + D_{n-2} + D_{n-3}$
M. Tribonacci–Lucas	$\{E_n\} = \{T_n(4, 4, 10; 1, 1, 1)\}$	$E_n = E_{n-1} + E_{n-2} + E_{n-3}$
A. Tribonacci–Lucas	$\{F_n\} = \{T_n(4, 2, 0; 1, 1, 1)\}$	$F_n = F_{n-1} + F_{n-2} + F_{n-3}$
Padovan (Cordonnier)	$\{G_n\} = \{T_n(1, 1, 1; 0, 1, 1)\}$	$G_n = G_{n-2} + G_{n-3}$
Perrin	$\{H_n\} = \{T_n(3, 0, 2; 0, 1, 1)\}$	$H_n = H_{n-2} + H_{n-3}$
Van der Laan	$\{I_n\} = \{T_n(1, 0, 1; 0, 1, 1)\}$	$I_n = I_{n-2} + I_{n-3}$
Padovan–Perrin	$\{J_n\} = \{T_n(0, 0, 1; 0, 1, 1)\}$	$J_n = J_{n-2} + J_{n-3}$
M. Padovan	$\{K_n\} = \{T_n(3, 1, 3; 0, 1, 1)\}$	$K_n = K_{n-2} + K_{n-3}$
A. Padovan	$\{L_n\} = \{T_n(0, 1, 0; 0, 1, 1)\}$	$L_n = L_{n-2} + L_{n-3}$
Pell–Padovan	$\{M_n\} = \{T_n(1, 1, 1; 0, 2, 1)\}$	$M_n = 2M_{n-2} + M_{n-3}$
Pell–Perrin	$\{N_n\} = \{T_n(3, 0, 2; 0, 2, 1)\}$	$N_n = 2N_{n-2} + N_{n-3}$
T. Fibonacci–Pell	$\{O_n\} = \{T_n(1, 0, 2; 0, 2, 1)\}$	$O_n = 2O_{n-2} + O_{n-3}$
T. Lucas–Pell	$\{P_n\} = \{T_n(3, 0, 4; 0, 2, 1)\}$	$P_n = 2P_{n-2} + P_{n-3}$
A. Pell–Padovan	$\{R_n\} = \{T_n(0, 1, 0; 0, 2, 1)\}$	$R_n = 2R_{n-2} + R_{n-3}$
T. Pell	$\{S_n\} = \{T_n(0, 1, 2; 2, 1, 1)\}$	$S_n = 2S_{n-1} + S_{n-2} + S_{n-3}$
T. Pell–Lucas	$\{U_n\} = \{T_n(3, 2, 6; 2, 1, 1)\}$	$U_n = 2U_{n-1} + U_{n-2} + U_{n-3}$
T. modified Pell	$\{V_n\} = \{T_n(0, 1, 1; 2, 1, 1)\}$	$V_n = 2V_{n-1} + V_{n-2} + V_{n-3}$
T. Pell–Perrin	$\{W_n\} = \{T_n(3, 0, 2; 2, 1, 1)\}$	$W_n = 2W_{n-1} + W_{n-2} + W_{n-3}$
T. Jacobsthal	$\{X_n\} = \{T_n(0, 1, 1; 1, 1, 2)\}$	$X_n = X_{n-1} + X_{n-2} + 2X_{n-3}$
T. Jacobsthal–Lucas	$\{Y_n\} = \{T_n(2, 1, 5; 1, 1, 2)\}$	$Y_n = Y_{n-1} + Y_{n-2} + 2Y_{n-3}$
M. T. Jacobsthal	$\{Z_n\} = \{T_n(3, 1, 3; 1, 1, 2)\}$	$Z_n = Z_{n-1} + Z_{n-2} + 2Z_{n-3}$
T. Jacobsthal–Perrin	$\{\Gamma_n\} = \{T_n(3, 0, 2; 1, 1, 2)\}$	$\Gamma_n = \Gamma_{n-1} + \Gamma_{n-2} + 2\Gamma_{n-3}$
Jacobsthal–Padovan	$\{\chi_n\} = \{T_n(1, 1, 1; 0, 1, 2)\}$	$\chi_n = \chi_{n-2} + 2\chi_{n-3}$
Jacobsthal–Perrin	$\{\Delta_n\} = \{T_n(3, 0, 2; 0, 1, 2)\}$	$\Delta_n = \Delta_{n-2} + 2\Delta_{n-3}$
A. Jacobsthal–Padovan	$\{\omega_n\} = \{T_n(0, 1, 0; 0, 1, 2)\}$	$\omega_n = \omega_{n-2} + 2\omega_{n-3}$
M. Jacobsthal–Padovan	$\{\Omega_n\} = \{T_n(3, 1, 3; 0, 1, 2)\}$	$\Omega_n = \Omega_{n-2} + 2\Omega_{n-3}$
Narayana	$\{\vartheta_n\} = \{T_n(0, 1, 1; 1, 0, 1)\}$	$\vartheta_n = \vartheta_{n-1} + \vartheta_{n-3}$
Narayana–Lucas	$\{\tau_n\} = \{T_n(3, 1, 1; 1, 0, 1)\}$	$\tau_n = \tau_{n-1} + \tau_{n-3}$
Narayana–Perrin	$\{\sigma_n\} = \{T_n(3, 0, 2; 1, 0, 1)\}$	$\sigma_n = \sigma_{n-1} + \sigma_{n-3}$
3-primes	$\{\kappa_n\} = \{T_n(0, 1, 2; 2, 3, 5)\}$	$\kappa_n = 2\kappa_{n-1} + 3\kappa_{n-2} + 5\kappa_{n-3}$
Lucas 3-primes	$\{\theta_n\} = \{T_n(3, 2, 10; 2, 3, 5)\}$	$\theta_n = 2\theta_{n-1} + 3\theta_{n-2} + 5\theta_{n-3}$
M. 3-primes	$\{\gamma_n\} = \{T_n(0, 1, 1; 2, 3, 5)\}$	$\gamma_n = 2\gamma_{n-1} + 3\gamma_{n-2} + 5\gamma_{n-3}$
Reverse 3-primes	$\{\nabla_n\} = \{T_n(0, 1, 5; 5, 3, 2)\}$	$\nabla_n = 5\nabla_{n-1} + 3\nabla_{n-2} + 2\nabla_{n-3}$
Reverse Lucas 3-primes	$\{\Lambda_n\} = \{T_n(3, 5, 31; 5, 3, 2)\}$	$\Lambda_n = 5\Lambda_{n-1} + 3\Lambda_{n-2} + 2\Lambda_{n-3}$
Reverse M. 3-primes	$\{\phi_n\} = \{T_n(0, 1, 4; 5, 3, 2)\}$	$\phi_n = 5\phi_{n-1} + 3\phi_{n-2} + 2\phi_{n-3}$

*M.:Modified, A.:Adjusted, T.:Third order

TABLE 2. A brief classification for generalized Tribonacci numbers.

Name	$\{T_n\} = \{T_n(T_0, T_1, T_2; r, s, t)\}$	Recurrence Relation
G. Tribonacci (usual)	$\{\mathcal{A}_n\} = \{T_n(T_0, T_1, T_2; 1, 1, 1)\}$	$\mathcal{A}_n = \mathcal{A}_{n-1} + \mathcal{A}_{n-2} + \mathcal{A}_{n-3}$
G. Padovan	$\{\mathcal{G}_n\} = \{T_n(T_0, T_1, T_2; 0, 1, 1)\}$	$\mathcal{G}_n = \mathcal{G}_{n-2} + \mathcal{G}_{n-3}$
G. Pell–Padovan	$\{\mathcal{M}_n\} = \{T_n(T_0, T_1, T_2; 0, 2, 1)\}$	$\mathcal{M}_n = 2\mathcal{M}_{n-2} + \mathcal{M}_{n-3}$
G. T. Pell	$\{\mathcal{S}_n\} = \{T_n(T_0, T_1, T_2; 2, 1, 1)\}$	$\mathcal{S}_n = 2\mathcal{S}_{n-1} + \mathcal{S}_{n-2} + \mathcal{S}_{n-3}$
G. T. Jacobsthal	$\{\mathcal{X}_n\} = \{T_n(T_0, T_1, T_2; 1, 1, 2)\}$	$\mathcal{X}_n = \mathcal{X}_{n-1} + \mathcal{X}_{n-2} + 2\mathcal{X}_{n-3}$
G. Jacobsthal–Padovan	$\{\boldsymbol{\chi}_n\} = \{T_n(T_0, T_1, T_2; 0, 1, 2)\}$	$\boldsymbol{\chi}_n = \boldsymbol{\chi}_{n-2} + 2\boldsymbol{\chi}_{n-3}$
G. Narayana	$\{\boldsymbol{\vartheta}_n\} = \{T_n(T_0, T_1, T_2; 1, 0, 1)\}$	$\boldsymbol{\vartheta}_n = \boldsymbol{\vartheta}_{n-1} + \boldsymbol{\vartheta}_{n-3}$
G. 3-primes	$\{\boldsymbol{\kappa}_n\} = \{T_n(T_0, T_1, T_2; 2, 3, 5)\}$	$\boldsymbol{\kappa}_n = 2\boldsymbol{\kappa}_{n-1} + 3\boldsymbol{\kappa}_{n-1} + 5\boldsymbol{\kappa}_{n-3}$
G. Reverse 3-primes	$\{\boldsymbol{\nabla}_n\} = \{T_n(T_0, T_1, T_2; 5, 3, 2)\}$	$\boldsymbol{\nabla}_n = 5\boldsymbol{\nabla}_{n-1} + 3\boldsymbol{\nabla}_{n-1} + 2\boldsymbol{\nabla}_{n-3}$

*G.:Generalized, T.:Third Order

3. The generalized Tribonacci dual quaternions ($\mathcal{GTDQ}$)

In this original section, $\mathcal{GTDQ}$ are defined and some particular properties of them are discussed.

Definition 1. Let T_n be nth generalized Tribonacci number (see (2)). Then the nth $\mathcal{GTDQ}$ is denoted by $\mathscr{T}_n$ and identified as

$$\mathscr{T}_n = T_n + T_{n+1}i + T_{n+2}j + T_{n+3}k, \quad \forall n \geq 0 \tag{7}$$

with the initial values

$$\begin{cases} \mathscr{T}_0 = & a + bi + cj + (rc + sb + ta)k, \\ \mathscr{T}_1 = & b + ci + (rc + sb + ta)j + \left[\left(r^2 + s\right)c + (rs + t)\,b + rta\right]k, \\ \mathscr{T}_2 = & c + (rc + sb + ta)\,i + \left[\left(r^2 + s\right)c + (rs + t)\,b + rta\right]j \\ & + \left[\left(r^3 + 2rs + t\right)c + \left(r^2 s + s^2 + rt\right)b + \left(r^2 t + st\right)a\right]k. \end{cases}$$

Here i, j, k satisfy the rules in (1).

Theorem 1 (Recurrence Relation)**.** *Let $\mathscr{T}_n$ be the nth $\mathcal{GTDQ}$. Then, the following recurrence relation holds:*

$$\mathscr{T}_n = r\mathscr{T}_{n-1} + s\mathscr{T}_{n-2} + t\mathscr{T}_{n-3}, \quad \forall n \geq 3. \tag{8}$$

Proof. By equations (2) and (7), the proof is completed as

$$\begin{aligned} r\mathscr{T}_{n-1} + s\mathscr{T}_{n-2} + t\mathscr{T}_{n-3} = \; & rT_{n-1} + sT_{n-2} + tT_{n-3} + (rT_n + sT_{n-1} + tT_{n-2})i \\ & + (rT_{n+1} + sT_n + tT_{n-1})j + (rT_{n+2} + sT_{n+1} + tT_n)k \\ = \; & T_n + T_{n+1}i + T_{n+2}j + T_{n+3}k \\ = \; & \mathscr{T}_n. \end{aligned}$$

□

Similarly, $\mathcal{GTDQ}$ can be also extended to negative subscripts by the equation

$$\mathscr{T}_{-n} = T_{-n} + T_{-(n-1)}i + T_{-(n-2)}j + T_{-(n-3)}k, \quad \forall n \in \mathbb{Z}^+. \tag{9}$$

Hence the recurrence relation for them can be obtained as:

$$\mathscr{T}_{-n} = -\tfrac{s}{t}\mathscr{T}_{-(n-1)} - \tfrac{r}{t}\mathscr{T}_{-(n-2)} + \tfrac{1}{t}\mathscr{T}_{-(n-3)}, \quad \forall n \in \mathbb{Z}^+, \tag{10}$$

where $t \neq 0$. Thus, some values of the $\mathcal{GTDQ}$ with negative subscripts can be calculated as:

$$\begin{cases} \mathscr{T}_{-1} = & \frac{c}{t} - \frac{r}{t}b - \frac{s}{t}a + ai + bj + ck, \\ \mathscr{T}_{-2} = & \frac{b}{t} - \frac{r}{t}a - \frac{s}{t}\left(\frac{c}{t} - \frac{s}{t}a - \frac{r}{t}b\right) + \left(\frac{c}{t} - \frac{r}{t}b - \frac{s}{t}a\right)i + aj + bk, \\ \mathscr{T}_{-3} = & \frac{a}{t} - \frac{r}{t}\left(\frac{c}{t} - \frac{r}{t}b - \frac{s}{t}a\right) - \frac{s}{t}\left[\frac{b}{t} - \frac{r}{t}a - \frac{s}{t}\left(\frac{c}{t} - \frac{s}{t}a - \frac{r}{t}b\right)\right] \\ & + \left[\frac{b}{t} - \frac{r}{t}a - \frac{s}{t}\left(\frac{c}{t} - \frac{s}{t}a - \frac{r}{t}b\right)\right]i + \left(\frac{c}{t} - \frac{s}{t}a - \frac{r}{t}b\right)j + ak. \end{cases}$$

Note 1. It should be noted that equations (7) and (8) are satisfied for every $n \in \mathbb{Z}$.

Algorithm 1 Finding the nth term of the $\mathcal{GTDQ}$

1: Begin
2: Input $\mathscr{T}_0, \mathscr{T}_1$ and $\mathscr{T}_2$
3: Compose $\mathscr{T}_n$ with respect to (8) for every $n \geq 3$
4: Count up $\mathscr{T}_n$
5: Output $\mathscr{T}_n = T_n + T_{n+1}i + T_{n+2}j + T_{n+3}k$
6: Final

Algorithm 2 Finding the $-n$th term of the $\mathcal{GTDQ}$

1: Begin
2: Input $\mathscr{T}_{-1}, \mathscr{T}_{-2}$ and $\mathscr{T}_{-3}$
3: Form $\mathscr{T}_n$ according to (10)
4: Calculate $\mathscr{T}_{-n}$
5: Output $\mathscr{T}_{-n} = T_{-n} + T_{-n+1}i + T_{-n+2}j + T_{-n+3}k$
6: Final

In this step, using numerical Algorithm 1 and Algorithm 2, one can calculate nth and $-nth$ terms of $\mathcal{GTDQ}$ with respect to the recurrence relations (8) and (10).

After this part, since putting $n \to -n$ gives straightforward calculations (see Note 1), it is not necessary to write the following definitions and recurrence relations for negative subscripts $\mathcal{GTDQ}$ for the sake of brevity.

Now, let us discuss some fundamental algebraic properties of $\mathscr{T}_n$. Let $\mathscr{T}_n = T_n + T_{n+1}i + T_{n+2}j + T_{n+3}k$ and $\mathscr{T}_m = T_m + T_{m+1}i + T_{m+2}j + T_{m+3}k$ be the nth and mth $\mathcal{GTDQ}$ for every $n, m \in \mathbb{Z}$, respectively. Then

- $\mathscr{T}_n = \mathscr{T}_m \Leftrightarrow T_n = T_m,\ T_{n+1} = T_{m+1},\ T_{n+2} = T_{m+2},\ T_{n+3} = T_{m+3},$
- $\mathscr{T}_n \pm \mathscr{T}_m = (T_n \pm T_m) + (T_{n+1} \pm T_{m+1})i + (T_{n+2} \pm T_{m+2})j$
  $+ (T_{n+3} \pm T_{m+3})k,$
- $\lambda\mathscr{T}_n = \lambda T_n + \lambda T_{n+1}i + \lambda T_{n+2}j + \lambda T_{n+3}k, \quad \lambda \in \mathbb{R},$
- $\mathscr{T}_n\mathscr{T}_m = T_nT_m + (T_nT_{m+1} + T_{n+1}T_m)i + (T_nT_{m+2} + T_{n+2}T_m)j$
  $+ (T_nT_{m+3}T_{n+3}T_m)k,$
- $\overline{\mathscr{T}}_n = T_n - T_{n+1}i - T_{n+2}j - T_{n+3}k,$

where $\overline{\mathscr{T}}_n$ denotes the conjugate of $\mathscr{T}_n$. It should be noted that (1) is taken into account for multiplication. Additionally, the scalar part of $\mathscr{T}_n$ is $S_{\mathscr{T}_n} = T_n$. The vector part of $\mathscr{T}_n$ is $\overrightarrow{V}_{\mathscr{T}_n} = T_{n+1}i + T_{n+2}j + T_{n+3}k$. This implies that $S_{\mathscr{T}_n \pm \mathscr{T}_m} = T_n \pm T_m = S_{\mathscr{T}_n} \pm S_{\mathscr{T}_m}$ and $\overrightarrow{V}_{\mathscr{T}_n \pm \mathscr{T}_m} = \overrightarrow{V}_{\mathscr{T}_n} \pm \overrightarrow{V}_{\mathscr{T}_m}$.

In light of (7), recurrence relation (10), Table 1 and Table 2, we present some special subfamilies of the $\mathcal{GTDQ}$ in Table 3. This table can be classified particularly by bearing in mind Table 1 regarding recurrence relations and the initial values. For shortness, a particular element of them can be examined in Table 4. Moreover, in Table 5 and Table 6, we give some initial values of the special cases of the $\mathcal{GTDQ}$ and negative subscripted $\mathcal{GTDQ}$, respectively.

TABLE 3. Some special subfamilies of the $\mathcal{GTDQ}$.

Name	Definition	Recurrence Relation
G. Tribonacci usual D.Q.	$\widehat{\mathcal{A}}_n = \mathcal{A}_n + \mathcal{A}_{n+1}i + \mathcal{A}_{n+2}j + \mathcal{A}_{n+3}k$	$\widehat{\mathcal{A}}_n = \widehat{\mathcal{A}}_{n-1} + \widehat{\mathcal{A}}_{n-2} + \widehat{\mathcal{A}}_{n-3}$
G. Padovan D.Q.	$\widehat{\mathcal{G}}_n = \mathcal{G}_n + \mathcal{G}_{n+1}i + \mathcal{G}_{n+2}j + \mathcal{G}_{n+3}k$	$\widehat{\mathcal{G}}_n = \widehat{\mathcal{G}}_{n-2} + \widehat{\mathcal{G}}_{n-3}$
G. Pell–Padovan D.Q.	$\widehat{\mathcal{M}}_n = \mathcal{M}_n + \mathcal{M}_{n+1}i + \mathcal{M}_{n+2}j + \mathcal{M}_{n+3}k$	$\widehat{\mathcal{M}}_n = 2\widehat{\mathcal{M}}_{n-2} + \widehat{\mathcal{M}}_{n-3}$
G. T. Pell D.Q.	$\widehat{\mathcal{S}}_n = \mathcal{S}_n + \mathcal{S}_{n+1}i + \mathcal{S}_{n+2}j + \mathcal{S}_{n+3}k$	$\widehat{\mathcal{S}}_n = 2\widehat{\mathcal{S}}_{n-1} + \widehat{\mathcal{S}}_{n-2} + \widehat{\mathcal{S}}_{n-3}$
G. T. Jacobsthal D.Q.	$\widehat{\mathcal{X}}_n = \mathcal{X}_n + \mathcal{X}_{n+1}i + \mathcal{X}_{n+2}j + \mathcal{X}_{n+3}k$	$\widehat{\mathcal{X}}_n = \widehat{\mathcal{X}}_{n-1} + \widehat{\mathcal{X}}_{n-2} + 2\widehat{\mathcal{X}}_{n-3}$
G. Jacobsthal–Pad. D.Q.	$\widehat{\boldsymbol{\chi}}_n = \boldsymbol{\chi}_n + \boldsymbol{\chi}_{n+1}i + \boldsymbol{\chi}_{n+2}j + \boldsymbol{\chi}_{n+3}k$	$\widehat{\boldsymbol{\chi}}_n = \widehat{\boldsymbol{\chi}}_{n-2} + 2\widehat{\boldsymbol{\chi}}_{n-3}$
G. Narayana D.Q.	$\widehat{\boldsymbol{\vartheta}}_n = \boldsymbol{\vartheta}_n + \boldsymbol{\vartheta}_{n+1}i + \boldsymbol{\vartheta}_{n+2}j + \boldsymbol{\vartheta}_{n+3}k$	$\widehat{\boldsymbol{\vartheta}}_n = \widehat{\boldsymbol{\vartheta}}_{n-1} + \widehat{\boldsymbol{\vartheta}}_{n-3}$
G. 3-primes D.Q.	$\widehat{\boldsymbol{\kappa}}_n = \boldsymbol{\kappa}_n + \boldsymbol{\kappa}_{n+1}i + \boldsymbol{\kappa}_{n+2}j + \boldsymbol{\kappa}_{n+3}k$	$\widehat{\boldsymbol{\kappa}}_n = 2\widehat{\boldsymbol{\kappa}}_{n-1} + 3\widehat{\boldsymbol{\kappa}}_{n-2} + 5\widehat{\boldsymbol{\kappa}}_{n-3}$
G. Reverse 3-prim. D.Q.	$\widehat{\boldsymbol{\nabla}}_n = \boldsymbol{\nabla}_n + \boldsymbol{\nabla}_{n+1}i + \boldsymbol{\nabla}_{n+2}j + \boldsymbol{\nabla}_{n+3}k$	$\widehat{\boldsymbol{\nabla}}_n = 5\widehat{\boldsymbol{\nabla}}_{n-1} + 3\widehat{\boldsymbol{\nabla}}_{n-2} + 2\widehat{\boldsymbol{\nabla}}_{n-3}$

*G.:Generalized, T.:Third Order, D.Q.: Dual Quaternion

TABLE 4. Particular elements of subfamilies given in Table 3.

Name	Definition	Recurrence Relation
Tribonacci-Perrin D.Q.	$\mathscr{C}_n = C_n + C_{n+1}i + C_{n+2}j + C_{n+3}k$	$\mathscr{C}_n = \mathscr{C}_{n-1} + \mathscr{C}_{n-2} + \mathscr{C}_{n-3}$
Van der Laan D.Q.	$\mathscr{I}_n = I_n + I_{n+1}i + I_{n+2}j + I_{n+3}k$	$\mathscr{I}_n = \mathscr{I}_{n-2} + \mathscr{I}_{n-3}$
T. Lucas–Pell D.Q.	$\mathscr{P}_n = P_n + P_{n+1}i + P_{n+2}j + P_{n+3}k$	$\mathscr{P}_n = 2\mathscr{P}_{n-2} + \mathscr{P}_{n-3}$
T. Pell–Lucas D.Q.	$\mathscr{U}_n = U_n + U_{n+1}i + U_{n+2}j + U_{n+3}k$	$\mathscr{U}_n = 2\mathscr{U}_{n-1} + \mathscr{U}_{n-2} + \mathscr{U}_{n-3}$
T. Jacobsthal–Lucas D.Q.	$\mathscr{Y}_n = Y_n + Y_{n+1}i + Y_{n+2}j + Y_{n+3}k$	$\mathscr{Y}_n = \mathscr{Y}_{n-1} + \mathscr{Y}_{n-2} + 2\mathscr{Y}_{n-3}$
Jacobsthal–Perrin D.Q.	$\widetilde{\Delta}_n = \Delta_n + \Delta_{n+1}i + \Delta_{n+2}j + \Delta_{n+3}k$	$\widetilde{\Delta}_n = \widetilde{\Delta}_{n-2} + 2\widetilde{\Delta}_{n-3}$
Narayana–Lucas D.Q.	$\widetilde{\tau}_n = \tau_n + \tau_{n+1}i + \tau_{n+2}j + \tau_{n+3}k$	$\widetilde{\tau}_n = \widetilde{\tau}_{n-1} + \widetilde{\tau}_{n-3}$
Modified 3-primes D.Q.	$\widetilde{\gamma}_n = \gamma_n + \gamma_{n+1}i + \gamma_{n+2}j + \gamma_{n+3}k$	$\widetilde{\gamma}_n = 2\widetilde{\gamma}_{n-1} + 3\widetilde{\gamma}_{n-2} + 5\widetilde{\gamma}_{n-3}$
Reverse Lucas 3-prim. D.Q.	$\widetilde{\Lambda}_n = \Lambda_n + \Lambda_{n+1}i + \Lambda_{n+2}j + \Lambda_{n+3}k$	$\widetilde{\Lambda}_n = 5\widetilde{\Lambda}_{n-1} + 3\widetilde{\Lambda}_{n-2} + 2\widetilde{\Lambda}_{n-3}$

*T.:Third Order, D.Q.: Dual Quaternion

TABLE 5. Initial values of particular elements.

For	$n=0$	$n=1$	$n=2$
$\mathscr{C}_n$	$3+2j+5k$	$2i+5j+7k$	$2+5i+7j+14k$
$\mathscr{I}_n$	$1+j+k$	$i+j+k$	$1+i+j+2k$
$\mathscr{P}_n$	$3+4j+3k$	$4i+3j+8k$	$4+3i+8j+10k$
$\mathscr{U}_n$	$3+2i+6j+17k$	$2+6i+17j+42k$	$6+17i+42j+107k$
$\mathscr{Y}_n$	$2+i+5j+10k$	$1+5i+10j+17k$	$5+10i+17j+37k$
$\widetilde{\Delta}_n$	$3+2j+6k$	$2i+6j+2k$	$2+6i+2j+10k$
$\widetilde{\tau}_n$	$3+i+j+4k$	$1+i+4j+5k$	$1+4i+5j+6k$
$\widetilde{\gamma}_n$	$i+j+5k$	$1+i+5j+18k$	$1+5i+18j+56k$
$\widetilde{\Lambda}_n$	$3+5i+31j+176k$	$5+31i+176j+983k$	$31+176i+983j+5505k$

Table 6. Some values of particular elements with negative subscripts.

For	$n=-1$	$n=-2$	$n=-3$
$\mathscr{C}_{-n}$	$-1+3i+2k$	$-2-i+3j$	$6-2i-j+3k$
$\mathscr{I}_{-n}$	$i+k$	j	$1+k$
$\mathscr{P}_{-n}$	$-2+3i+4k$	$4-2i+3j$	$-5+4i-2j+3k$
$\mathscr{U}_{-n}$	$-1+3i+2j+6k$	$-3-i+3j+2k$	$8-3i-j+3k$
$\mathscr{Y}_{-n}$	$1+2i+j+5k$	$-1+i+2j+k$	$1-i+j+2k$
$\widetilde{\Delta}_{-n}$	$-\frac{1}{2}+3i+2k$	$\frac{1}{4}-\frac{1}{2}i+3j$	$\frac{11}{8}+\frac{1}{4}i-\frac{1}{2}j+3k$
$\widetilde{\tau}_{-n}$	$3i+j+k$	$-2+3j+k$	$3-2i+3k$
$\widetilde{\gamma}_{-n}$	$-\frac{1}{5}+j+k$	$\frac{8}{25}-\frac{1}{5}i+k$	$-\frac{14}{125}+\frac{8}{25}i-\frac{1}{5}j$
$\widetilde{\Lambda}_{-n}$	$-\frac{3}{2}+3i+5j+31k$	$-\frac{11}{4}-\frac{3}{2}i+3j+5k$	$\frac{75}{8}-\frac{11}{4}i-\frac{3}{2}j+3k$

Now, we obtain some new features about $\mathcal{GTDQ}$ in the following theorems.

Theorem 2. *For every $n \in \mathbb{Z}$, the following properties hold:*

(i) $\mathscr{T}_n - \mathscr{T}_{n+1}i - \mathscr{T}_{n+2}j - \mathscr{T}_{n+3}k = T_n$,

(ii) $\mathscr{T}_n + \mathscr{T}_{n+1}i + \mathscr{T}_{n+2}j + \mathscr{T}_{n+3}k = 2\mathscr{T}_n - T_n$,

(iii) $\mathscr{T}_n \overline{\mathscr{T}}_n = T_n^2$,

(iv) $\mathscr{T}_n + \overline{\mathscr{T}_n} = 2T_n$,

(v) $\mathscr{T}_n - \overline{\mathscr{T}_n} = 2\mathscr{T}_n - 2T_n$,

(vi) $\mathscr{T}_n^2 = T_n^2 + 2T_nT_{n+1}i + 2T_nT_{n+2}j + 2T_nT_{n+3}k = 2T_n\mathscr{T}_n - T_n^2$,

(vii) $\overline{\mathscr{T}}_n^{\,2} = T_n^2 - 2T_nT_{n+1}i - 2T_nT_{n+2}j - 2T_nT_{n+3}k = 3T_n^2 - 2T_n\mathscr{T}_n$.

Proof. (ii) Considering equations (1) and (7), we obtain

$$\begin{aligned} \mathscr{T}_n + \mathscr{T}_{n+1}i + \mathscr{T}_{n+2}j + \mathscr{T}_{n+3}k = \;& T_n + T_{n+1}i + T_{n+2}j + T_{n+3}k \\ & +(T_{n+1} + T_{n+2}i + T_{n+3}j + T_{n+4}k)i \\ & +(T_{n+2} + T_{n+3}i + T_{n+4}j + T_{n+5}k)j \\ & +(T_{n+3} + T_{n+4}i + T_{n+5}j + T_{n+6}k)k \\ =\;& 2\mathscr{T}_n - T_n. \end{aligned}$$

(vi) From the equations (1), (7) and multiplication of two $\mathcal{GTDQ}$, we get

$$\begin{aligned} \mathscr{T}_n^2 &= (T_n + T_{n+1}i + T_{n+2}j + T_{n+3}k)(T_n + T_{n+1}i + T_{n+2}j + T_{n+3}k) \\ &= T_n^2 + 2T_nT_{n+1}i + 2T_nT_{n+2}j + 2T_nT_{n+3}k \\ &= 2T_n\mathscr{T}_n - T_n^2. \end{aligned}$$

(vii) From (1), conjugate of $\mathscr{T}_n$ and multiplication of two $\mathcal{GTDQ}$, we get

$$\begin{aligned} \overline{\mathscr{T}}_n^2 &= (T_n - T_{n+1}i - T_{n+2}j - T_{n+3}k)(T_n - T_{n+1}i - T_{n+2}j - T_{n+3}k) \\ &= T_n^2 - 2T_nT_{n+1}i - 2T_nT_{n+2}j - 2T_nT_{n+3}k \\ &= 3T_n^2 - 2T_n\mathscr{T}_n. \end{aligned}$$

The other parts are clear from the equations (1), (7) and conjugate of $\mathscr{T}_n$. □

Theorem 3. *For every $n \in \mathbb{Z}$, the following properties hold:*

(i) $\mathscr{T}_n\mathscr{T}_m - \overline{\mathscr{T}}_n\overline{\mathscr{T}}_m = 2(T_n\mathscr{T}_m - T_m\mathscr{T}_n - 2T_nT_m)$,
(ii) $\mathscr{T}_n\mathscr{T}_m + \overline{\mathscr{T}}_n\overline{\mathscr{T}}_m = 2T_nT_m$,
(iii) $\mathscr{T}_n\overline{\mathscr{T}}_m - \overline{\mathscr{T}}_n\mathscr{T}_m = 2\,(T_m\mathscr{T}_n - T_n\mathscr{T}_m)$,
(iv) $\mathscr{T}_n\overline{\mathscr{T}}_m + \overline{\mathscr{T}}_n\mathscr{T}_m = 2T_nT_m$.

Proof. Considering the equations (1), (7), conjugation and multiplication properties for $\mathcal{GTDQ}$, the proofs are completed:

(ii)
$$\begin{aligned}&\mathscr{T}_n\mathscr{T}_m + \overline{\mathscr{T}}_n\overline{\mathscr{T}}_m\\ &= (T_n + T_{n+1}i + T_{n+2}j + T_{n+3}k)(T_m + T_{m+1}i + T_{m+2}j + T_{m+3}k)\\ &\quad +(T_n - T_{n+1}i - T_{n+2}j - T_{n+3}k)(T_m - T_{m+1}i - T_{m+2}j - T_{m+3}k)\\ &= 2T_nT_m.\end{aligned}$$

(iii)
$$\begin{aligned}&\mathscr{T}_n\overline{\mathscr{T}}_m - \overline{\mathscr{T}}_n\mathscr{T}_m\\ &= (T_n + T_{n+1}i + T_{n+2}j + T_{n+3}k)(T_m - T_{m+1}i - T_{m+2}j - T_{m+3}k)\\ &\quad -(T_n - T_{n+1}i - T_{n+2}j - T_{n+3}k)(T_m + T_{m+1}i + T_{m+2}j + T_{m+3}k)\\ &= 2(T_m\mathscr{T}_n - T_n\mathscr{T}_m).\end{aligned}$$

□

Theorem 4. *For every $n \in \mathbb{Z}$, the Binet formula for $\mathcal{GTDQ}$ is*

$$\mathscr{T}_n = \frac{\widetilde{P}x_1^n\widetilde{x}_1}{(x_1-x_2)(x_1-x_3)} + \frac{\widetilde{R}x_2^n\widetilde{x}_2}{(x_2-x_1)(x_2-x_3)} + \frac{\widetilde{S}x_3^n\widetilde{x}_3}{(x_3-x_1)(x_3-x_2)},$$

where $\widetilde{x}_1 = 1+x_1i+x_1^2j+x_1^3k$, $\widetilde{x}_2 = 1+x_2i+x_2^2j+x_2^3k$, $\widetilde{x}_3 = 1+x_3i+x_3^2j+x_3^3k$ (*see* $\widetilde{P}, \widetilde{R}, \widetilde{S}$ *in* (6)).

Proof. From the equations (5) and (7) we obtain

$$\begin{aligned}\mathscr{T}_n =\ & T_n + T_{n+1}i + T_{n+2}j + T_{n+3}k\\ =\ & \frac{\widetilde{P}x_1^n}{(x_1-x_2)(x_1-x_3)} + \frac{\widetilde{R}x_2^n}{(x_2-x_1)(x_2-x_3)} + \frac{\widetilde{S}x_3^n}{(x_3-x_1)(x_3-x_2)}\\ & + \left(\frac{\widetilde{P}x_1^{n+1}}{(x_1-x_2)(x_1-x_3)} + \frac{\widetilde{R}x_2^{n+1}}{(x_2-x_1)(x_2-x_3)} + \frac{\widetilde{S}x_3^{n+1}}{(x_3-x_1)(x_3-x_2)}\right)i\\ & + \left(\frac{\widetilde{P}x_1^{n+2}}{(x_1-x_2)(x_1-x_3)} + \frac{\widetilde{R}x_2^{n+2}}{(x_2-x_1)(x_2-x_3)} + \frac{\widetilde{S}x_3^{n+2}}{(x_3-x_1)(x_3-x_2)}\right)j\\ & + \left(\frac{\widetilde{P}x_1^{n+3}}{(x_1-x_2)(x_1-x_3)} + \frac{\widetilde{R}x_2^{n+3}}{(x_2-x_1)(x_2-x_3)} + \frac{\widetilde{S}x_3^{n+3}}{(x_3-x_1)(x_3-x_2)}\right)k\\ =\ & \frac{\widetilde{P}x_1^n\widetilde{x}_1}{(x_1-x_2)(x_1-x_3)} + \frac{\widetilde{R}x_2^n\widetilde{x}_2}{(x_2-x_1)(x_2-x_3)} + \frac{\widetilde{S}x_3^n\widetilde{x}_3}{(x_3-x_1)(x_3-x_2)}.\end{aligned}$$

□

Theorem 5. *The generating functions for non-negative and negative subscripted $\mathcal{GTDQ}$ are given as*

$$\sum_{n=0}^{\infty}\mathscr{T}_nx^n = \frac{\mathscr{T}_0 + (\mathscr{T}_1 - r\mathscr{T}_0)x + (\mathscr{T}_2 - r\mathscr{T}_1 - s\mathscr{T}_0)x^2}{1 - rx - sx^2 - tx^3}, \tag{11}$$

$$\sum_{n=0}^{\infty}\mathscr{T}_{-n}x^n = \frac{\mathscr{T}_0 + (\mathscr{T}_{-1} + \frac{s}{t}\mathscr{T}_0)x + (\mathscr{T}_{-2} + \frac{s}{t}\mathscr{T}_{-1} + \frac{r}{t}\mathscr{T}_0)x^2}{1 - \frac{x^3}{t} + \frac{s}{t}x + \frac{r}{t}x^2}. \tag{12}$$

Proof. Let the function $\sum_{n=0}^{\infty}\mathscr{T}_n x^n=\mathscr{T}_0+\mathscr{T}_1x+\mathscr{T}_2x^2+\ldots+\mathscr{T}_nx^n+\ldots$ be a generating function of $\mathscr{T}_n$. Multiplying both sides of this equation with rx, sx^2, tx^3, and then applying (8) gives (11) as

$$(1-rx-sx^2-tx^3)\sum_{n=0}^{\infty}\mathscr{T}_n x^n=\mathscr{T}_0+(\mathscr{T}_1-r\mathscr{T}_0)x+(\mathscr{T}_2-r\mathscr{T}_1-s\mathscr{T}_0)x^2.$$

With a similar idea, let $\sum_{n=0}^{\infty}\mathscr{T}_{-n}x^n=\mathscr{T}_0+\mathscr{T}_{-1}x+\mathscr{T}_{-2}x^2+\ldots+\mathscr{T}_{-n}x^n+\ldots$ be a generating function of $\mathcal{GTDQ}$ with negative subscripts. Then considering (10), we can write

$$\begin{aligned}\sum_{n=0}^{\infty}\mathscr{T}_{-n}x^n &= \mathscr{T}_0+\mathscr{T}_{-1}x+\mathscr{T}_{-2}x^2+\sum_{n=3}^{\infty}\mathscr{T}_{-n}x^n\\
&= \mathscr{T}_0+\mathscr{T}_{-1}x+\mathscr{T}_{-2}x^2+\sum_{n=3}^{\infty}\left(-\tfrac{s}{t}\mathscr{T}_{-(n-1)}-\tfrac{r}{t}\mathscr{T}_{-(n-2)}+\tfrac{1}{t}\mathscr{T}_{-(n-3)}\right)x^n\\
&= \mathscr{T}_0+\mathscr{T}_{-1}x+\mathscr{T}_{-2}x^2+\tfrac{1}{t}\sum_{n=0}^{\infty}\mathscr{T}_{-n}x^{n+3}-\tfrac{s}{t}\sum_{n=2}^{\infty}\mathscr{T}_{-n}x^{n+1}\\
&\quad -\tfrac{r}{t}\sum_{n=1}^{\infty}\mathscr{T}_{-n}x^{n+2}\\
&= \mathscr{T}_0+\mathscr{T}_{-1}x+\mathscr{T}_{-2}x^2+\tfrac{1}{t}\sum_{n=0}^{\infty}\mathscr{T}_{-n}x^{n+3}\\
&\quad -\tfrac{s}{t}\left(\sum_{n=0}^{\infty}\mathscr{T}_{-n}x^{n+1}-\mathscr{T}_{-1}x^2-\mathscr{T}_0x\right)-\tfrac{r}{t}\left(\sum_{n=0}^{\infty}\mathscr{T}_{-n}x^{n+2}-\mathscr{T}_0x^2\right)\\
&= \mathscr{T}_0+\left(\mathscr{T}_{-1}+\tfrac{s}{t}\mathscr{T}_0\right)x+\left(\mathscr{T}_{-2}+\tfrac{s}{t}\mathscr{T}_{-1}+\tfrac{r}{t}\mathscr{T}_0\right)x^2+\tfrac{1}{t}x^3\sum_{n=0}^{\infty}\mathscr{T}_{-n}x^n\\
&\quad -\tfrac{s}{t}x\sum_{n=0}^{\infty}\mathscr{T}_{-n}x^n-\tfrac{r}{t}x^2\sum_{n=0}^{\infty}\mathscr{T}_{-n}x^n.\end{aligned}$$

Hence (12) can be obtained by rearranging the last equation. □

Theorem 6. *The exponential generating functions of $\mathcal{GTDQ}$ with non-negative and negative subscripts are the following:*

$$\sum_{n=0}^{\infty}\mathscr{T}_n\frac{y^n}{n!}=\frac{\widetilde{P}\widetilde{x}_1e^{x_1y}}{(x_1-x_2)(x_1-x_3)}+\frac{\widetilde{R}\widetilde{x}_2e^{x_2y}}{(x_2-x_1)(x_2-x_3)}+\frac{\widetilde{S}\widetilde{x}_3e^{x_3y}}{(x_3-x_1)(x_3-x_2)},\tag{13}$$

$$\sum_{n=0}^{\infty}\mathscr{T}_{-n}\frac{y^n}{n!}=\frac{\widetilde{P}\widetilde{x}_1e^{\frac{y}{x_1}}}{(x_1-x_2)(x_1-x_3)}+\frac{\widetilde{R}\widetilde{x}_2e^{\frac{y}{x_2}}}{(x_2-x_1)(x_2-x_3)}+\frac{\widetilde{S}\widetilde{x}_3e^{\frac{y}{x_3}}}{(x_3-x_1)(x_3-x_2)}.\tag{14}$$

Proof. Considering (4), we have:

$$\begin{aligned}\sum_{n=0}^{\infty}\mathscr{T}_n\tfrac{y^n}{n!} &= \sum_{n=0}^{\infty}\left(\tfrac{\widetilde{P}x_1^n\widetilde{x}_1}{(x_1-x_2)(x_1-x_3)}+\tfrac{\widetilde{R}x_2^n\widetilde{x}_2}{(x_2-x_1)(x_2-x_3)}+\tfrac{\widetilde{S}x_3^n\widetilde{x}_3}{(x_3-x_1)(x_3-x_2)}\right)\tfrac{y^n}{n!}\\
&= \tfrac{\widetilde{P}\widetilde{x}_1}{(x_1-x_2)(x_1-x_3)}\sum_{n=0}^{\infty}\tfrac{(x_1y)^n}{n!}+\tfrac{\widetilde{R}\widetilde{x}_2}{(x_2-x_1)(x_2-x_3)}\sum_{n=0}^{\infty}\tfrac{(x_2y)^n}{n!}\\
&\quad +\tfrac{\widetilde{S}\widetilde{x}_3}{(x_3-x_1)(x_3-x_2)}\sum_{n=0}^{\infty}\tfrac{(x_3y)^n}{n!}\\
&= \tfrac{\widetilde{P}\widetilde{x}_1e^{x_1y}}{(x_1-x_2)(x_1-x_3)}+\tfrac{\widetilde{R}\widetilde{x}_2e^{x_2y}}{(x_2-x_1)(x_2-x_3)}+\tfrac{\widetilde{S}\widetilde{x}_3e^{x_3y}}{(x_3-x_1)(x_3-x_2)}.\end{aligned}$$

Similarly, by putting $n \to -n$ in (4), (14) is obtained owing to the fact that it holds for all integers. $\square$

Inspired by the study [44], we obtain the sum formulae for $\mathcal{GTDQ}$ in Theorem 7 and Theorem 9. The proof is straightforward, so we omit it.

Theorem 7. *For every $m, n \in \mathbb{N}$, the following sum formulae for $\mathcal{GTDQ}$ hold:*

$$\text{(i)}\ \sum_{n=0}^{m} \mathscr{T}_n = \frac{\begin{array}{c}\mathscr{T}_{m+3} + (1-r)\mathscr{T}_{m+2} + (1-r-s)\mathscr{T}_{m+1} - \mathscr{T}_2 \\ +(r-1)\mathscr{T}_1 + (r+s-1)\mathscr{T}_0\end{array}}{r+s+t-1},$$

$$\text{(ii)}\ \sum_{n=0}^{m} \mathscr{T}_{2n} = \frac{\begin{array}{c}(1-s)\mathscr{T}_{2m+2} + (t+rs)\mathscr{T}_{2m+1} + (t^2+rt)\mathscr{T}_{2m} \\ +(s-1)\mathscr{T}_2 + (-t-rs)\mathscr{T}_1 + (r^2-s^2+rt+2s-1)\mathscr{T}_0\end{array}}{(r+s+t-1)(r-s+t+1)},$$

$$\text{(iii)}\ \sum_{n=0}^{m} \mathscr{T}_{2n+1} = \frac{\begin{array}{c}(r+t)\mathscr{T}_{2m+2} + (s-s^2+t^2+rt)\mathscr{T}_{2m+1} + (t-st)\mathscr{T}_{2m} \\ +(-r-t)\mathscr{T}_2 + (-1+s+r^2+rt)\mathscr{T}_1 + (-t+st)\mathscr{T}_0\end{array}}{(r-s+t+1)(r+s+t-1)},$$

where the denominators are not equal to zero.

Special Case 1. *If $s = 1$, we reach the following sum formulae for special cases of part* (ii) *and part* (iii) *of Theorem* 7 *for non-zero denominators:*

$$\text{(i)}\ \sum_{n=0}^{m} \mathscr{T}_{2n} = \frac{\mathscr{T}_{2m+1} + t\mathscr{T}_{2m} - \mathscr{T}_1 + r\mathscr{T}_0}{r+t},$$

$$\text{(ii)}\ \sum_{n=0}^{m} \mathscr{T}_{2n+1} = \frac{\mathscr{T}_{2m+2} + t\mathscr{T}_{2m+1} - \mathscr{T}_2 + r\mathscr{T}_1}{r+t}.$$

Thanks to Cerda-Morales [2], we adapt Theorem 8.

Theorem 8. *For every $m, n \in \mathbb{N}$, the following sum property is satisfied for $\mathcal{GTDQ}$:*

$$\sum_{n=0}^{m} \mathscr{T}_n = \frac{\mathscr{T}_{m+2} + (1-r)\mathscr{T}_{m+1} + t\mathscr{T}_m + \eta}{\delta},$$

where

$$\begin{cases} \delta = r+s+t-1, \\ \lambda = (r+s-1)\,a + (r-1)\,b - c, \\ \eta = \lambda + (\lambda - \delta a)\,i + [\lambda - \delta\,(a+b)]\,j + [\lambda - \delta\,(a+b+c)]\,k. \end{cases}$$

Proof. By means of (7) and the study [2] (see Lemma 2.3 on page 6 in [2]), we get

$$\begin{aligned}
\sum_{n=0}^{m} \mathscr{T}_n &= \sum_{n=0}^{m} \left(T_n + T_{n+1}i + T_{n+2}j + T_{n+3}k\right) \\
&= \sum_{n=0}^{m} T_n + \sum_{n=0}^{m} T_{n+1}i + \sum_{n=0}^{m} T_{n+2}j + \sum_{n=0}^{m} T_{n+3}k \\
&= \frac{1}{\delta}\left[\begin{array}{l} T_{m+2} + (1-r)T_{m+1} + tT_m + \lambda \\ + \left[T_{m+3} + (1-r)T_{m+2} + tT_{m+1} + \lambda - \delta a\right] i \\ + \left[T_{m+4} + (1-r)T_{m+3} + tT_{m+2} + \lambda - \delta\,(a+b)\right] j \\ + \left[T_{m+5} + (1-r)T_{m+4} + tT_{m+3} + \lambda - \delta\,(a+b+c)\right] k \end{array}\right] \\
&= \frac{\mathscr{T}_{m+2} + (1-r)\,\mathscr{T}_{m+1} + t\mathscr{T}_m + \eta}{\delta}.
\end{aligned}$$

□

Theorem 9. *For every $m \in \mathbb{Z}^+$, we have the following summation formulae for $\mathcal{GTDQ}$ with negative indices:*

(i) $$\sum_{n=1}^{m} \mathscr{T}_{-n} = \frac{\begin{array}{c} -(r+s+t)\mathscr{T}_{-m-1} - (s+t)\mathscr{T}_{-m-2} - t\mathscr{T}_{-m-3} + \mathscr{T}_2 \\ +(1-r)\mathscr{T}_1 + (1-r-s)\mathscr{T}_0 \end{array}}{r+s+t-1},$$

(ii) $$\sum_{n=1}^{m} \mathscr{T}_{-2n} = \frac{\begin{array}{c} -(r+t)\mathscr{T}_{-2m+1} + (r^2+rt+s-1)\mathscr{T}_{-2m} + (st-t)\mathscr{T}_{-2m-1} \\ +(1-s)\mathscr{T}_2 + (t+rs)\mathscr{T}_1 + (1-rt-2s-r^2+s^2)\mathscr{T}_0 \end{array}}{(r+s+t-1)(r-s+t+1)},$$

(iii) $$\sum_{n=1}^{m} \mathscr{T}_{-2n+1} = \frac{\begin{array}{c} (s-1)\mathscr{T}_{-2m+1} - (t+rs)\mathscr{T}_{-2m} - (t^2+rt)\mathscr{T}_{-2m-1} \\ +(r+t)\mathscr{T}_2 + (1-r^2-rt-s)\mathscr{T}_1 + (t-st)\mathscr{T}_0 \end{array}}{(r-s+t+1)(r+s+t-1)},$$

where the denominators are not equal to zero.

Special Case 2. *If $s \neq 1$ and $r+t=0$, we attain the following sum formulae for special cases of part* (ii) *and part* (iii) *of Theorem* 9*:*

(i) $$\sum_{n=0}^{m} \mathscr{T}_{-2n} = \frac{-\mathscr{T}_{-2m} - t\mathscr{T}_{-2m-1} + \mathscr{T}_2 + t\mathscr{T}_1 + (1-s)\mathscr{T}_0}{s-1},$$

(ii) $$\sum_{n=0}^{m} \mathscr{T}_{-2n+1} = \frac{-\mathscr{T}_{-2m+1} - t\mathscr{T}_{-2m} + \mathscr{T}_1 + t\mathscr{T}_0}{s-1}.$$

In the following theorem, we will discuss some matrix formulae regarding $\mathcal{GTDQ}$ with non-negative and negative subscripts.

Theorem 10. *For every $n \in \mathbb{Z}^+$, the following matrix properties are obtained:*

(i) $$\begin{pmatrix} \mathscr{T}_{n+2} \\ \mathscr{T}_{n+1} \\ \mathscr{T}_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \mathscr{T}_2 \\ \mathscr{T}_1 \\ \mathscr{T}_0 \end{pmatrix},$$

(ii) $$\begin{pmatrix} \mathscr{T}_{-n} \\ \mathscr{T}_{-n-1} \\ \mathscr{T}_{-n-2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{t} & -\frac{r}{t} & -\frac{s}{t} \end{pmatrix}^n \begin{pmatrix} \mathscr{T}_0 \\ \mathscr{T}_{-1} \\ \mathscr{T}_{-2} \end{pmatrix}.$$

Proof. By mathematical induction, the proof is completed easily. □

Thanks to Kızılateş et al. [28], we shall obtain a new determinant property which will be used to find the nth and $-(n+1)$th terms of $\mathcal{GTDQ}$, given as the following theorem.

Theorem 11. *For every $n \geq 0$, the following equalities hold:*

$$\mathscr{T}_n = \begin{vmatrix} \mathscr{T}_0 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \mathscr{T}_1 & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \mathscr{T}_2 & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & t & s & r & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & r & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & s & r \end{vmatrix}_{(n+1)\times(n+1)}, \tag{15}$$

$$\mathscr{T}_{-(n+1)} = \begin{vmatrix} \mathscr{T}_{-1} & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \mathscr{T}_{-2} & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \mathscr{T}_{-3} & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{t} & -\frac{r}{t} & -\frac{s}{t} & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & -\frac{s}{t} & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & -\frac{r}{t} & -\frac{s}{t} \end{vmatrix}_{(n+1)\times(n+1)}. \tag{16}$$

Proof. The proof is straightforward by (8) and the study [28] (see Theorem 5 on page 5 in [28]). □

Inspired by the studies [6] and [7], we can give Theorem 12 without a proof for the sake of brevity.

Theorem 12. *Let $\mathscr{T}_n$ and $\mathscr{T}_{-n}$ be the nth and $-n$th $\mathcal{GTDQ}$ with non-negative and negative subscripts, respectively. For every $n \geq 0$, the following equalities hold for determinants with size $n+1$:*

$$\mathscr{T}_n = \begin{vmatrix} \mathscr{T}_0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ r\mathscr{T}_0 - \mathscr{T}_1 & r & \frac{1}{\mathscr{T}_0} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & r\mathscr{T}_1 - \mathscr{T}_2 & r & t & 0 & 0 & \dots & 0 & 0 \\ 0 & \mathscr{T}_0 & -\frac{s}{t} & r & t & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{1}{t} & -\frac{s}{t} & r & t & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & r & t \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & -\frac{s}{t} & r \end{vmatrix} \tag{17}$$

where $\mathscr{T}_0 \neq 0$,

$$\mathscr{T}_{-(n+1)} = \begin{vmatrix} \mathscr{T}_{-1} & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ r\mathscr{T}_{-1} - \mathscr{T}_{-2} & r & \frac{1}{\mathscr{T}_{-1}} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & r\mathscr{T}_{-2} - \mathscr{T}_{-3} & r & \frac{1}{t} & 0 & 0 & \dots & 0 & 0 \\ 0 & \mathscr{T}_{-1} & r & -\frac{s}{t} & \frac{1}{t} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & r & -\frac{s}{t} & \frac{1}{t} & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & -\frac{s}{t} & \frac{1}{t} \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & r & -\frac{s}{t} \end{vmatrix} \tag{18}$$

where $\mathscr{T}_{-1} \neq 0$.

One can see numerical Algorithm 3 and Algorithm 4 related to the determinant equation in Theorem 11 to find the nth and $-(n+1)$th terms of $\mathcal{GTDQ}$. Additionally, Algorithm 5 and Algorithm 6 are linked to Theorem 12 to find the terms of $\mathcal{GTDQ}$.

Algorithm 3 Finding the nth term of the $\mathcal{GTDQ}$

1: Begin
2: Input $\mathscr{T}_0, \mathscr{T}_1$ and $\mathscr{T}_2$
3: Form $\mathscr{T}_n$ with respect to (15)
4: Compute $\mathscr{T}_n$
5: Output $\mathscr{T}_n = T_n + T_{n+1}i + T_{n+2}j + T_{n+3}k$
6: Final

Algorithm 4 Finding the $-(n+1)$th term of the $\mathcal{GTDQ}$

1: Begin
2: Input $\mathscr{T}_{-1}, \mathscr{T}_{-2}$ and $\mathscr{T}_{-3}$
3: Form $\mathscr{T}_{-(n+1)}$ with respect to (16)
4: Compute $\mathscr{T}_{-(n+1)}$
5: Output $\mathscr{T}_{-(n+1)} = T_{-(n+1)} + T_{-(n+1)+1}i + T_{-(n+1)+2}j + T_{-(n+1)+3}k$
6: Final

Algorithm 5 Finding the nth term of the $\mathcal{GTDQ}$

1: Begin
2: Input
3: $\mathscr{T}_0, \mathscr{T}_1$ and $\mathscr{T}_2$
4: Form $\mathscr{T}_n$ according to (17)
5: Compute $\mathscr{T}_n$
6: Output $\mathscr{T}_n = T_n + T_{n+1}i + T_{n+2}j + T_{n+3}k$
7: Final

Algorithm 6 Finding the $-(n+1)$th term of the $\mathcal{GTDQ}$

1: Begin
2: Input $\mathscr{T}_{-1}, \mathscr{T}_{-2}$ and $\mathscr{T}_{-3}$
3: Form $\mathscr{T}_{-(n+1)}$ according to (18)
4: Calculate $\mathscr{T}_{-(n+1)}$
5: Output $\mathscr{T}_{-(n+1)} = T_{-(n+1)} + T_{-(n+1)+1}i + T_{-(n+1)+2}j + T_{-(n+1)+3}k$
6: Final

4. Reviewing results for special cases: Narayana and 3-primes dual quaternions

Let us examine the obtained theorems and definitions for Narayana dual quaternions and 3-primes dual quaternions in the following Corollary 1 and Corollary 2 as special cases. The other special cases can be examined in detail. One can also see the results for Padovan, Perrin and Pell–Padovan dual quaternions in the study [23].

Corollary 1. *Consider the nth Narayana dual quaternion $\widetilde{\vartheta}_n$ with the recurrence relation $\widetilde{\vartheta}_n = \widetilde{\vartheta}_{n-1} + \widetilde{\vartheta}_{n-3}$ and the initial values:*

$$\begin{cases} \widetilde{\vartheta}_0 = i+j+k, & \widetilde{\vartheta}_1 = 1+i+j+2k, & \widetilde{\vartheta}_2 = 1+i+2j+3k, \\ \widetilde{\vartheta}_{-1} = j+k, & \widetilde{\vartheta}_{-2} = 1+k, & \widetilde{\vartheta}_{-3} = i. \end{cases}$$

Then the following holds for Narayana dual quaternions.

(i) *The Binet formula:*

$$\widetilde{\vartheta}_n = \frac{x_1^{n+1}\widetilde{x}_1}{(x_1-x_2)(x_1-x_3)} + \frac{x_2^{n+1}\widetilde{x}_2}{(x_2-x_1)(x_2-x_3)} + \frac{x_3^{n+1}\widetilde{x}_3}{(x_3-x_1)(x_3-x_2)} .$$

(ii) *The generating functions:*

$$\begin{aligned} \sum_{n=0}^{\infty} \widetilde{\vartheta}_n x^n &= \frac{\widetilde{\vartheta}_0+(\widetilde{\vartheta}_1-\widetilde{\vartheta}_0)x+(\widetilde{\vartheta}_2-\widetilde{\vartheta}_1)x^2}{1-x-x^3}, \\ \sum_{n=0}^{\infty} \widetilde{\vartheta}_{-n} x^n &= \frac{\widetilde{\vartheta}_0+\widetilde{\vartheta}_{-1}x+(\widetilde{\vartheta}_{-2}+\widetilde{\vartheta}_0)x^2}{1+x^2-x^3}. \end{aligned}$$

(iii) *The exponential generating functions:*

$$\begin{aligned} \sum_{n=0}^{\infty} \widetilde{\vartheta}_n \frac{y^n}{n!} &= \frac{x_1\widetilde{x}_1 e^{x_1 y}}{(x_1-x_2)(x_1-x_3)} + \frac{x_2\widetilde{x}_2 e^{x_2 y}}{(x_2-x_1)(x_2-x_3)} + \frac{x_3\widetilde{x}_3 e^{x_3 y}}{(x_3-x_1)(x_3-x_2)}, \\ \sum_{n=0}^{\infty} \widetilde{\vartheta}_{-n} \frac{y^n}{n!} &= \frac{x_1\widetilde{x}_1 e^{\frac{y}{x_1}}}{(x_1-x_2)(x_1-x_3)} + \frac{x_2\widetilde{x}_2 e^{\frac{y}{x_2}}}{(x_2-x_1)(x_2-x_3)} + \frac{x_3\widetilde{x}_3 e^{\frac{y}{x_3}}}{(x_3-x_1)(x_3-x_2)}. \end{aligned}$$

(iv) *The summation formulae for any $m \in \mathbb{N}$:*

(a) $\sum\limits_{n=0}^{m} \widetilde{\vartheta}_n = \widetilde{\vartheta}_{m+3} - \widetilde{\vartheta}_2,$

(b) $\sum\limits_{n=0}^{m} \widetilde{\vartheta}_{2n} = \frac{1}{3}\left(\widetilde{\vartheta}_{2m+2} + \widetilde{\vartheta}_{2m+1} + 2\widetilde{\vartheta}_{2m} - \widetilde{\vartheta}_2 - \widetilde{\vartheta}_1 + \widetilde{\vartheta}_0\right),$

(c) $\sum\limits_{n=0}^{m} \widetilde{\vartheta}_{2n+1} = \frac{1}{3}\left(2\widetilde{\vartheta}_{2m+2} + 2\widetilde{\vartheta}_{2m+1} + \widetilde{\vartheta}_{2m} - 2\widetilde{\vartheta}_{2} + \widetilde{\vartheta}_{1} - \widetilde{\vartheta}_{0}\right),$

(d) $\sum\limits_{n=1}^{m} \widetilde{\vartheta}_{-n} = -2\widetilde{\vartheta}_{-m-1} - \widetilde{\vartheta}_{-m-2} - \widetilde{\vartheta}_{-n-3} + \widetilde{\vartheta}_{2},$

(e) $\sum\limits_{n=1}^{m} \widetilde{\vartheta}_{-2n} = \frac{1}{3}\left(-2\widetilde{\vartheta}_{-2m+1} + \widetilde{\vartheta}_{-2m} - \widetilde{\vartheta}_{-2m-1} + \widetilde{\vartheta}_{2} + \widetilde{\vartheta}_{1} - \widetilde{\vartheta}_{0}\right),$

(f) $\sum\limits_{n=1}^{m} \widetilde{\vartheta}_{-2n+1} = \frac{1}{3}\left(-\widetilde{\vartheta}_{-2m+1} - \widetilde{\vartheta}_{-2m} - 2\widetilde{\vartheta}_{-2m-1} + 2\widetilde{\vartheta}_{2} - \widetilde{\vartheta}_{1} + \widetilde{\vartheta}_{0}\right),$

(g) $\sum\limits_{n=0}^{m} \widetilde{\vartheta}_{n} = \widetilde{\vartheta}_{m+2} + \widetilde{\vartheta}_{m} + (-1 - i - 2j - 3k).$

(v) *The matrix equalities:*

(a) $$\begin{pmatrix} \widetilde{\vartheta}_{n+2} \\ \widetilde{\vartheta}_{n+1} \\ \widetilde{\vartheta}_{n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{n} \begin{pmatrix} \widetilde{\vartheta}_{2} \\ \widetilde{\vartheta}_{1} \\ \widetilde{\vartheta}_{0} \end{pmatrix},$$

(b) $$\begin{pmatrix} \widetilde{\vartheta}_{-n} \\ \widetilde{\vartheta}_{-n-1} \\ \widetilde{\vartheta}_{-n-2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}^{n} \begin{pmatrix} \widetilde{\vartheta}_{0} \\ \widetilde{\vartheta}_{-1} \\ \widetilde{\vartheta}_{-2} \end{pmatrix}.$$

(vi) *The determinant equalities for any $n \in \mathbb{N}$:*

(a) $$\widetilde{\vartheta}_{n} = \begin{vmatrix} \widetilde{\vartheta}_{0} & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \widetilde{\vartheta}_{1} & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \widetilde{\vartheta}_{2} & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 1 \end{vmatrix}_{(n+1)\times(n+1)}$$

(b) $$\widetilde{\vartheta}_{-(n+1)} = \begin{vmatrix} \widetilde{\vartheta}_{-1} & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \widetilde{\vartheta}_{-2} & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \widetilde{\vartheta}_{-3} & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & -1 & 0 \end{vmatrix}_{(n+1)\times(n+1)}$$

(vii) *Since $\widetilde{\vartheta}_{0} = \widetilde{\vartheta}_{-1} = 0$, we can not construct the methods given in equations (17) and (18). Hence Theorem 12 is not applicable for Narayana dual quaternions.*

Corollary 2. *Consider the nth 3-primes dual quaternion $\widetilde{\kappa}_{n}$ with the recurrence relation $\widetilde{\kappa}_{n} = 2\widetilde{\kappa}_{n-1} + 3\widetilde{\kappa}_{n-2} + 5\widetilde{\kappa}_{n-3}$ and the initial values*

$$\begin{cases} \widetilde{\kappa}_{0} = & i + 2j + 7k, & \widetilde{\kappa}_{1} = & 1 + 2i + 7j + 25k, & \widetilde{\kappa}_{2} = & 2 + 7i + 25j + 81k, \\ \widetilde{\kappa}_{-1} = & j + 2k, & \widetilde{\kappa}_{-2} = & \frac{1}{5} + k, & \widetilde{\kappa}_{-3} = & -\frac{3}{25} + \frac{1}{5}i. \end{cases}$$

Then the following holds for 3-primes dual quaternions.

(i) *The Binet formula:*

$$\widetilde{\kappa}_n = \frac{x_1^{n+1}\widetilde{x}_1}{(x_1-x_2)(x_1-x_3)} + \frac{x_2^{n+1}\widetilde{x}_2}{(x_2-x_1)(x_2-x_3)} + \frac{x_3^{n+1}\widetilde{x}_3}{(x_3-x_1)(x_3-x_2)}.$$

(ii) *The generating functions:*

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{\kappa}_n x^n &= \frac{\widetilde{\kappa}_0 + (\widetilde{\kappa}_1 - 2\widetilde{\kappa}_0)x + (\widetilde{\kappa}_2 - 2\widetilde{\kappa}_1 - 3\widetilde{\kappa}_0)x^2}{1-2x-3x^2-5x^3},\\ \sum_{n=0}^{\infty}\widetilde{\kappa}_{-n} x^n &= \frac{\widetilde{\kappa}_0 + (\widetilde{\kappa}_{-1} + \frac{3}{5}\widetilde{\kappa}_0)x + (\widetilde{\kappa}_{-2} + \frac{3}{5}\widetilde{\kappa}_{-1} + \frac{2}{5}\widetilde{\kappa}_0)x^2}{1+\frac{3}{5}x+\frac{2}{5}x^2-\frac{1}{5}x^3}.\end{aligned}$$

(iii) *The exponential generating functions:*

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{\kappa}_n\frac{y^n}{n!} &= \frac{x_1\widetilde{x}_1e^{x_1y}}{(x_1-x_2)(x_1-x_3)} + \frac{x_2\widetilde{x}_2e^{x_2y}}{(x_2-x_1)(x_2-x_3)} + \frac{x_3\widetilde{x}_3e^{x_3y}}{(x_3-x_1)(x_3-x_2)},\\ \sum_{n=0}^{\infty}\widetilde{\kappa}_{-n}\frac{y^n}{n!} &= \frac{x_1\widetilde{x}_1e^{\frac{y}{x_1}}}{(x_1-x_2)(x_1-x_3)} + \frac{x_2\widetilde{x}_2e^{\frac{y}{x_2}}}{(x_2-x_1)(x_2-x_3)} + \frac{x_3\widetilde{x}_3e^{\frac{y}{x_3}}}{(x_3-x_1)(x_3-x_2)}.\end{aligned}$$

(iv) *The summation formulae for any $m \in \mathbb{N}$:*

(a) $\sum_{n=0}^{m}\widetilde{\kappa}_n = \frac{1}{9}\left(\widetilde{\kappa}_{m+3} - \widetilde{\kappa}_{m+2} - 4\widetilde{\kappa}_{m+1} - \widetilde{\kappa}_2 + \widetilde{\kappa}_1 + 4\widetilde{\kappa}_0\right),$

(b) $\sum_{n=0}^{m}\widetilde{\kappa}_{2n} = \frac{1}{45}\left(-2\widetilde{\kappa}_{2m+2} + 11\widetilde{\kappa}_{2m+1} + 35\widetilde{\kappa}_{2m} + 2\widetilde{\kappa}_2 - 11\widetilde{\kappa}_1 + 10\widetilde{\kappa}_0\right),$

(c) $\sum_{n=0}^{m}\widetilde{\kappa}_{2n+1} = \frac{1}{45}\left(7\widetilde{\kappa}_{2m+2} + 29\widetilde{\kappa}_{2m+1} - 10\widetilde{\kappa}_{2m} - 7\widetilde{\kappa}_2 + 16\widetilde{\kappa}_1 + 10\widetilde{\kappa}_0\right),$

(d) $\sum_{n=1}^{m}\widetilde{\kappa}_{-n} = \frac{1}{9}\left(-10\widetilde{\kappa}_{-m-1} - 8\widetilde{\kappa}_{-m-2} - 5\widetilde{\kappa}_{-m-3} + \widetilde{\kappa}_2 - \widetilde{\kappa}_1 - 4\widetilde{\kappa}_0\right),$

(e) $\sum_{n=1}^{m}\widetilde{\kappa}_{-2n} = \frac{1}{45}\left(-7\widetilde{\kappa}_{-2m+1} + 16\widetilde{\kappa}_{-2m} + 10\widetilde{\kappa}_{-2m-1} - 2\widetilde{\kappa}_2 + 11\widetilde{\kappa}_1 - 10\widetilde{\kappa}_0\right),$

(f) $\sum_{n=1}^{m}\widetilde{\kappa}_{2n+1} = \frac{1}{45}\left(2\widetilde{\kappa}_{-2m+1} - 11\widetilde{\kappa}_{-2m} - 35\widetilde{\kappa}_{-2m-1} + 7\widetilde{\kappa}_2 - 16\widetilde{\kappa}_1 - 10\widetilde{\kappa}_0\right),$

(g) $\sum_{n=0}^{m}\widetilde{\kappa}_n = \dfrac{\widetilde{\kappa}_{m+2} - \widetilde{\kappa}_{m+1} + 5\widetilde{\kappa}_m + (-1-i-10j-28k)}{9}.$

(v) *The matrix equalities:*

(a) $\begin{pmatrix}\widetilde{\kappa}_{n+2}\\ \widetilde{\kappa}_{n+1}\\ \widetilde{\kappa}_n\end{pmatrix} = \begin{pmatrix}2&3&5\\1&0&0\\0&1&0\end{pmatrix}^n\begin{pmatrix}\widetilde{\kappa}_2\\ \widetilde{\kappa}_1\\ \widetilde{\kappa}_0\end{pmatrix},$

(b) $\begin{pmatrix}\widetilde{\kappa}_{-n}\\ \widetilde{\kappa}_{-n-1}\\ \widetilde{\kappa}_{-n-2}\end{pmatrix} = \begin{pmatrix}0&1&0\\0&0&1\\\frac{1}{5}&-\frac{2}{5}&-\frac{3}{5}\end{pmatrix}^n\begin{pmatrix}\widetilde{\kappa}_0\\ \widetilde{\kappa}_{-1}\\ \widetilde{\kappa}_{-2}\end{pmatrix}.$

(vi) *The determinant equalities for any $n \in \mathbb{N}$:*

$$\text{(a)}\ \widetilde{\kappa}_n = \begin{vmatrix} \widetilde{\kappa}_0 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \widetilde{\kappa}_1 & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \widetilde{\kappa}_2 & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 5 & 3 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 3 & 2 \end{vmatrix}_{(n+1)\times(n+1)}$$

$$\text{(b)}\ \widetilde{\kappa}_{-(n+1)} = \begin{vmatrix} \widetilde{\kappa}_{-1} & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \widetilde{\kappa}_{-2} & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \widetilde{\kappa}_{-3} & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{5} & -\frac{2}{5} & -\frac{3}{5} & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & -\frac{3}{5} & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & -\frac{2}{5} & -\frac{3}{5} \end{vmatrix}_{(n+1)\times(n+1)}$$

(vii) *Since* $\widetilde{\kappa}_0 = \widetilde{\kappa}_{-1} = 0$, *we cannot construct the methods given in equations* (17) *and* (18). *Hence, Theorem* 12 *is not applicable for* 3*-primes dual quaternions.*

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