ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 27, Number 2, December 2023
Available online at https://ojs.utlib.ee/index.php/ACUTM

Second cohomology group and quadratic extensions of metric Hom-Jacobi–Jordan algebras

Nejib Saadaoui

Abstract. In this paper, we introduce and study the low dimensional cohomology of metric Hom-Jacobi–Jordan algebras. We establish one-to-one correspondence between the equivalence classes of abelian quadratic extensions of a Hom-Jacobi–Jordan algebra and its second cohomology group.

Introduction

The Jacobi–Jordan algebras were recently introduced in [4] as vector spaces A over a field $\mathbb{K}$, equipped with a bilinear map $\cdot \colon A \times A \longrightarrow A$, satisfying the Jacobi identity and instead of the skew-symmetry condition valid for Lie algebras, the commutativity condition $x \cdot y = y \cdot x$, for all $x, y \in A$, is imposed. This class of algebras appears under different names in the reflecting literature (Jordan–Lie algebras in [18], mock-Lie algebras in [20], etc.). Wörz-Busekros in [19] relates these types of algebras with Bernstein algebras. One crucial remark is that Jacobi–Jordan algebras are examples of the more popular and well-referenced Jordan algebras [1, 15] introduced in order to achieve an axiomatization for the algebra of observables in quantum mechanics. In [4], the authors achieved the classification of these algebras up to dimension 6 over an algebraically closed field of characteristic different from 2 and 3.

Hom-type algebras appeared naturally when studying q-deformations of some algebras of vector fields, like Witt and Virasoro algebras. It turns out that the Jacobi identity is no longer satisfied, these new structures involving a bracket and a linear map satisfy a twisted version of the Jacobi identity

Received July 6, 2023.
2020 *Mathematics Subject Classification.* 16W10, 16S70, 17B60, 17B61, 20J06.
Key words and phrases. Metric-Hom-Jacobi–Jordan algebra, cohomology, quadratic extension, twofold extension.
https://doi.org/10.12697/ACUTM.2023.27.19
Corresponding author: Nejib Saadaoui

and define a so called Hom-Lie algebras which form a wider class, see [2, 7, 8, 12, 17].

The quadratic Lie algebras, also called metrizable or orthogonal (see [9, 10]), are intensively studied. One of the fundamental results of constructing and characterizing quadratic Lie algebras is due to Medina and Revoy (see [14]) using double extensions, while the concept of T^*-extension is due to Bordemann, see [11]. The T^*-extension concerns non-associative algebras with a nondegenerate associative symmetric bilinear form, such algebras are called metrizable algebras. In [11], the metrizable nilpotent associative algebras and metrizable solvable Lie algebras are described. A study of graded quadratic Lie algebras can be found in [5]. The Hom-Lie case for quadratic algebras is introduced and studied by S. Benayadi and A. Makhlouf in [3]. The Hom-Jacobi–Jordan case is introduced by Cyrille in [6]. In this paper, we are interested in studying the second group of cohomology of metric Hom-Jacobi–Jordan algebras and its relation with quadratic extensions.

This paper is organized as follows. In the first section, we briefly recall some facts about Hom-Jacobi–Jordan algebras and we give the isomorphism classification of 2-dimensional multiplicative Hom-Jacobi–Jordan algebras. Section 2 is devoted to giving some examples of representations of Hom-Jacobi–Jordan algebras. In section 3, we introduce metric Hom-Jacobi–Jordan algebras. In section 4, we provide the second cohomology group of a metric Hom-Jacobi–Jordan algebra with coefficients in a given representation. Section 5 deals with quadratic extensions of metric Hom-Jacobi–Jordan algebras. We show that the second cohomology group classifies quadratic extensions of a metric Hom-Jacobi–Jordan algebra.

Throughout the paper, all considered complex vector spaces are finite-dimensional.

1. Hom-Jacobi–Jordan algebras

In this section, we recall some facts about Hom-Jacobi–Jordan algebras and we provide their classifications in a 2-dimensional multiplicative setting.

Definition 1.1 ([6])**.** A *Hom-Jacobi–Jordan algebra* is a triple $(J, [\cdot,\cdot], \alpha)$, where J is a vector space equipped with a symmetric bilinear map $[\cdot,\cdot]\colon J \times J \to J$ and a linear map $\alpha\colon J \to J$ such that

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0 \tag{1}$$

for all x, y, z in J. This identity is called the *Hom-Jacobi identity.*

We recover Jacobi–Jordan algebras when the linear map α is the identity map. A Hom-Jacobi–Jordan-algebra is called *multiplicative* if α is an algebraic morphism with

$$\alpha([x, y]) = [\alpha(x), \alpha(y)] \tag{2}$$

for any $x,\ y \in J$. Two Hom-Jacobi–Jordan algebras $(J, [\cdot,\cdot], \alpha)$ and $(J', [\cdot,\cdot]', \alpha')$ are said to be *isomorphic* if there exists an algebra isomorphism $\phi\colon J \to J'$ compatible with α and α', i.e

$$\phi([x,y]) = [\phi(x), \phi(y)]' \text{ and } \phi \circ \alpha = \alpha' \circ \phi. \tag{3}$$

The center of a Hom-Jacobi–Jordan algebra $(J, [\cdot,\cdot], \alpha)$ is the subspace

$$\mathfrak{Z}(J) = \{x \in J \mid [x,y] = 0, \forall y \in J\}.$$

A subspace I of J is said to be an *ideal* if, for $x \in I$ and $y \in J$, we have $[x,y] \in I$ and $\alpha(x) \in I$.

The following lemma describes the class of complex 2-dimensional multiplicative Hom-Jacobi–Jordan algebras when the matrix of α is of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

Lemma 1.1. *Let $(J, [\cdot,\cdot], \alpha)$ be a 2-dimensional multiplicative Hom-Jacobi–Jordan algebra with ordered basis $\{u_1, u_2\}$. Take $\alpha(u_1) = au_1$ and $\alpha(u_2) = bu_2$. Then there exists a basis $\{e_1, e_2\}$ of J in which $(J, [\cdot,\cdot], \alpha)$ has one of the following forms:*

(1) $J_1^1(0, b, 0) : [e_1, e_1] = e_1$ *and* $\alpha(e_1) = 0,\ \alpha(e_2) = be_2$,
(2) $J_2^1(a, a^2, 0) : [e_1, e_1] = e_2$ *and* $\alpha(e_1) = ae_1,\ \alpha(e_2) = a^2e_2$,

where the omitted products are zero.

Proof. Let sp be the set of eigenvalues of α. We have $\alpha(u_i) = a_iu_i$, $i = 1, 2$. Thus, using (2), we take $\alpha([u_i, u_j]) = a_ia_j[u_i, u_j]$. Then $a_ia_j \in sp(\alpha)$, or $[u_i, u_j] = 0$.

If $a_1 = a_2$, we obtain $\alpha = id_J$. Then J is the classical 2-dimensional Jacobi-Jordan algebra given in [4] by $[e_1, e_1] = e_2$.

If $a_1 \neq a_2$, the set of eigenvalues of α is given by $sp(\alpha) = \{a_1, a_2\}$. The eigenspace of the eigenvalue a_1 is generated by u_1 and the eigenspace of the eigenvalue a_2 is generated by u_2. The rest of the proof can be obtained easily by solving firstly the equation (1) and then using (3). □

The following lemma describes the class of complex 2-dimensional multiplicative Hom-Jacobi–Jordan algebras, where $\alpha = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$.

Lemma 1.2. *Let $(J, [\cdot,\cdot], \alpha)$ be a 2-dimensional multiplicative Hom-Jacobi–Jordan algebra with ordered basis $\{u_1, u_2\}$. Take $\alpha(u_1) = au_1$ and $\alpha(u_2) = u_1 + au_2$. Then there exists a basis $\{e_1, e_2\}$ of J in which $(J, [\cdot,\cdot], \alpha)$ has one of the following forms:*

(1) $J_1^2(0, 0, 1) : [e_2, e_2] = e_1$ *and* $\alpha(e_1) = 0,\ \alpha(e_2) = e_1$,
(2) $J_2^2(0, 0, c) : [e_2, e_1] = [e_1, e_2] = e_1,\ [e_2, e_2] = e_1$ *and* $\alpha(e_1) = 0, \alpha(e_2) = ce_1$,
(3) $J_3^2(0, 0, 1) : [e_2, e_1] = [e_1, e_2] = e_1$ *and* $\alpha(e_1) = 0,\ \alpha(e_2) = e_1$,

(4) $J_4^2(1,1,1) : [e_2, e_2] = e_1$ *and* $\alpha(e_1) = e_1,\ \alpha(e_2) = e_1 + e_2$,
where the omitted products are zero.

Proof. The proof follows by straightforward computations similar to the proof of Lemma 1.1. □

Combining the previous lemmas we get the following theorem.

Theorem 1.3. *All the classes of* 2*-dimensional multiplicative Hom-Jacobi–Jordan algebra are given in Lemma 1.1 and Lemma 1.2 up to isomorphism.*

2. Representation of Hom-Jacobi–Jordan algebras

In this section, we give some examples of representations that we will need in the remainder of the paper.

Definition 2.1. Let J and V be two vector spaces. A k-linear map $f\colon \underbrace{J \times J \ldots \times J}_{k \text{ times}} \to V$ is said to be *symmetric* if

$$f(x_{\sigma(1)}, \cdots, x_{\sigma(k)}) = f(x_1, \cdots, x_k) \text{ for all } \sigma \in \mathfrak{S}_k,$$

where $\mathfrak{S}_k$ is the group of permutations of $\{1, \cdots, k\}$. For $k \in \mathbb{N}$, the set of symmetric k-linear maps is denoted by $S^k(J, V)$.

Definition 2.2 ([6])**.** A representation of a Hom-Jacobi–Jordan algebra $(J, [\cdot,\cdot], \alpha)$ on a vector space V with respect to $\beta \in End(V)$ is a linear map $\rho\colon J \to End(V)$ satisfying

$$\rho(\alpha(x)) \circ \beta = \beta \circ \rho(x), \tag{4}$$

$$\rho\left([x, y]\right) \circ \beta = -\rho\left(\alpha(x)\right) \rho(y) - \rho\left(\alpha(y)\right) \circ \rho(x) \tag{5}$$

for all $x,\ y \in J$. We denote such a representation by (V, ρ, β).

Definition 2.3. Let (V, ρ, β) be a representation of a Hom-Jacobi–Jordan $(J, [\cdot,\cdot], \alpha)$. The set of k*-Hom-cochains* on J with coefficients in V, denoted by $C^k_{\alpha,\beta}(J,\ V)$, is given by

$$C^k_{\alpha,\beta}(J,\ V) = \left\{ f \in S^k(J, V) \mid \beta \circ f = f \circ \alpha \right\}.$$

Definition 2.4. The 1*-coboundary operator* of a Hom-Jacobi–Jordan algebra J is the map

$$d^1 : C^1_{\alpha,\beta}(J, V) \to C^2_{\alpha,\beta}(J, V), \qquad f \mapsto d^1 f,$$

defined by

$$d^1(f)(x, y) \quad = \quad f([x, y])) - \rho(x) f(y) - \rho(y) f(x). \tag{6}$$

Definition 2.5. The 2-*coboundary operator* of a Hom-Jacobi–Jordan algebra J is the map

$$d^2 : C^2_{\alpha,\beta}(J,V) \to C^3_{\alpha,\beta}(J,V), \qquad f \mapsto d^2 f,$$

defined by

$$\begin{aligned} d^2(f)(x,y,z) =& f\left([x,y],\alpha(z)\right) + f\left([x,z],\alpha(y)\right) + f\left(\alpha(x),[y,z]\right) \\ &+ \rho\left(\alpha(x)\right) f(y,z) + \rho\left(\alpha(y)\right) f(x,z) + \rho\left(\alpha(z)\right) f(x,y). \end{aligned} \tag{7}$$

Theorem 2.1 ([16]). *We have* $d^2 \circ d^1 = 0$.

The 2-cocycles space is defined as $Z^2_{\alpha,\beta}(J,V) = \ker(d^2)$, the 2-coboundary space is defined as $B^2_{\alpha,\beta}(J,V) = Im(d^1)$ and the 2^{nd} cohomology space is the quotient $H^2_{\alpha,\beta}(J,V) = Z^2_{\alpha,\beta}(J,V)/B^2_{\alpha,\beta}(J,V)$.

Let J and V be two vector spaces and let $[\cdot,\cdot]$, $\theta\colon J^2 \to V$, $\lambda\colon J \times V \to V$ be bilinear symmetric maps. Define a bracket $[\cdot,\cdot]_M$ and a morphism α_M on $M = J \oplus V$ by

$$\begin{aligned} [x+v, y+w]_M &= [x,y] + \lambda(x,w) + \lambda(y,v) + \theta(x,y), \\ \alpha_M(x+v) &= \alpha(x) + \beta(v). \end{aligned}$$

Theorem 2.2 ([16]). *With the above notations,* $(M,[\cdot,\cdot]_M,\alpha_M)$ *is a Hom-Jacobi–Jordan algebra if and only if the following conditions hold:*

(1) $(J,[\cdot,\cdot],\alpha)$ *is a Hom-Jacobi–Jordan algebra;*
(2) *the linear map* $\rho\colon J \to End(V)$, $x \longmapsto \lambda(x,\cdot)$, *defines a representation of* J *on* V*;*
(3) θ *is a* 2*-cocycle of the Hom-Jacobi–Jordan algebra* $(J,[\cdot,\cdot],\alpha)$ *with coefficients in the representation* (V,ρ,β) *(i.e.,* $\theta \in Z^2_{\alpha,\beta}(J,V)$*).*

If, in addition, $(M,[\cdot,\cdot]_M,\alpha_M)$ *is multiplicative, then* θ *is a* 2*-Hom-cochain and the Hom-Jacobi–Jordan algebra* $(J,[\cdot,\cdot],\alpha)$ *is also multiplicative.*

Definition 2.6. Let (V,ρ,β) be a representation of a multiplicative Hom-Jacobi–Jordan algebra $(J,[\cdot,\cdot],\alpha)$ and θ be a 2-cocycle of J on V. The multiplicative Hom-Jacobi–Jordan algebra $(M,[\cdot,\cdot]_M,\alpha_M)$ is called an *abelian extension* of J by V by means of θ.

2.1. Representation on $V' = End(J,V)$.

Let $V' = End(J,V)$ be the vector space of linear maps $f\colon J \to V$. We define the linear maps $\alpha'\colon V' \to V'$ and $\rho'\colon J \to End(V')$ as follows

$$\alpha'(Z) = Z\left(\alpha(\cdot)\right), \tag{8}$$

$$\rho'(x)Z = Z\left([x,\cdot]\right). \tag{9}$$

If we compute the right-hand side of the identity (5), then we obtain

$$-\rho'\left(\alpha(x)\right)\rho'(y)Z - \rho'\left(\alpha(y)\right)\rho'(x)Z = -Z\big([y,[\alpha(x),\cdot]]\big) - Z\big([x,[\alpha(y),\cdot]]\big).$$

The left hand side of (5) gives

$$\rho'\left([x,y]\right)\alpha'(Z) = Z\big(\alpha\big(\,[[x,y],\cdot]\,\big)\big).$$

Therefore we obtain the following result.

Proposition 2.3. *The triple (V',ρ',α') is a representation of J if and only if*

$$\alpha\big(\,[[x,y],\cdot]\,\big) = -\,[y,[\alpha(x),\cdot]] - [x,[\alpha(y),\cdot]] \tag{10}$$

for all $x,y\in J$. In this case, (V',ρ',α') is called the generalized coadjoint representation.

Associated to the generalized coadjoint representation ρ', the coboundary operators $d^1\colon C^1_{\alpha,\beta}\to C^2_{\alpha,\beta}$ and $d^2\colon C^2_{\alpha,\beta}\to C^3_{\alpha,\beta}$ defined in (6) and (7), respectively, are given by

$$d'^1\colon C^1_{\alpha,\alpha'}\to C^2_{\alpha,\alpha'};\ d'^1(f)(x,y) = f([x,y]) - f(y)\big([x,\cdot]\big) - f(x)\big([y,\cdot]\big)$$

and $d'^2\colon C^2_{\alpha,\alpha'}\to C^3_{\alpha,\alpha'}$;

$$\begin{aligned} d'^2 g(x,y,z) =& g([x,y],\alpha(z)) + g([x,z],\alpha(y)) + g([y,z],\alpha(x)) \\ &+ g(x,y)([\alpha(z),\cdot]) + g(x,z)([\alpha(y),\cdot]) + g(y,z)([\alpha(x),\cdot]). \end{aligned}$$

Hence, by Theorem 2.1, we deduce that

$$d'^2\circ d'^1 = 0. \tag{11}$$

In the particular case in which $V=\mathbb{R}$, we obtain the dual space J^* and we denote

$$\begin{aligned} C^2_r(J,\mathbb{R}) &= \{f \text{ bilinear form } \mid f(x,\cdot)\in C^1_{\alpha,\alpha'}(J,J^*), \forall x\in J\};\\ C^3_r(J,\mathbb{R}) &= \{f \text{ trilinear form } \mid f(x,y,\cdot)\in C^2_{\alpha,\alpha'}(J,J^*), \forall x,y\in J\};\\ C^4_r(J,\mathbb{R}) &= \{f \text{ 4-linear form } \mid f(x,y,z,\cdot)\in S^3(J,J^*), \forall x,y,z\in J\}. \end{aligned}$$

Let us define $d^2_r\colon C^2_r(J,\mathbb{R})\to C^3_r(J,\mathbb{R})$ and $d^3_r\colon C^3_r(J,\mathbb{R})\to C^4_r(J,\mathbb{R})$, respectively, by

$$d^2_r f(x,y,t) = f([x,y],t) - f(y,[x,t]) - f(x,[y,t]) \tag{12}$$

and

$$\begin{aligned} d^3_r\gamma(x,y,z,t) =& \gamma([x,y],\alpha(z),t) + \gamma([x,z],\alpha(y),t) + \gamma([y,z],\alpha(x),t) \\ &+ \gamma\left(x,y,[\alpha(z),t]\right) + \gamma\left(y,z,[\alpha(x),t]\right) + \gamma\left(x,z,[\alpha(y),t]\right) \end{aligned} \tag{13}$$

Theorem 2.4. *With the above notation, we have $d^3_r\circ d^2_r = 0$.*

Proof. We have $d^2_r f(x,y,t) = d'^1 f(x,y)(t)$ and $d^3_r f(x,y,z,t) = d'^2 f(x,y,z)(t)$. By (11), we obtain $d^3_r\circ d^2_r = 0$. □

The following proposition comes directly from Proposition 2.3.

Proposition 2.5. *Let* (V,ρ,β) *be a representation of a Hom-Jacobi–Jordan algebra* $(J,[\cdot,\cdot],\alpha)$ *and* θ *be a* 2*-cocycle of* J *on* V*. Let* $(M,[\cdot,\cdot]_M,\alpha_M)$ *be the extension of* J *by* V *by means of* θ*. Then the triple* (V'',ρ'',β'')*, where* $V''=End(M,V)$*,* $\rho''\colon M\to End(V'')$ *is given by* $\rho''(x+v)f(\cdot))=f([x+v,\cdot]_M)$ *and* $\beta''\colon V''\to V''$ *is given by* $\beta''(f)=f\circ\alpha_M$*, defines a representation of the Hom-Jacobi–Jordan algebra* $(M,[\cdot,\cdot]_M,\alpha_M)$ *if and only if*

$$\alpha\big([[x,y],t]\big)=-[y,[\alpha(x),t]]-[x,[\alpha(y),t]]; \tag{14}$$

$$\beta\Big(\rho([x,y])v\Big)=-\rho(y)\rho(\alpha(x))v-\rho(x)\rho(\alpha(y))v; \tag{15}$$

$$\beta\Big(\rho(t)\theta(x,y)\Big)=-\rho(x)\theta(\alpha(y),t)-\rho(y)\theta(\alpha(x),t); \tag{16}$$

$$\beta\Big(\rho(t)\rho(x)v\Big)=-\rho([\alpha(x),t])v-\rho(x)\rho(t)\beta(v). \tag{17}$$

Let us define $d_c^1\colon C^1_{\alpha,\beta}(J,V)\to S^2(J,V)$ and $d_c^2\colon S^2(J,V)\to C^3(J,V)$, respectively, by

$$\begin{aligned}d^1(f)(x,y)=&f([x,y])-\rho(x)f(y)-\rho(y)f(x),\\ d_c^2(\theta)(x,y,z)=&\theta(x,[\alpha(y),z])+\theta(y,[z,\alpha(x)])+\beta\Big(\theta(z,[x,y])\Big)\\ &+\rho(x)\theta(\alpha(y),z)+\rho(y)\theta(z,\alpha(y))+\beta\Big(\rho(z)\theta(x,y)\Big),\end{aligned}$$

where $C^3(J,V)=\{\gamma\in Hom(J^3,V)\mid\gamma(x,y,t)=\gamma(y,x,t)\}$.

Theorem 2.6. *We have* $d_c^2\circ d^1=0$.

Proof. It is straightforward. □

2.2. Extensions of Hom-Jacobi–Jordan algebras. Let $(J,[\cdot,\cdot],\alpha)$ be a Hom-Jacobi–Jordan algebra, and let (V,ρ,β) be a representation of $(J,[\cdot,\cdot],\alpha)$. An *abelian extension* of a Hom-Jacobi–Jordan algebra J by V is an exact sequence

$$0\longrightarrow(V,\rho,\beta)\overset{i}{\longrightarrow}(M,[\cdot,\cdot]_M,\alpha_M)\overset{\pi}{\longrightarrow}(J,[\cdot,\cdot],\alpha)\longrightarrow0$$

satisfying $\alpha_M\circ i=i\circ\beta$ and $\alpha\circ\pi=\pi\circ\alpha_M$. We say that the extension is *central* if $[i(V),M]_M=0$. A section of an abelian extension $(M,[\cdot,\cdot]_M,\alpha_M)$ of a Hom-Jacobi–Jordan algebra $(J,[\cdot,\cdot],\alpha)$ by (V,ρ,β) is a linear map $s\colon J\to M$ such that $\pi\circ s=Id_J$. Two extensions

$$\begin{array}{ccccccccc}0&\longrightarrow&(V,\rho,\beta)&\overset{i}{\longrightarrow}&(M,[\cdot,\cdot]_M,\alpha_M)&\overset{\pi}{\longrightarrow}&(J,[\cdot,\cdot],\alpha)&\longrightarrow&0\\ &&\downarrow{\scriptstyle Id_V}&&\downarrow{\scriptstyle\Phi}&&\downarrow{\scriptstyle id_J}&&\\ 0&\longrightarrow&(V,\rho,\beta)&\overset{i'}{\longrightarrow}&(M',[\cdot,\cdot]_{M'},\alpha_{M'})&\overset{\pi'}{\longrightarrow}&(J,[\cdot,\cdot],\alpha)&\longrightarrow&0\end{array}$$

are equivalent if there exists an isomorphism of Jacobi–Jordan algebras $\Phi:M\to M'$, such that $\Phi\circ\ i=i'$ and $\pi'\ \circ\ \Phi=\pi$.

Theorem 2.7 ([16]). *Let (V, ρ, β) be a representation of a multiplicative Hom-Jacobi–Jordan algebra $(J, [\cdot,\cdot], \alpha)$ and θ be a 2-cocycle of J on V. Define a bracket $[\cdot,\cdot]_M$ and a morphism α_M on $M = J \oplus V$ by*

$$[x+v, y+w]_\theta = [x,y] + \rho(x)w + \rho(y)v + \theta(x,y),$$
$$\alpha_M(x+v) = \alpha(x) + \beta(v).$$

Define $i_0 \colon V \to M$ by $i_0(v) = v$ and $\pi_0 \colon M \to J$ by $\pi_0(x) = x$. The sequence

$$0 \longrightarrow (V,\rho,\beta) \xrightarrow{i_0} (M, [\cdot,\cdot]_\theta, \alpha_M) \xrightarrow{\pi_0} (J, [\cdot,\cdot], \alpha) \longrightarrow 0$$

defines an abelian extension of J by V.

Proposition 2.8 ([16]). *Let*

$$\mathcal{E} : 0 \longrightarrow (V,\rho,\beta) \xrightarrow{i} (M', [\cdot,\cdot]_{M'}, \alpha_{M'}) \xrightarrow{\pi} (J, [\cdot,\cdot], \alpha) \longrightarrow 0$$

be an abelian extension of J by V and s be a section of $\mathcal{E}$. Then we have $M' = s(J) \oplus i(V)$ and there exists a 2-cocycle $\theta \in Z^2_{\alpha,\beta}(J,V)$ such that, with the notation of the above theorem, the extension $\mathcal{E}$ is equivalent to

$$0 \longrightarrow (V,\rho,\beta) \xrightarrow{i_0} (M, [\cdot,\cdot]_\theta, \alpha_M) \xrightarrow{\pi_0} (J, [\cdot,\cdot], \alpha) \longrightarrow 0.$$

Theorem 2.9 ([16]). *Let (V, ρ, β) be a representation of a multiplicative Hom-Jacobi–Jordan algebra $(J, [\cdot,\cdot], \alpha)$. Then the abelian extensions of J by V are classified by $H^2_{\alpha,\beta}(J,V)$.*

3. Metric Hom-Jacobi–Jordan algebras

In this section, we introduce the notion of metric Hom-Jacobi–Jordan algebras and provide their properties.

Definition 3.1. A *metric Hom-Jacobi–Jordan algebra* is a 4-tuple $(J, [\cdot,\cdot], \alpha, B)$ consisting of a Hom-Jacobi–Jordan algebra $(J, [\cdot,\cdot], \alpha)$ and a nondegenerate symmetric bilinear form B satisfying:

$$B(x,[y,z])) = B([x,y],z) \text{ (invariance of } B\text{)}, \tag{18}$$
$$B(\alpha(x),y) = B(x,\alpha(y)) \text{ (Hom-invariance of } B\text{)}, \tag{19}$$

for any x, y, $z \in J$. We recover the metric Jacobi-Jordan algebra when $\alpha = id_J$.

We say that two metric Hom-Jacobi–Jordan algebras $(J, [\cdot,\cdot], \alpha, B)$ and $(J', [\cdot,\cdot]', \alpha', B')$ are *isometrically isomorphic* (or *i-isomorphic*, for short) if there exists a Hom-Jacobi–Jordan isomorphism f from J onto J' satisfying $B'(f(x), f(y)) = B(x,y)$ for all $x, y \in J$. In this case, f is called an *i-isomorphism.*

Definition 3.2. Let I be an ideal of a metric Hom-Jacobi–Jordan algebra $(J, [\cdot,\cdot], \alpha, B)$.

(1) The *orthogonal* $I^{\perp}$ of I, with respect to B, is defined by

$$I^{\perp} = \{x \in \mathfrak{J} \mid B(x,y) = 0 \, \forall y \in I\}.$$

(2) An ideal I is *isotropic* if $I \subset I^{\perp}$.

Let $(J, [\cdot,\cdot], \alpha, B)$ be a multiplicative metric Hom-Jacobi–Jordan algebra. Since B is non-degenerate and invariant, we obtain some properties described in the following results.

Proposition 3.1. (1) *The center* $\mathfrak{Z}(J)$ *is an ideal of* J.
(2) $\mathfrak{Z}(J) = [J,J]^{\perp}$ *and then* $\dim(\mathfrak{Z}(J)) + \dim([J,J]) = \dim(J)$.

Proposition 3.2. *Let* I *be an ideal of a metric Hom-Jacobi–Jordan algebra* $(J, [\cdot,\cdot], \alpha, B)$. *Then*
(1) $I^{\perp}$ *is an ideal of* J,
(2) *the centralizer* $\mathfrak{Z}(I)$ *of* I *contains* $I^{\perp}$.

For the rest of this paper, for any metric Hom-Jacobi–Jordan algebra, the generalized coadjoint representation identity (10) is satisfied.

Proposition 3.3. *A* 4*-tuple* $(J, [\cdot,\cdot], \alpha, B)$ *is a metric Hom-Jacobi–Jordan algebra if and only if* B *is a nondegenerate symmetric bilinear form satisfying* (19) *and* $d_r^3\gamma = 0$ *where* $\gamma(x,y,z) = B([x,y],z)$ *and* d_r^3 *is given by* (13).

Proof. Let B be a nondegenerate symmetric bilinear form satisfying (19). For all $x,y,z \in J$, we have

$$\begin{aligned} & d_r^3\gamma(x,y,z,t) \\ =& \gamma([x,y],\alpha(z),t) + \gamma([x,z],\alpha(y),t) + \gamma([y,z],\alpha(x),t) \\ & + \gamma(x,y,[\alpha(z),t]) + \gamma(y,z,[\alpha(x),t]) + \gamma(x,z,[\alpha(y),t]) \\ =& B([[x,y],\alpha(z)],t) + B([[x,z],\alpha(y)],t) + B([\alpha(x),[y,z]],t) \quad (20) \\ & + B([x,y],[\alpha(z),t]) + B([y,z],[\alpha(x),t]) + B([x,z],[\alpha(y),t]). \quad (21) \end{aligned}$$

If the identity (18) is satisfied, then we have

$$(21) = B(x,[y,[\alpha(z),t]]) + B([[y,z],t],\alpha(x)) + B(x,[z,[\alpha(y),t]])$$

By (19), we have $B([[y,z],t],\alpha(x)) = B(\alpha([[y,z],t]),x)$. Hence

$$(21) = B(x,[y,[\alpha(z),t]]) + B(x,\alpha([[y,z],t])) + B(x,[z,[\alpha(y),t]]).$$

Then, if (18) and (19) are satisfied, we obtain

$$\begin{aligned} & d_r^3\gamma(x,y,z,t) \\ =& B([[x,y],\alpha(z)],t) + B([[x,z],\alpha(y)],t) + B([\alpha(x),[y,z]],t) \quad (22) \\ & + B(x,[y,[\alpha(z),t]]) + B(x,\alpha([[y,z],t])) + B(x,[z,[\alpha(y),t]]). \quad (23) \end{aligned}$$

By the Hom-Jacobi identity, we deduce that (22)=0. On the other hand, by the generalized coadjoint representation identity, we obtain (23)=0. Therefore $d_r^3\gamma = 0$.

Now, we aim to show that $\gamma \in S^3(J,\mathbb{R})$. For all $x, y, z \in J$, by the equality (18), $[\cdot,\cdot]$ and B are symmetric and we have

$$B([x,y],z) = B([y,x],z) = B(y,[x,z]) = B([x,z],y),$$

which implies that

$$\gamma(x,y,z) = \gamma(y,x,z) = \gamma(x,z,y).$$

So

$$\gamma(x,z,y) = \gamma(z,x,y) = \gamma(x,y,z)$$

and

$$\gamma(y,z,x) = \gamma(z,y,x) = \gamma(y,x,z).$$

Therefore $\gamma \in S^3(J,\mathbb{R})$.

Conversely, we assume that $\gamma \in S^3(J,\mathbb{R})$ and $d_r^3\gamma = 0$. First, we verify the symmetric condition for $[\cdot,\cdot]$. By $\gamma \in S^3(J,\mathbb{R})$, we have $\gamma(x,y,z) = \gamma(y,x,z)$. Hence $B([x,y],z) = B([y,x],z)$. Since B is nondegenerate, one can deduce $[x,y] = [y,x]$.

Next, we verify the equality (18). For any $x, y, z \in J$, we have $\gamma(x,y,z) = \gamma(y,z,x)$, that is, $B([x,y],z) = B([y,z],x)$. Then $B([x,y],z) = B(x,[y,z])$. So (18) holds.

Now, we prove the Hom-Jacobi–Jordan identity. For all $x, y, z \in J$, by the equality (18), we have

$$(21) = B\big([[x,y],\alpha(z)],t\big) + B\big([[y,z],\alpha(x)],t\big) + B\big([[x,z],\alpha(y)],t\big).$$

Thus

$$d_r^3\gamma(x,y,z,t) = 2\Big(B\big([[x,y],\alpha(z)],t\big) + B\big([[y,z],\alpha(x)],t\big) + B\big([[x,z],\alpha(y)],t\big)\Big).$$

Since $d_r^3\gamma = 0$ and B is nondegenerate, we get the Hom-Jacobi identity.

Finally, we prove the coadjoint representation identity. Since (18) and (19) are satisfied, we have $d_r^3\gamma(x,y,z,t) = (22) + (23)$. Since $d_r^3\gamma(x,y,z,t) = 0$ and $(22) = 0$, we obtain $(23) = 0$. This finishes the proof. □

4. The second cohomology group of a metric Hom-Jacobi–Jordan algebra

The task of this section is to introduce the second cohomology group of a metric Hom-Jacobi–Jordan algebra, which we will use to describe the quadratic extensions.

4.1. Construction of 2-coboundary operators for a metric Hom-Jacobi–Jordan algebra. Let $M = J \oplus \mathfrak{a}$ be a Hom-Jacobi–Jordan algebra with structure $\alpha_M = \alpha + \beta$ where $\alpha\colon J \to J$, $\beta\colon \mathfrak{a} \to \mathfrak{a}$ and $[\cdot,\cdot]_M$ are such that $\mathfrak{a}$ is an abelian ideal of M. Then, by Theorem 2.2, $[\cdot,\cdot]_M = [\cdot,\cdot] + \rho + \theta$, where $(J, [\cdot,\cdot], \alpha)$ is a Hom-Jacobi–Jordan algebra, ρ is a representation of J on $\mathfrak{a}$, and θ is a 2-cocycle of J on $\mathfrak{a}$. Let $\mathfrak{n} = M \oplus J^*$, $[\cdot,\cdot]_{\mathfrak{n}}\colon \mathfrak{n}^2 \to \mathfrak{n}$ be a bilinear symmetric map satisfying $[J^*, J^*]_{\mathfrak{n}} = 0$ and $\alpha_{\mathfrak{n}}\colon \mathfrak{n} \to \mathfrak{n}$ a linear map given by $\alpha_{\mathfrak{n}}(x + v + Z) = \alpha_M(x + v) + \alpha'(Z)$ for all $x \in J$, $v \in V$, $Z \in J^*$.

We assume that $(\mathfrak{n}, [\cdot,\cdot]_{\mathfrak{n}}, \alpha_{\mathfrak{n}})$ is a Hom-Jacobi–Jordan algebra. Then (by Theorem 2.2) $[\cdot,\cdot]_{\mathfrak{n}} = [\cdot,\cdot]_M + \rho' + \gamma'$ where ρ' is a representation of M on J^* and γ' is a 2-cocycle of M on J^*. Hence, for all $x \in J$, $v \in V, Z_1, Z_2 \in J^*$,

$$[x, y]_{\mathfrak{n}} = [x, y] + \theta(x, y) + \gamma'(x, y); \tag{24}$$

$$[x, v]_{\mathfrak{n}} = \rho(x)v + \gamma'(x, v); \tag{25}$$

$$[v, w]_{\mathfrak{n}} = \gamma'(v, w); \tag{26}$$

$$[Z, x]_{\mathfrak{n}} = \rho'(x)Z; \tag{27}$$

$$[Z, v]_{\mathfrak{n}} = \rho'(v)Z; \tag{28}$$

$$[Z_1, Z_2]_{\mathfrak{n}} = 0. \tag{29}$$

Let $B\colon \mathfrak{n}^2 \to \mathbb{R}$ be a bilinear form such that $(\mathfrak{n}, [\cdot,\cdot]_{\mathfrak{n}}, \alpha_{\mathfrak{n}}, B)$ is a metric Hom-Jacobi–Jordan algebra, the ideals J and J^* are isotropic and

$$B(Z, x + v) = Z(x) \tag{30}$$

for all $Z \in J^*$, $x \in J$, $v \in \mathfrak{a}$.

Lemma 4.1. *Under the above notation, we have*

$$[Z, x]_{\mathfrak{n}} = Z([x, \cdot]) \text{ and } [Z, v]_{\mathfrak{n}} = 0$$

for all $Z \in J^*$, $x \in J, v \in \mathfrak{a}$.

Proof. Let $Z \in J^*$, $x \in J, v \in \mathfrak{a}$. We have $B(Z, v) = Z(v) = 0$. Then $B(Z, [x, y]_{\mathfrak{n}}) = Z([x, y])$. Moreover, by invariance of B, we have $B(Z, [x, y]_{\mathfrak{n}}) = B([Z, x]_{\mathfrak{n}}, y)$. Hence $\rho'(x)Z(y) = Z([x, y])$, which implies that $[Z, x]_{\mathfrak{n}} = Z([x, \cdot])$.

Now, we show that $[Z, v]_{\mathfrak{n}} = 0$. Since J^* is an ideal of $\mathfrak{n}$, according to Proposition 3.2, we have $(J^*)^\perp \subset \mathfrak{Z}(J^*)$. Then $\mathfrak{a} \subset \mathfrak{Z}(J^*)$, since $B(Z, v) = 0$. Therefore $[Z, v]_{\mathfrak{n}} = 0$. ☐

Proposition 4.2. *For all $v, w \in \mathfrak{a}$, we have*

$$B(\beta(v), w) = B(v, \beta(w)). \tag{31}$$

Proof. By (19) we have $B((\alpha + \beta + \alpha')(v), w) = B(v, (\alpha + \beta + \alpha')(w))$. Therefore $B(\beta(v), w) = B(v, \beta(w))$. ☐

Theorem 4.3. *If* $(\mathfrak{n}, [\cdot,\cdot]_{\mathfrak{n}}, \alpha_{\mathfrak{n}}, B)$ *is a metric Hom-Jacobi–Jordan algebra, then, for all* $x, y \in J$, $v, w \in \mathfrak{a}$, $Z \in J^*$, *we have*

$$\begin{aligned} [x,y]_{\mathfrak{n}} &= [x,y] + \theta(x,y) + \gamma(x,y,\cdot); \\ [x,v]_{\mathfrak{n}} &= \rho(x)v + B\left(\theta(\cdot,x), v\right); \\ [v,w]_{\mathfrak{n}} &= B\left(\rho(\cdot)v, w\right); \\ [Z,x]_{\mathfrak{n}} &= Z\left([x,\cdot]\right); \\ [Z_1, v+Z_2]_{\mathfrak{n}} &= 0, \end{aligned} \tag{32}$$

where $\gamma \in S^3(J,\mathbb{R})$.

Proof. Assume that $(\mathfrak{n}, [\cdot,\cdot]_{\mathfrak{n}}, \alpha_{\mathfrak{n}}, B)$ is a metric Hom-Jacobi–Jordan algebra. Let $\gamma(x,y,z) = \gamma'(x,y)(z)$. By the equality (18), we have $B\left([x,y]_{\mathfrak{n}}, z\right) = B\left(x, [y,z]_{\mathfrak{n}}\right)$. Thus, using (24), we have $\gamma'(x,y)(z) = \gamma'(y,z)(x)$. Hence $\gamma(x,y,z) = \gamma(y,z,x)$. Moreover, since $[x,y]_{\mathfrak{n}} = [y,x]_{\mathfrak{n}}$, we have $\gamma(x,y,z) = \gamma(y,x,z)$. By repeating this process, we obtain that $\gamma \in S^3(J,\mathbb{R})$.

Now we aim to prove that $\gamma'(x,v)(y) = B\left(\theta(y,x), v\right)$. By the equality (18), we have $B\left([y,x]_{\mathfrak{n}}, v\right) = B\left(y, [x,v]_{\mathfrak{n}}\right)$. Thus, using (24), (25) and (30), we obtain $\gamma'(x,v)(y) = B_{\mathfrak{a}}\left(\theta(y,x), v\right)$. For $\gamma'(v,w)$, by (18), we have $B\left([x,v]_{\mathfrak{n}}, w\right) = B\left(x, [v,w]_{\mathfrak{n}}\right)$. Thus, using (25), (26) and (30), we have $\gamma'(v,w)(x) = B_{\mathfrak{a}}(\rho(x)v, w)$. Hence

$$\gamma'(v,w) = B\left(\rho(\cdot)v, w\right). \tag{33}$$

□

Definition 4.1. A *Quadratic representation* of a Hom-Jacobi–Jordan algebra $(J, [\cdot,\cdot], \alpha)$ on a vector space $\mathfrak{a}$ with respect to $\beta \in End(\mathfrak{a})$ consists of a 4-tuple $(\mathfrak{a}, \rho, \beta, B_{\mathfrak{a}})$, where $\rho\colon J \to End(\mathfrak{a})$ is a representation of the Hom-Jacobi–Jordan algebra J on $\mathfrak{a}$ with respect to $\beta \in End(\mathfrak{a})$, and $B_{\mathfrak{a}}\colon \mathfrak{a}\times\mathfrak{a} \to \mathbb{R}$ a symmetric bilinear form, satisfying,

$$B_{\mathfrak{a}}\left(\rho(x)(v), w\right) = B_{\mathfrak{a}}\left(v, \rho(x)(w)\right) \tag{34}$$

for all $x, y \in J$ and $v, w \in \mathfrak{a}$.

Lemma 4.4. *If* $(\mathfrak{n}, [\cdot,\cdot]_{\mathfrak{n}}, \alpha_{\mathfrak{n}}, B_{\mathfrak{a}})$ *is a metric Hom-Jacobi–Jordan algebra, then* $(\mathfrak{a}, \rho, \beta, B_{\mathfrak{a}})$ *is a quadratic representation of* J *on* $\mathfrak{a}$.

Proof. Using (33) and the symmetry of the bracket $[\cdot,\cdot]_{\mathfrak{n}}$, we obtain $B_{\mathfrak{a}}\left(\rho(\cdot)v, w\right) = B_{\mathfrak{a}}\left(\rho(\cdot)w, v\right)$, which finishes the proof. □

Proposition 4.5. *Let* $(\mathfrak{n}, [\cdot,\cdot]_{\mathfrak{n}}, \alpha_{\mathfrak{n}}, B_{\mathfrak{a}})$ *be a metric Hom-Jacobi–Jordan algebra. For* $f, g \in C^2_{\alpha,\beta}(J,\mathfrak{a})$, *we have*

$$B_{\mathfrak{a}}\left(f(\alpha(x),\alpha(y)), g(z,t)\right) = B_{\mathfrak{a}}\left(f(x,y), g(\alpha(z),\alpha(t))\right)$$

for all $x, y, z, t \in J$.

Proof. Since $f, g \in C^2_{\alpha,\beta}(J, \mathfrak{a})$, we have, $f \circ \alpha = \beta \circ f$ and $g \circ \alpha = \beta \circ g$. According to Proposition 4.2, we have $B_{\mathfrak{a}}(\beta \circ f(x,y), g(x,z)) = B_{\mathfrak{a}}(f(x,y), \beta \circ g(x,z))$. Thus $B_{\mathfrak{a}}(f(\alpha(x),\alpha(y)), g(z,t)) = B_{\mathfrak{a}}(f(x,y), g(\alpha(z),\alpha(t)))$. □

Define a bilinear multiplication on $S^p(J,\mathfrak{a}) \times S^q(J,\mathfrak{a})$ by

$$B_{\mathfrak{a}}(f \wedge g)(x_1, \cdots, x_{p+q}) = \sum_{\sigma \in Sh(p,q)} B_{\mathfrak{a}}\big(f(x_{\sigma(1)}, \cdots, x_{\sigma(p)}), g(x_{\sigma(p+1)}, \cdots, x_{\sigma(p+q)})\big), \tag{35}$$

where $Sh(p,q)$ are the permutations in $\mathfrak{S}_{p+q}$ which are increasing on the first p and the last q elements.

Proposition 4.6. *If $(\mathfrak{n}, [\cdot,\cdot]_{\mathfrak{n}}, \alpha_{\mathfrak{n}}, B_{\mathfrak{a}})$ is a metric Hom-Jacobi–Jordan algebra, then the pair (θ, γ) satisfies the following properties*

$$d^2\theta(x,y,z) = 0,$$
$$d_r^3\gamma(x,y,z,\alpha(a)) + \frac{1}{2}B_{\mathfrak{a}}(\theta \wedge (\theta \circ \alpha))(x,y,z,a) = 0$$

for all $x, y, z, a \in J$.

Proof. We have that $(M, [\cdot,\cdot]_M, \alpha_M)$ is a Hom-Jacobi–Jordan algebra, (J^*, ρ', α') is a representation of the Hom-Jacobi–Jordan algebra M, $\mathfrak{n} = M \oplus J^*$ and $[\cdot,\cdot]_{\mathfrak{n}} = [\cdot,\cdot]_M + \gamma'$. By Theorem 2.2, it follows that $d^2\gamma' = 0$. For all $x, y, z, a \in J$, we have

$$\begin{aligned}
&d^2\gamma'(x,y,z)(t)\\
&= \gamma'([x,y]_M, \alpha_M(z))(t) + \gamma'([x,z]_M, \alpha_M(y))(t) + \gamma'([y,z]_M, \alpha_M(x))(t)\\
&\quad + \rho'(\alpha_M(z))\gamma'(x,y)(t) + \rho'(\alpha_M(x))\gamma'(y,z)(t) + \rho'(\alpha_M(y))\gamma'(x,z)(t),
\end{aligned}$$

where $t = \alpha(a)$. Since $[x,y]_M = [x,y] + \theta(x,y)$, $\gamma'(x,v)(y) = B_{\mathfrak{a}}(\theta(y,x), v)$ and $\gamma'(v,w) = B_{\mathfrak{a}}(\rho(\cdot)v, w)$, we obtain

$$d^2\gamma'(x,y,z)(t) = \gamma([x,y], \alpha(z), t) + \gamma([x,z], \alpha(y), t) + \gamma([y,z], \alpha(x), t) \tag{36}$$
$$+ \gamma(x, y, [\alpha(z), t]) + \gamma(y, z, [\alpha(x), t]) + \gamma(x, z, [\alpha(y), t]) \tag{37}$$
$$\begin{aligned}
&+ B_{\mathfrak{a}}(\theta(\alpha(a), \alpha(z)), \theta(x,y)) + B_{\mathfrak{a}}(\theta(\alpha(a), \alpha(y)), \theta(x,z))\\
&+ B_{\mathfrak{a}}(\theta(\alpha(a), \alpha(x)), \theta(y,z)).
\end{aligned}$$

Using (36) + (37) = $d_r^3\gamma(x,y,z,t)$ and Proposition 4.5, we obtain

$$d^2\gamma'(x,y,z)(t) = d_r^3\gamma(x,y,z,t) + \frac{1}{2}B_{\mathfrak{a}}(\theta \wedge (\theta \circ \alpha))(x,y,z,a).$$

Hence $d_r^3\gamma(x,y,z,\alpha(a)) + \frac{1}{2}B_{\mathfrak{a}}(\theta \wedge (\theta \circ \alpha))(x,y,z,a) = 0$. □

Bringing these results together, we provide the following definitions.

Definition 4.2. The pair (θ, γ) is called a *quadratic* 2-*cochain* if $\theta \in C^2_{\alpha,\beta}(J, \mathfrak{a})$ and $\gamma \in C^3_r(J, \mathbb{R})$. Denote by $C^2_Q(J, \mathfrak{a})$ the set of quadratic 2-cochains.
We define a map $d^2_Q\colon C^2_Q(J, \mathfrak{a}) \to C^3_r(J, \mathfrak{a}) \times C^4(J, \mathbb{R})$ as follows:

$$d^2_Q (\theta, \gamma) (x, y, z)(t) = \left(d^2\theta(x, y, z), d^3_r\gamma(x, y, z, t) + \frac{1}{2} B_{\mathfrak{a}} (\theta \wedge (\theta \circ \alpha)) (x, y, z, a)\right), \tag{38}$$

where $t = \alpha(a)$. (θ, γ) is called a *quadratic* 2-*Hom-cocycle* of J on $\mathfrak{a}$ if and only if $d^2_Q(\theta, \gamma) = 0$. We denote by $Z^2_Q(J, \mathfrak{a})$ the set of all quadratic 2-cocycles on $\mathfrak{a}$.

4.2. Construction of 1-coboundary operators of a metric Hom-Jacobi–Jordan algebra. In this section we aim to construct a map d^1_Q satisfying $d^2_Q \circ d^1_Q = 0$ and then the second cohomology group of a metric Hom-Jacobi–Jordan algebra.

Proposition 4.7. *Let $f \in C^2_{\alpha,\beta}(J, \mathfrak{a})$ and $g \in C^1_{\alpha,\beta}(J, \mathfrak{a})$. We have*

$$\begin{aligned} d^3_r B_{\mathfrak{a}}(f \wedge g)(x, y, z, t) = & B_{\mathfrak{a}} \left(d^2 f(x, y, z), g(t)\right) + B_{\mathfrak{a}} \left(d^2_c f(x, y, t), g(z)\right) \\ & + B_{\mathfrak{a}} \left(d^2_c f(x, z, t), g(y)\right) + B_{\mathfrak{a}} \left(d^2_c f(y, z, t), g(x)\right) \\ & + B_{\mathfrak{a}} \left((f \circ \alpha) \wedge d^1 g\right) (x, y, z, a) \end{aligned}$$

for any $x, y, z, a \in J$ and $t = \alpha(a)$.

Proof. Let $f \in C^2_{\alpha,\beta}(J, \mathfrak{a})$ and $g \in C^1_{\alpha,\beta}(J, \mathfrak{a})$. We take $\gamma = B_{\mathfrak{a}} (f \wedge g)$. For any $x, y, z, a \in J$ and $t = \alpha(a)$, we have

$$\begin{aligned} & d^3_r\gamma(x, y, z, t) \\ = & \gamma([x, y], \alpha(z), t) + \gamma([x, z], \alpha(y), t) + \gamma([y, z], \alpha(x), t) \\ & + \gamma (x, y, [\alpha(z), t]) + \gamma (y, z, [\alpha(x), t]) + \gamma (x, z, [\alpha(y), t]) \\ = & \circlearrowleft_{x,y,z} \left(\gamma([x, y], \alpha(z), t) + \circlearrowleft_{x,y,z} \gamma (x, y, [\alpha(z), t])\right) \\ = & \circlearrowleft_{x,y,z} \left(B_{\mathfrak{a}}(f([x, y], \alpha(z)), g(t)) + B_{\mathfrak{a}}(f([x, y], t), g(\alpha(z)))\right) + B_{\mathfrak{a}}(f(\alpha(a), \alpha(z)), g([x, y])) \end{aligned} \tag{39}$$

$$+ \circlearrowleft_{x,y,z} \left(B_{\mathfrak{a}}(f(x, y), g([\alpha(z), t])) + B_{\mathfrak{a}} (f(x, [\alpha(z), t]), g(y) + B_{\mathfrak{a}} (f(y, [\alpha(z), t]), g(x)))\right), \tag{40}$$

where $\circlearrowleft_{x,y,z}$ denotes a summation over the cyclic permutation on x, y, and z.
By Proposition 4.2 and taking into account that $g \in C^1_{\alpha,\beta}(J, \mathfrak{a})$, we have

$$\circlearrowleft_{x,y,z} B_{\mathfrak{a}}(f([x, y], t), g(\alpha(z))) = \circlearrowleft_{x,y,z} B_{\mathfrak{a}} (\beta(f([x, y], t)), g(z)) .$$

Hence

$$\begin{aligned} (39) = & B_{\mathfrak{a}}\Big(d^2 f(x, y, z), g(t)\Big) - \circlearrowleft_{x,y,z} B_{\mathfrak{a}}\Big(\rho (\alpha(x)) f(y, z), g(t)\Big) \\ & + \circlearrowleft_{x,y,z} B_{\mathfrak{a}} (\beta(f([x, y], t)), g(z)) + \circlearrowleft_{x,y,z} B_{\mathfrak{a}}(f(\alpha(a), \alpha(z)), g([x, y])). \end{aligned}$$

For (40), we have

$$\circlearrowleft_{x,y,z} B_{\mathfrak{a}}\left(f(x,y), g([\alpha(z),t])\right) = \circlearrowleft_{x,y,z} B_{\mathfrak{a}}\left(f(\alpha(x)), \alpha(y), g([z,a])\right).$$

Then

$$\begin{aligned}
&(39)+(40)\\
=&B_{\mathfrak{a}}\Big(d^2 f(x,y,z), g(t)\Big) - \circlearrowleft_{x,y,z} B\Big(\rho\left(\alpha(x)\right) f(y,z), g(t)\Big)\\
&+\circlearrowleft_{x,y,z}\Big(B_{\mathfrak{a}}\left(\beta(f([x,y],t)), g(z)\right) + B_{\mathfrak{a}}\left(f(x,[\alpha(z),t]), g(y)\right) + B_{\mathfrak{a}}\left(f(y,[\alpha(z),t]), g(x)\right)\Big)\\
&+\circlearrowleft_{x,y,z} B_{\mathfrak{a}}(f(\alpha(a),\alpha(z)), g([x,y])) + \circlearrowleft_{x,y,z} B_{\mathfrak{a}}(f(\alpha(x),\alpha(y)), g([z,a])).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\beta\left(f([x,y],t)\right) + f(y,[\alpha(x),t]) + f(x,[\alpha(y),t])\\
&\quad = d_c^2 f(x,y,t) - \rho(y) f(\alpha(x),t) - \rho(x) f(\alpha(y),t) - \beta\left(\rho(t) f(x,y)\right),
\end{aligned}$$

and

$$\begin{aligned}
B_{\mathfrak{a}}\left(\rho(y) f(\alpha(x),t), g(z)\right) &= B_{\mathfrak{a}}\left(f(\alpha(x),t), \rho(y) g(z)\right)\\
&= B_{\mathfrak{a}}\left(f(\alpha(x),\alpha(a)), \rho(y) g(x)\right).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
B_{\mathfrak{a}}\left(\beta\left(\rho(t) f(y,z)\right), g(x)\right) &= B_{\mathfrak{a}}\left(\rho(\alpha(a)) f(y,z), \beta\left(g(x)\right)\right)\\
&= B_{\mathfrak{a}}\left(f(y,z), \rho(\alpha(a))\beta\left(g(x)\right)\right)\\
&= B_{\mathfrak{a}}\left(f(y,z), \beta\left(\rho(a) g(x)\right)\right)\\
&= B_{\mathfrak{a}}\left(\beta\left(f(y,z)\right), \rho(a) g(x)\right)\\
&= B_{\mathfrak{a}}\left(f(\alpha(y),\alpha(z)), \rho(a) g(x)\right).
\end{aligned}$$

Therefore, by straightforward computations, we obtain

$$\begin{aligned}
d_r^3\gamma(x,y,z,t) =& B_{\mathfrak{a}}(d^2 f(x,y,z)), g(t)) + B_{\mathfrak{a}}(d_c^2 f(x,y,t)), g(z))\\
&+ B_{\mathfrak{a}}(d_c^2 f(x,z,t)), g(y)) + B_{\mathfrak{a}}(d_c^2 f(y,z,t)), g(x))\\
&+ B_{\mathfrak{a}}((f\circ\alpha)\wedge d_c^1 g)(x,y,z,a).
\end{aligned}$$

□

Remark 4.1. If $\alpha = id_J$ and $\beta = id_{\mathfrak{a}}$, we have

$$d_r^3(f\wedge g) = B_{\mathfrak{a}}(d^2 f\wedge g) + B_{\mathfrak{a}}\left(f\wedge d^1 g\right).$$

Lemma 4.8. *Let (θ,γ) and (θ',γ') be two quadratic 2-cochains. Then $d_Q^2(\theta,\gamma) = d_Q^2(\theta',\gamma')$ if and only if there exists a 1-Hom-cochain τ such that the following equalities hold:*

$$\theta' = \theta + d^1\tau, \tag{41}$$

$$d_r^3\gamma' = d_r^3\gamma - \frac{1}{2} d_r^3 B_{\mathfrak{a}}(\tau\wedge d^1\tau) - d_r^3 B_{\mathfrak{a}}(\tau\wedge\theta) + B_{\mathfrak{a}}(d'^2\theta\wedge\tau), \tag{42}$$

where $d'^2\theta(x,y,z)=d^2\theta(x,y,z)$ *and* $d'^2\theta(x,y,\cdot)=d^2_c\theta(x,y,\cdot)$.

Proof. Let (θ,γ) and (θ',γ') be two quadratic 2-cochain such that $d^2_Q(\theta,\gamma)=d^2_Q(\theta',\gamma')$. Then

$$d^2\theta=d^2\theta' \tag{43}$$

and

$$d^3_r\gamma+\frac{1}{2}B_{\mathfrak{a}}(\theta\wedge(\theta\circ\alpha))=d^3_r\gamma'+\frac{1}{2}B_{\mathfrak{a}}(\theta'\wedge(\theta'\circ\alpha)). \tag{44}$$

Equality (43) implies that there exist a 1-Hom-cochain τ which satisfies

$$\theta'=\theta+d^1\tau. \tag{45}$$

Thus, using (44), we have

$$\begin{aligned} d^3_r\gamma &= d^3_r\gamma'+\frac{1}{2}B_{\mathfrak{a}}\Big(\,(\theta+d^1\tau)\wedge((\theta+d^1\tau)\circ\alpha)\Big)-\frac{1}{2}B_{\mathfrak{a}}(\theta\wedge(\theta\circ\alpha))\\ &= d^3\gamma'+\frac{1}{2}B\Big(\theta\wedge(d^1\tau\circ\alpha)\Big)+\frac{1}{2}B\Big(d^1\tau\wedge(\theta\circ\alpha)\Big)+\frac{1}{2}B\Big(d^1\tau\wedge(d^1\tau\circ\alpha)\Big). \end{aligned} \tag{46}$$

Hence, by Proposition 4.5, we obtain $B_{\mathfrak{a}}\left(\theta\wedge(d^1\tau\circ\alpha)\right)=B_{\mathfrak{a}}\left(d^1\tau\wedge(\theta\circ\alpha)\right)$. Therefore

$$d^3\gamma=d^3\gamma'+B(d^1\tau\wedge(\theta\circ\alpha))+\frac{1}{2}B(d^1\tau\wedge(d^1\tau\circ\alpha)).$$

Replacing f, g by $d^1\tau$, τ in Proposition 4.7 and since by $d^2\circ d^1(\tau)=0$, we have

$$d^3_rB_{\mathfrak{a}}(d^1_c\tau\wedge\tau)(x,y,z,t)=B_{\mathfrak{a}}((d^1_c\tau\circ\alpha)\wedge d^1_c\tau)(x,y,z,a). \tag{47}$$

Replacing f, g by θ, τ in Proposition 4.7, we have

$$d^3_rB_{\mathfrak{a}}(\theta\wedge\tau)(x,y,z,t)=B_{\mathfrak{a}}((\theta\circ\alpha)\wedge d^1_c\tau)(x,y,z,a)+B_{\mathfrak{a}}(d'^2\theta\wedge\tau)(x,y,z,t),$$

where $d'^2\theta(x,y,z)=d^2\theta(x,y,z)$ and $d'^2\theta(x,y,t)=d^2_c\theta(x,y,t)$. Therefore

$$d^3\gamma=d^3\gamma'+d^3_rB_{\mathfrak{a}}(\theta\wedge\tau)+\frac{1}{2}d^3_rB_{\mathfrak{a}}(d^1_c\tau\wedge\tau)-B_{\mathfrak{a}}(d'^2\theta\wedge\tau)(x,y,z,t).$$

Hence

$$d^3\gamma'=d^3\gamma-\frac{1}{2}d^3B_{\mathfrak{a}}(\tau\wedge d^1\tau)-d^3B_{\mathfrak{a}}\,(\theta\wedge\tau)+B_{\mathfrak{a}}(d'^2\theta\wedge\tau)(x,y,z,t). \tag{48}$$

□

Using the previous lemma and Proposition 4.7, we obtain the following result.

Theorem 4.9. *Let* (θ, γ) *and* (θ', γ') *two quadratic* 2*-cochains. Then* $d_Q^2(\theta, \gamma) = d_Q^2(\theta', \gamma')$ *if and only if there exist* $\tau \in C^1_{\alpha,\beta}(J, \mathfrak{a})$*,* $\sigma \in C_r^2(J, \mathbb{R})$ *and* $\sigma' \in C_r^3(J, \mathbb{R})$ *such that, the following equalities hold:*

$$\theta' = \theta + d^1\tau, \tag{49}$$

$$d_r^3\sigma' = -B_{\mathfrak{a}}(d'^2\theta \wedge \tau), \tag{50}$$

$$\gamma' = \gamma + d_r^2\sigma + \sigma' - B(\tau \wedge (\theta + \frac{1}{2}d^1\tau)), \tag{51}$$

where $d'^2\theta(x, y, z) = d^2\theta(x, y, z)$ *and* $d'^2\theta(x, y, \cdot) = d_c^2\theta(x, y, \cdot)$*.*

Using the previous observations, we give the following definitions.

Definition 4.3. Define a map $d_Q^1 \colon C_Q^1(\mathfrak{J}, \mathfrak{a}) \to C_Q^2(\mathfrak{J}, \mathfrak{a})$ by

$$d_Q^1(\tau, \sigma) = \left(d^1\tau, d_r^2\sigma - \frac{1}{2}B\left(\tau \wedge d^1\tau\right)\right).$$

A quadratic 2-cochain (θ, γ) is called a *quadratic* 2*-cobord* if and only if there exists a quadratic 1-cochain (τ, σ) satisfies $d_Q^1(\tau, \sigma) = (\theta, \gamma)$. Denote by $B_Q^2(\mathfrak{J}, \mathfrak{a})$ the space of all quadratic 2-cobords.

Proposition 4.10. *Any quadratic* 2*-cobord is a quadratic* 2*-cocycle (i.e.,* $d_Q^2 \circ d_Q^1 = 0$*).*

Proof. We set $\theta = d^1\tau$ and $\gamma = d^2\sigma - \frac{1}{2}B_{\mathfrak{a}}(d^1\tau \wedge \tau)$. Using (47), we have $d^3\gamma = -\frac{1}{2}B_{\mathfrak{a}}\left(d^1\tau \wedge (d^1\tau \circ \alpha)\right)$. Hence, by (38)

$$\begin{aligned} d_Q^2(\theta, \gamma) &= (d^2\theta, d_r^3 \circ d_r^2\sigma - \frac{1}{2}B_{\mathfrak{a}}\left(d^1\tau \wedge (d^1\tau \circ \alpha)\right) + \frac{1}{2}B\left(d^1\tau \wedge (d^1\tau \circ \alpha)\right)) \\ &= (0, 0). \end{aligned}$$

□

4.3. The second cohomology group. Due to the nonlinearity of d_Q^1 and d_Q^2 we need to construct an equivalence relation in order to define the second cohomology group. We define a group structure on $C_Q^1(\mathfrak{J}, \mathfrak{a})$ by

$$(f, g) * (f', g') = (f + f', g + g' + \frac{1}{2}B_{\mathfrak{a}}\left((f + f') \wedge (f + f') \wedge \alpha)\right).$$

Let $(\gamma, \theta) \in Z_Q^2(\mathfrak{J}, \mathfrak{a})$ and $(\tau, \sigma) \in C_Q^1(\mathfrak{J}, \mathfrak{a})$. Then the formula

$$(\theta, \gamma) \bullet (\tau, \sigma) = (\theta + d^1\tau, \gamma + d^2\sigma + B\left((\theta + \frac{1}{2}d^1\tau) \wedge (\tau \circ \alpha)\right)$$

defines a right action of the group $C_Q^1(\mathfrak{J}, \mathfrak{a})$ on $Z_Q^2(\mathfrak{J}, \mathfrak{a})$. We have $(\theta, \gamma) \cong (\theta', \gamma')$ if and only if there exist $(\tau, \sigma) \in C_Q^1(\mathfrak{J}, \mathfrak{a})$ such that $(\gamma', \theta') = (\gamma, \theta) \bullet (\tau, \sigma)$.

Definition 4.4. The 2^{nd} *quadratic cohomology group* of the metric Hom-Jacobi–Jordan algebra $\mathfrak{J}$ on $\mathfrak{a} \times \mathfrak{J}^*$, with the action "$\bullet$" is the quotient

$$H_Q^{\bullet 2}(\mathfrak{J}, \mathfrak{a}) = Z_Q^2(\mathfrak{J}, \mathfrak{a})/C_Q^1(\mathfrak{J}, \mathfrak{a}),$$

where $Z_Q^2(\mathfrak{J}, \mathfrak{a}) = \{(\theta, \gamma) \mid d_Q^2((\theta, \gamma)) = 0\}$.

Proposition 4.11. *Let* $\mathfrak{d}_{\theta,\gamma} := (\mathfrak{n}, [\cdot,\cdot]_{\theta,\gamma}, \alpha_{\mathfrak{n}})$ *and* $\mathfrak{d}_{\theta',\gamma'} := \big(\mathfrak{n}, [\cdot,\cdot]_{\theta',\gamma'}, \alpha_{\mathfrak{n}}\big)$ *be two extensions such that* $d_Q^2(\theta, \gamma) = d_Q^2(\theta', \gamma')$*. Then the extensions* $\mathfrak{d}_{\theta,\gamma}$ *and* $\mathfrak{d}_{\theta',\gamma'}$ *are equivalent.*

Proof. Using Theorem 4.9, we have

$$\theta' = \theta + d^1\tau \text{ and } \gamma' = \gamma + d_r^2\sigma - B\left(\tau \wedge (\theta + \frac{1}{2}d^1\tau)\right).$$

Define the linear map $\Phi\colon J \oplus \mathfrak{a} \oplus J^* \to J \oplus \mathfrak{a} \oplus J^*$ by

$$\Phi(x + v + Z) = x + \underbrace{v - \tau(x)}_{\in \mathfrak{a}} \underbrace{-\sigma(x, \cdot) + Z - \frac{1}{2}B_{\mathfrak{a}}\big(\tau(x), \tau(\cdot)\big) + B_{\mathfrak{a}}(v, \tau(\cdot))}_{\in J^*}.$$

We have

$$\begin{aligned}
&\Phi(\alpha(x) + \beta(v) + \alpha'(Z))\\
&= \alpha(x) + \beta(v) - \tau(\alpha(x)) - \sigma(\alpha(x), \cdot) + \alpha'(Z) - \frac{1}{2}B_{\mathfrak{a}}\big(\tau(\alpha(x)), \tau(\cdot)\big) + B_{\mathfrak{a}}(\beta(v), \tau(\cdot))\\
&= \alpha(x) + \beta(v) - \beta(\tau(x) - \sigma(x, \alpha(\cdot))) + \alpha'(Z) - \frac{1}{2}B_{\mathfrak{a}}\big(\beta(\tau(x)), \tau(\cdot)\big) + B_{\mathfrak{a}}(\beta(v), \tau(\cdot))\\
&= \alpha(x) + \beta(v) - \beta(\tau(x)) + \alpha'(Z) - \alpha'\left(\sigma(x, \cdot)\right) - \frac{1}{2}B_{\mathfrak{a}}\big(\tau(x), \beta(\tau(\cdot))\big) + B_{\mathfrak{a}}(v, \beta(\tau(\cdot)))\\
&= \alpha(x) + \beta(v) - \beta(\tau(x)) - \alpha'\left(\sigma(x, \cdot)\right) + \alpha'(Z) - \frac{1}{2}B_{\mathfrak{a}}\big(\tau(x), \tau(\alpha(\cdot))\big) + B_{\mathfrak{a}}(v, \tau(\alpha(\cdot)))\\
&= \alpha(x) + \beta\left(v - \tau(x)\right) + \alpha'\left(-\sigma(x, \cdot) + Z - \frac{1}{2}B_{\mathfrak{a}}\big(\tau(x), \tau(\cdot)\big) + B_{\mathfrak{a}}(v, \tau(\cdot))\right).
\end{aligned}$$

Hence $\Phi \circ (\alpha + \beta + \alpha') = (\alpha + \beta + \alpha') \circ \Phi$.

We have

$$\begin{aligned}
[x, y]_{\theta,\gamma} &= [x, y] + \theta(x, y) + \gamma(x, y, \cdot);\\
[x, v]_{\theta,\gamma} &= \rho(x)v + B_{\mathfrak{a}}\left(\theta(\cdot, x), v\right);\\
[v, w]_{\theta,\gamma} &= B_{\mathfrak{a}}\left(\rho(\cdot)v, w\right);\\
[Z, x]_{\theta,\gamma} &= Z\left([x, \cdot]\right);\\
[Z_1, v + Z_2]_{\theta,\gamma} &= 0.
\end{aligned}$$

Hence the structure $[\cdot,\cdot]_{\theta',\gamma'}$ of the Hom-Jacobi-algebra $\mathfrak{d}_{\theta',\gamma'}$ is given by

$$\begin{aligned}
[x, y]_{\theta',\gamma'} = {} & [x, y] + \theta(x, y) + d^1\tau(x, y) + \gamma(x, y, \cdot)\\
& + d^2\sigma(x, y, \cdot) - B((\theta + \frac{1}{2}d^1\tau) \wedge \tau)(x, y, \cdot);
\end{aligned}$$

$$\begin{aligned}
[x,v]_{\theta',\gamma'} &= \rho(x)v + B_{\mathfrak{a}}\big(\theta(\cdot,x) + d^1\tau(\cdot,x), v\big)\,;\\
[v,w]_{\theta',\gamma'} &= B_{\mathfrak{a}}\left(\rho(\cdot)v, w\right);\\
[Z,x]_{\theta',\gamma'} &= Z\left([x,\cdot]\right);\\
[Z_1, v+Z_2]_{\theta',\gamma'} &= 0.
\end{aligned}$$

We have

$$\begin{aligned}
\Phi\big([x,y]_{\theta',\gamma'}\big) = {} & [x,y] + \theta(x,y) + d^1\tau(x,y) + \gamma(x,y,\cdot)\\
& + d^2\sigma(x,y,\cdot) - B_{\mathfrak{a}}\big((\theta + \tfrac{1}{2}d^1\tau)\wedge\tau\big)(x,y,\cdot)\\
& - \tau\left([x,y]\right) - \sigma\left([x,y],\cdot\right) + \frac{1}{2}B_{\mathfrak{a}}\big(\tau([x,y],\tau(\cdot)\big)\\
& + B_{\mathfrak{a}}\big((\theta(x,y) + d^1\tau(x,y),\tau(\cdot)\big)\,.
\end{aligned}$$

Hence, by (12) , (35) and (34), we obtain

$$\begin{aligned}
\Phi\big([x,y]_{\theta',\gamma'}\big) = {} & [x,y] + \theta(x,y) + \gamma(x,y,\cdot) - \rho(x)\tau(y) - \rho(y)\tau(x)\\
& - \sigma\left(y,[x,\cdot]\right) - \sigma\left(x,[y,\cdot]\right)\\
& - B_{\mathfrak{a}}\left(\theta(x,\cdot),\tau(y)\right) - B_{\mathfrak{a}}\left(\theta(y,\cdot),\tau(x)\right)\\
& - \frac{1}{2}B_{\mathfrak{a}}\Big(\tau\left([x,\cdot]\right),\tau(y)\Big) - \frac{1}{2}B_{\mathfrak{a}}\Big(\tau\left([y,\cdot]\right),\tau(x)\Big) + B_{\mathfrak{a}}\Big(\rho(\cdot)\tau(x),\tau(y)\Big).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& [\Phi(x),\Phi(y)]_{\theta,\gamma}\\
= {} & \Big[x - \tau(x) - \sigma(x,\cdot) - \frac{1}{2}B_{\mathfrak{a}}\big(\tau(x),\tau(\cdot)\big), y - \tau(y) - \sigma(y,\cdot) - \frac{1}{2}B_{\mathfrak{a}}\big(\tau(y),\tau(\cdot)\big)\Big]_{\theta,\gamma}\\
= {} & [x,y] + \theta(x,y) + \gamma(x,y,\cdot) - \rho(x)\tau(y) - B_{\mathfrak{a}}\left(\theta(\cdot,x),\tau(y)\right)\\
& - \sigma(y,[x,\cdot]) - \frac{1}{2}B_{\mathfrak{a}}\Big(\tau(y),\tau([x,\cdot])\Big) - \rho(y)\tau(x) - B_{\mathfrak{a}}\left(\theta(\cdot,y),\tau(x)\right)\\
& + B_{\mathfrak{a}}\left(\rho(\cdot)\tau(x),\tau(y)\right) - \sigma(x,[y,\cdot]) - \frac{1}{2}B_{\mathfrak{a}}\Big(\tau(x),\tau\left([y,\cdot]\right)\Big).
\end{aligned}$$

Therefore $\Phi\big([x,y]_{\theta',\gamma'}\big) = [\Phi(x),\Phi(y)]_{\theta,\gamma}$.

Similarly, we show that $\Phi\big([x,w]_{\theta',\gamma'}\big) = [\Phi(x),\Phi(w)]_{\theta,\gamma}$, $\Phi\big([x,Z]_{\theta',\gamma'}\big) = [\Phi(x),\Phi(Z)]_{\theta,\gamma}$, $\Phi\big([v,w]_{\theta',\gamma'}\big) = [\Phi(v),\Phi(w)]_{\theta,\gamma}$.

□

Remark 4.2. We have $B\left(\Phi(x),\Phi(y)\right) = 2\sigma(x,y)$ and $B(x,y) = 0$.

Let G the subgroup of $C^1_Q(\mathfrak{J},\mathfrak{a})$ generated by the set

$$\{(\tau,\sigma) \in C^1_Q(\mathfrak{J},\mathfrak{a}) \mid d^2\sigma = 0\}.$$

Hence, we have a new 2^{nd} quadratic cohomology group of the metric Hom-Jacobi–Jordan algebra $\mathfrak{J}$ on $\mathfrak{a} \times \mathfrak{J}^*$, with the action "$\bullet$". That is

$$H^2_Q(\mathfrak{J}, \mathfrak{a}) = Z^2_Q(\mathfrak{J}, \mathfrak{a})/G.$$

5. Quadratic extensions

In this section, we study quadratic extensions of Hom-Jacobi–Jordan algebras and we show that they are classified by the cohomology group $H^2_Q(\mathfrak{J}, \mathfrak{a})$. Let $(\mathfrak{J}, [\cdot,\cdot]_{\mathfrak{J}}, \alpha_{\mathfrak{J}}, B)$ be a metric of Hom-Jacobi–Jordan algebra and I an isotropic ideal of $\mathfrak{J}$. For all $x, y \in \mathfrak{J}$, we denote $[\pi_n(x), \pi_n(y)]_{\overline{\mathfrak{J}}} = \pi_n([x,y])$, $\overline{\alpha_{\mathfrak{J}}}(\pi_n(x)) = \pi_n \circ \alpha_{\mathfrak{J}}(x)$ and $\overline{B}(\pi_n(x), \pi_n(y)) = B(x,y)$ where π_n is the natural projection $\mathfrak{J} \to \mathfrak{J}/I$. If $i\colon \mathfrak{a} \to \mathfrak{J}$ is a homomorphism, we denote $\overline{i} = \pi_n \circ i$.

Definition 5.1. Let $(J, [\cdot,\cdot], \alpha)$ be a Hom-Jacobi–Jordan algebra, let I be an isotropic ideal in J and $(\mathfrak{a}, \rho, \beta, B_{\mathfrak{a}})$ a quadratic representation of J. *A quadratic extension* $(\mathfrak{J}, I, i, \pi)$ *of* J *by* $\mathfrak{a}$ *is an exact sequence*

$$0 \longrightarrow (\mathfrak{a}, \rho, \beta) \xrightarrow{\overline{i}} \left(\mathfrak{J}/I, [\cdot,\cdot]_{\overline{\mathfrak{J}}}, \overline{\alpha_{\mathfrak{J}}}, \overline{B}\right) \xrightarrow{\pi} (J, [\cdot,\cdot], \alpha) \longrightarrow 0$$

such that $(\mathfrak{J}, [\cdot,\cdot]_{\mathfrak{J}}, \alpha_{\mathfrak{J}}, B)$ is a metric Hom-Jacobi–Jordan algebra, $\overline{\alpha_{\mathfrak{J}}} \circ \overline{i} = \overline{i} \circ \beta$, $\alpha \circ \pi = \pi \circ \overline{\alpha_{\mathfrak{J}}}$, $\overline{i}(\mathfrak{a}) = I^{\perp}/I$ and $\overline{i}\colon \mathfrak{a} \to I^{\perp}/I$ is an isometry.

Proposition 5.1. *Let*

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\overline{i}} \mathfrak{J}/I \xrightarrow{\pi} J \longrightarrow 0, \tag{52}$$

be an extension of J by $\mathfrak{a}$ such that $i\colon \mathfrak{a} \to i(\mathfrak{a})$ is an isometry. Then the quadruple $(\mathfrak{J}, I, i, \pi)$ defines a quadratic extension if and only if the following sequence defines an extension of $\mathfrak{J}/I$ by J^*:

$$0 \longrightarrow J^* \xrightarrow{\tilde{\pi}^*} \mathfrak{J} \xrightarrow{\pi_n} \mathfrak{J}/I \longrightarrow 0, \tag{53}$$

where π_n is the natural projection $\mathfrak{J} \to \mathfrak{J}/I$, $\tilde{\pi} = \pi \circ \pi_n$, $\tilde{\pi}^*$ the dual map of $\tilde{\pi}$ where we identify J^* with J.

Proof. We have that

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\overline{i}} \mathfrak{J}/I \xrightarrow{\pi} J \longrightarrow 0,$$

is an extension of J by $\mathfrak{a}$ such that $i\colon \mathfrak{a} \to i(\mathfrak{a})$ is an isometry. Then

$$\overline{\alpha_{\mathfrak{J}}} \circ \overline{i} = \overline{i} \circ \beta, \tag{54}$$

$$\alpha \circ \pi = \pi \circ \overline{\alpha_{\mathfrak{J}}}, \tag{55}$$

$$\overline{i}(\mathfrak{a}) = \ker \pi, \tag{56}$$

$$B(i(v), i(w)) = B(v, w). \tag{57}$$

We assume that $(\mathfrak{J}, I, i, \pi)$ is a quadratic extension. Then $Im(i) = I^{\perp}/I$.

First, we show that $\alpha_{\mathfrak{J}}^* \circ \tilde{\pi}^* = \tilde{\pi}^* \circ \alpha^*$. We have

$$\alpha \circ \pi = \pi \circ \overline{\alpha_{\mathfrak{J}}} = \pi \circ \pi_n \circ \alpha_{\mathfrak{J}} = \tilde{\pi} \circ \alpha_{\mathfrak{J}}.$$

Hence $(\alpha \circ \pi)^* = (\tilde{\pi} \circ \alpha_{\mathfrak{J}})^*$. Then $\pi^* \circ \alpha^* = \alpha_{\mathfrak{J}}^* \circ \tilde{\pi}^*$.

Now, we show that $Im(\tilde{\pi}^*) = \ker(\pi_n)$. By $\ker \pi = i(\mathfrak{a}) = I^{\perp}/I$ and $\tilde{\pi} = \pi \circ \pi_n$ we obtain $\ker(\tilde{\pi}) = I^{\perp}$. Since $Im(\tilde{\pi}^*) = (\ker(\tilde{\pi}))^{\perp}$, one can deduce $Im(\tilde{\pi}^*) = I$. So $Im(\tilde{\pi}^*) = \ker(\pi_n)$ and the sequence

$$0 \longrightarrow J^* \xrightarrow{\tilde{\pi}^*} \mathfrak{J}^* \cong \mathfrak{J} \xrightarrow{\pi_n} \mathfrak{J}/I \longrightarrow 0,$$

defines an extension of $\mathfrak{J}/I$ by J^*.

Conversely, we assume that the sequence

$$0 \longrightarrow J^* \xrightarrow{\tilde{\pi}^*} \mathfrak{J}^* \cong \mathfrak{J} \xrightarrow{\pi_n} \mathfrak{J}/I \longrightarrow 0$$

defines an extension. Then $\alpha_{\mathfrak{J}}^* \circ \tilde{\pi}^* = \pi^* \circ \alpha^*$, $\overline{\alpha_{\mathfrak{J}}} \circ \pi_n = \pi_n \circ \alpha_{\mathfrak{J}}$ and $Im(\tilde{\pi}^*) = \ker(\pi_n)$. We have $Im(\tilde{\pi}^*) = (\ker(\tilde{\pi}))^{\perp}$, $Im(\tilde{\pi}^*) = \ker(\pi_n)$ and $\ker(\pi_n) = I$. Hence, $\ker(\tilde{\pi}) = I^{\perp}$ and $I \subset I^{\perp}$. Then $\ker(\pi) = I^{\perp}/I$. By (56), we have $Im(\bar{i}) = \ker(\pi) = I^{\perp}/I$. Moreover, we have (54), (55) and (57). Therefore, $(\mathfrak{J}, I, i, \pi)$ is a quadratic extension. □

5.1. Twofold extensions. Twofold extensions of Lie algebras were studied in [10] (also called Standard models in [9]). In the following, we define and study twofold extensions of Hom-Jacobi–Jordan algebras.

Let $(J, [\cdot,\cdot], \alpha)$ be a Hom-Jacobi–Jordan algebra and let $(\mathfrak{a}, \rho, \beta, B_{\mathfrak{a}})$ be a quadratic representation of J. For each $(\theta, \gamma) \in Z_Q^2(J, \mathfrak{a})$, we want to define structures of a metric Hom-Jacobi–Jordan algebra on the vector space $\mathfrak{d}_{\theta,\gamma} := J \oplus \mathfrak{a} \oplus J^*$. Let $\alpha_{\mathfrak{d}_{\theta,\gamma}} = \alpha + \beta + \alpha^*$. We define a bracket on $\mathfrak{d}_{\theta,\gamma}$ by

$$\begin{aligned}
[x, y]_{\theta,\gamma} &= [x, y] + \theta(x, y) + \gamma(x, y, \cdot);\\
[x, v]_{\theta,\gamma} &= \rho(x)v + B_{\mathfrak{a}}(\theta(\cdot, x), v);\\
[v, w]_{\theta,\gamma} &= B_{\mathfrak{a}}(\rho(\cdot)v, w);\\
[Z, x]_{\theta,\gamma} &= Z([x, \cdot]);\\
[Z_1, v + Z_2]_{\theta,\gamma} &= 0.
\end{aligned}$$

We define a symmetric bilinear form B on $\mathfrak{d}_{\theta,\gamma}$ by

$$B(x + v + Z_1, y + w + Z_2) = Z_1(y) + Z_2(x) + B_{\mathfrak{a}}(v, w)$$

for all $x, y \in J$, $v, w \in \mathfrak{a}$, $Z_1, Z_2 \in J^*$. We define a linear map $i_0\colon \mathfrak{a}_{\theta,\gamma} \to \mathfrak{d}_{\theta,\gamma}/J^*$ by $i_0(v) = v + J^*$ and a linear map $\pi_0\colon \mathfrak{d}_{\theta,\gamma}/J^* \to J$ by $\pi_0(x + v + J^*) = x$.

Proposition 5.2. *With the above notations, the quadruple* $(\mathfrak{d}_{\theta,\gamma}, J^*, i_0, \pi_0)$ *defines a quadratic extension.*

Proof. We only prove that $\left(\mathfrak{d}_{\theta,\gamma},[\cdot,\cdot]_{\theta,\gamma},\alpha_{\mathfrak{d}_{\theta,\gamma}},B\right)$ is a metric Hom-Jordan–Jacobi algebra. Denote $\mathfrak{d}_{\theta,\gamma}=\mathfrak{n}$ and define a trilinear form $\gamma_{\mathfrak{n}}$ on $\mathfrak{n}$ by $\gamma_{\mathfrak{n}}(a,b,c)=B([a,b]_{\theta,\gamma},c)$ for all $a,b,c\in\mathfrak{n}$. Using Theorem 3.3, it is sufficient to show that $\gamma_{\mathfrak{n}}$ is symmetric and $d_r^3\gamma_{\mathfrak{n}}=0$.

We have

$$\gamma_{\mathfrak{n}}(x,y,z)=B\left([x,y]_{\theta,\gamma},z\right)=B\left([x,y]+\theta(x,y)+\gamma(x,y,\cdot),z\right)=\gamma(x,y,z).$$

Since γ is symmetric, we obtain that the restriction of $\gamma_{\mathfrak{n}}$ to J^3 is symmetric. For all $x,y\in J$, $v\in\mathfrak{a}$, we have

$$\gamma_{\mathfrak{n}}(x,y,v)=B\left([x,y]_{\theta,\gamma},v\right)=B_{\mathfrak{a}}(\theta(x,y),v);$$
$$\gamma_{\mathfrak{n}}(x,v,y)=B\left([x,v]_{\theta,\gamma},y\right)=B_{\mathfrak{a}}(\theta(x,y),v).$$

Therefore, using the fact that $[x,y]_{\theta,\gamma}=[y,x]_{\theta,\gamma}$ and $[x,v]_{\theta,\gamma}=[v,x]_{\theta,\gamma}$, one can deduce that the restriction of $\gamma_{\mathfrak{n}}$ to $J^2\times V$ is symmetric.

For all $x\in J$, $v,w\in\mathfrak{a}$, we have

$$\gamma_{\mathfrak{n}}(x,v,w)=B\left([x,v]_{\theta,\gamma},w\right)=B_{\mathfrak{a}}(\rho(x)w,v),$$
$$\gamma_{\mathfrak{n}}(v,w,x)=B\left([v,w]_{\theta,\gamma},x\right)=B_{\mathfrak{a}}\left(\rho(x)v,w\right),$$

and since $(\mathfrak{a},\rho,\beta,B_{\mathfrak{a}})$ is a quadratic representation of J on $\mathfrak{a}$, the restriction of $\gamma_{\mathfrak{n}}$ to $J\times V^2$ is symmetric.

For all $u,v,w\in\mathfrak{a}$, we have

$$\gamma_{\mathfrak{n}}(v,w,u)=B\left([v,w]_{\theta,\gamma},u\right)=B\left(B_{\mathfrak{a}}\left(\rho(\cdot)v,w\right),u\right)=0.$$

Thus, the restriction of $\gamma_{\mathfrak{n}}$ to V^3 is symmetric too.

For all $x,y,z,a\in J$ and for $t=\alpha(a)$, we have

$$\begin{aligned}
&d_r^3\gamma_{\mathfrak{n}}(x,y,z,t)\\
=&\gamma\left([x,y],\alpha(z),t\right)+\gamma\left([x,z],\alpha(y),t\right)+\gamma\left([y,z],\alpha(x),t\right) &(58)\\
&+\gamma(x,y,[\alpha(z),t])+\gamma(x,z,[\alpha(y),t])+\gamma(y,z,[\alpha(x),t]) &(59)\\
&+\gamma(\alpha(z),t,[x,y])+\gamma(\alpha(y),t,[x,z])+\gamma(\alpha(x),t,[y,z]) &(60)\\
&+\gamma\left([\alpha(z),t],x,y\right)+\gamma\left([\alpha(y),t],x,z\right)+\gamma\left([\alpha(x),t],y,z\right) &(61)\\
&+B_{\mathfrak{a}}\left(\theta(y,x),\theta(\alpha(z),t)\right)+B_{\mathfrak{a}}\left(\theta(z,x),\theta(\alpha(y),t)\right)+B_{\mathfrak{a}}\left(\theta(z,y),\theta(\alpha(x),t)\right) &(62)\\
&+B_{\mathfrak{a}}\left(\theta(t,\alpha(z)),\theta(x,y)\right)+B_{\mathfrak{a}}\left(\theta(t,\alpha(y)),\theta(x,z)\right)+B_{\mathfrak{a}}\left(\theta(t,\alpha(x)),\theta(y,z)\right). &(63)
\end{aligned}$$

Since γ is symmetric, we get

$$(58)+(59)=d_r\gamma(x,y,z,t)\text{ and }(60)+(61)=d_r\gamma(x,y,z,t).$$

Since θ is a 2-Hom-cochain, by Proposition 4.5, we obtain

$$(62)+(63)=B_{\mathfrak{a}}\left(\theta\wedge(\theta\circ\alpha)\right)(x,y,z,a).$$

Thus $d_r^3\gamma_{\mathfrak{n}}(x,y,z,t) = 2d_r\gamma(x,y,z,t) + B_{\mathfrak{a}}(\theta\wedge(\theta\circ\alpha))(x,y,z,a)$. Then, since (θ,γ) is a quadratic 2-cocycle, we obtain $d_r^3\gamma_{\mathfrak{n}}(x,y,z,t)=0$. By straightforward computations, for all $x,y,z\in J$, $v\in\mathfrak{a}$, we have

$$\begin{aligned}
&\frac{1}{2}d_r^3\gamma_{\mathfrak{n}}(x,y,z,v)\\
=&B_{\mathfrak{a}}(\theta([x,y],\alpha(z)),v)+B_{\mathfrak{a}}(\theta([x,z],\alpha(y)),v)+B_{\mathfrak{a}}(\theta([y,z],\alpha(x)),v)\\
&+B(\rho(\alpha(z))\theta(x,y),v)+B(\rho(\alpha(x))\theta(y,z),v)+B(\rho(\alpha(y))\theta(x,z),v)\\
=&\frac{1}{2}B_{\mathfrak{a}}\left(d^2\theta(x,y,z),v\right).
\end{aligned}$$

Therefore $d^3\gamma_{\mathfrak{n}}(x,y,z,v)=0$ by (θ,γ) is a quadratic 2-cocycle.

Similarly, for any $x,y\in J$, $u,v\in\mathfrak{a}$, we get

$$\begin{aligned}
&\frac{1}{2}d_r^3\gamma_{\mathfrak{n}}(x,y,u,v)\\
=&B_{\mathfrak{n}}\Big(u,\beta(\rho([x,y])v)\Big)+B_{\mathfrak{a}}(u,\rho(x)\rho(\alpha(y))v)+B_{\mathfrak{a}}(u,\rho(y)\rho(\alpha(x))v)
\end{aligned}$$

Therefore, by (15), we have $d_r^3\gamma_{\mathfrak{n}}(x,y,u,v)=0$. For all $x\in J$, $u,v,w,s\in\mathfrak{a}$, $Z\in J^*$, by $B(Z,u)=0$, we have $d^3\gamma_{\mathfrak{n}}(u,v,w,x)=0$, $d^3\gamma_{\mathfrak{n}}(u,v,x,w)=0$ and $d^3\gamma_{\mathfrak{n}}(u,v,w,s)=0$. The rest of the proof is straightforward. □

Definition 5.2. We denote the quadratic extension$(\mathfrak{d}_{\theta,\gamma},J^*,i_0,\pi_0)$, constructed in Proposition 5.2, by $\mathfrak{d}_{\theta,\gamma}(\mathfrak{a},J,\rho)$ and call it a *twofold extension.*

5.2. Classification by cohomology. In this subsection, we show that quadratic extensions are classified by the cohomology group $H_Q^2(\mathfrak{J},\mathfrak{a})$.

Definition 5.3. Two quadratic extensions $(\mathfrak{J}_1,I_1,i_1,\pi_1)$, $(\mathfrak{J}_2,I_2,i_2,\pi_2)$ of J by $\mathfrak{a}$ are called to be *equivalent* if there exists an isomorphism of metric Lie algebras $\Phi\colon\mathfrak{J}_1\to\mathfrak{J}_2$ which maps i_1 onto i_2 and satisfies $\overline{\Phi}\circ i_1=i_2$ and $\pi_2\circ\overline{\Phi}=\pi_1$, where $\overline{\Phi}\colon\mathfrak{J}_1/I_1\to\mathfrak{J}_2/I_2$ is the induced map.

Proposition 5.3. *Any quadratic extension* $(\mathfrak{J},I,i,\pi)$ *is equivalent to a twofold extension* $(\mathfrak{d}_{\theta,\gamma},J^*,i_0,\pi_0)$.

Proof. Let

$$\mathcal{E}:0\longrightarrow\mathfrak{a}\xrightarrow{\bar{i}}\mathfrak{J}/I\xrightarrow{\pi}J\longrightarrow 0$$

be the extension of J defined in (52) and s a section of $\mathcal{E}$. Then, by Proposition 2.8, we have $\mathfrak{J}/I=s(J)\oplus\bar{i}(\mathfrak{a})$ and the extension $\mathcal{E}$ is equivalent to

$$0\longrightarrow(\mathfrak{a},\rho,\beta)\xrightarrow{i_0}(M,[\cdot,\cdot]_\theta,\alpha_M)\xrightarrow{\pi_0}(J,[\cdot,\cdot],\alpha)\longrightarrow 0,$$

where θ is a 2-cocyle of J on $\mathfrak{a}$ and $M=J\oplus\mathfrak{a}$.

Now, let

$$\mathcal{E}^*:0\longrightarrow J^*\xrightarrow{\tilde{\pi}^*}\mathfrak{J}\xrightarrow{\pi_n}\mathfrak{J}/I\longrightarrow 0$$

be the extension defined in (53) and s' a section of $\mathcal{E}^*$. Then, by Proposition 2.8, we have $\mathfrak{J} = s'(J/I) \oplus \tilde{\pi}^*(J^*)$ and the extension $\mathcal{E}^*$ is equivalent to

$$0 \longrightarrow \left(J^*, \rho', \beta'\right) \xrightarrow{i_0} \left(M', [\cdot,\cdot]_{\gamma'}, \alpha_{M'}\right) \xrightarrow{\pi_0} \left(\mathfrak{J}/I, [\cdot,\cdot]_{\overline{\mathfrak{J}}}, \overline{\alpha_{\mathfrak{J}}}\right) \longrightarrow 0$$

where γ' is a 2-cocycle of $\mathfrak{J}/I$ on J^* and $M' = \mathfrak{J}/I \oplus J^*$.

We have $\mathfrak{J} = s'(\mathfrak{J}/I) \oplus \tilde{\pi}^*(J^*) = s'(s(J) \oplus i(\mathfrak{a})) \oplus \tilde{\pi}^*(J^*)$. We can write $\pi_n \colon s'(\mathfrak{J}/I) \to \mathfrak{J}/I$ and $\pi \colon s(J) \to J$. Hence $\tilde{\pi}^*(J^*) = (s's(J))^*$.

Using $\mathfrak{J} = s'(J/I) \oplus \tilde{\pi}^*(J^*)$ and $\tilde{\pi}^*(J^*) = (s's(J))^*$, we obtain $\mathfrak{J} = s's(J) \oplus s'i(\mathfrak{a}) \oplus (s's(J))^*$. Then, using Proposition 4.3, for all $x \in J$, $v \in \mathfrak{a}$, $Z \in \mathfrak{J}^*$, we have

$$\begin{aligned}
[s's(x), s's(y)]_{\mathfrak{J}} &= [s's(x), s's(y)]_{s's(J)} + \theta(s's(x), s's(y)) + \gamma(s's(x), s's(y), \cdot);\\
[s's(x), s'i(v)]_{\mathfrak{J}} &= \rho(s's(x))v + B_\rho(s'i(v), \theta(s's(x), \cdot));\\
[s'i(v), s'i(w)]_{\mathfrak{J}} &= B_{\mathfrak{a}}(\rho(\cdot)(s'i(v)), s'i(w));\\
[Z, s's(x)]_{\mathfrak{J}} &= Z([s's(x), \cdot]);\\
[Z_1, s'i(v) + Z_2]_{\mathfrak{J}} &= 0.
\end{aligned}$$

Now, we define a linear map $\Psi \colon J \oplus \mathfrak{a} \oplus J^* \to \mathfrak{J}$ by $\Psi(x + v + Z) = s's(x) + s'i(v) + (s's)^*(Z)$ and a bilinear map $[\cdot,\cdot]_{\mathfrak{d}} \colon J \oplus \mathfrak{a} \oplus J^* \to J \oplus \mathfrak{a} \oplus J^*$ by

$$[x+v+Z, y+w+Z']_{\mathfrak{d}} = \Psi^{-1}\left([s's(x)+s'i(v)+(s's)^*(Z), s's(y)+s'i(w)+(s's)^*(Z')]_{\mathfrak{J}}\right).$$

Then

$$\begin{aligned}
&\left[\Psi(x+v+Z), \Psi(y+w+Z')\right]_{\mathfrak{J}}\\
&= \left[s's(x) + s'i(v) + (s's)^*(Z), s's(y) + s'i(w) + (s's)^*(Z')\right]_{\mathfrak{J}}\\
&= \Psi\left([x+v+Z, y+w+Z']_{\mathfrak{d}}\right).
\end{aligned}$$

Moreover, we have $\overline{\Psi} \circ i_0(v) = i(v)$ and $\pi \circ \overline{\Psi}(\overline{x}) = \pi \circ s(x) = x = \pi_0(x)$. $\square$

Lemma 5.4. *Let $\mathfrak{d}_{\theta,\gamma} := \mathfrak{d}_{\theta,\gamma}(\mathfrak{a}, J, \rho)$ and $\mathfrak{d}_{\theta',\gamma'} := \mathfrak{d}_{\theta',\gamma'}(\mathfrak{a}, J, \rho)$ be two twofold extensions such that $(\theta,\gamma) \cong (\theta',\gamma')$. Then the twofold extensions $\mathfrak{d}_{\theta,\gamma} := \mathfrak{d}_{\theta,\gamma}(\mathfrak{a}, J, \rho)$ and $\mathfrak{d}_{\theta',\gamma'} := \mathfrak{d}_{\theta',\gamma'}(\mathfrak{a}, J, \rho)$ are equivalent.*

Proof. Using Theorem 4.9, we have $\theta' = \theta + d^1_r\tau$ and $\gamma' = \gamma + d^2_r\sigma - B\left(\tau \wedge (\theta + \frac{1}{2}d^1\tau)\right)$ where $(\tau,\sigma) \in G$. Then, $d^2_r\sigma = 0$. Define a linear map $\Phi \colon J \oplus \mathfrak{a} \oplus J^* \to J \oplus \mathfrak{a} \oplus J^*$ by

$$\Phi(x+v+Z) = x + \underbrace{v - \tau(x)}_{\in \mathfrak{a}} + \underbrace{Z - \frac{1}{2}B_{\mathfrak{a}}\big(\tau(x), \tau(\cdot)\big) + B_{\mathfrak{a}}(v, \tau(\cdot))}_{\in J^*}.$$

Then Φ is an isomorphism of metric Hom-Jacobi-algebras (see the proof of

Proposition 4.11). Finally, we show that Φ is isometric:

$$\begin{aligned}B(\Phi(x),\Phi(y)) &= B\left(x-\tau(x)-\frac{1}{2}B_{\mathfrak{a}}\big(\tau(x),\tau(\cdot)\big), y-\tau(y)-\frac{1}{2}B_{\mathfrak{a}}\big(\tau(y),\tau(\cdot)\big)\right)\\ &= B_{\mathfrak{a}}\left(\tau(x),\tau(y)\right)-\frac{1}{2}B_{\mathfrak{a}}\big(\tau(y),\tau(x)\big)-\frac{1}{2}B_{\mathfrak{a}}\big(\tau(x),\tau(y)\big)\\ &= 0 = B(x,y),\end{aligned}$$

$$\begin{aligned}B(\Phi(x),\Phi(v)) &= B(x-\tau(x)-\frac{1}{2}B_{\mathfrak{a}}\big(\tau(x),\tau(\cdot)\big), v+B_{\mathfrak{a}}(v,\tau(\cdot)))\\ &= -B_{\mathfrak{a}}(\tau(x),v)+B_{\mathfrak{a}}(v,\tau(x))=0\end{aligned}$$

$$\begin{aligned}B(\Phi(u),\Phi(v)) &= B(u+B_{\mathfrak{a}}(u,\tau(\cdot)), v+B_{\mathfrak{a}}(v,\tau(\cdot)))\\ &= B_{\mathfrak{a}}(u,v).\end{aligned}$$

□

Lemma 5.5. *Let $\mathfrak{d}_{\alpha,\gamma} := \mathfrak{d}_{\theta,\gamma}(\mathfrak{a},J,\rho)$ and $\mathfrak{d}_{\theta',\gamma'} := \mathfrak{d}_{\theta',\gamma'}(\mathfrak{a},J,\rho)$ be two equivalent twofold extensions. Then the quadratic 2-cocycle $(\theta-\theta',\gamma-\gamma')$ is trivial.*

Proof. Let $\Phi(x) = f(x)+\tau(x)+\zeta(x)$ where $f\colon J\to J$, $\tau\colon J\to\mathfrak{a}$ and $\zeta\colon J\to J^*$. Using $\pi \circ \Phi' = \pi$, we obtain $f(x)=x$. Then

$$\Phi(x) = x+\tau(x)+\zeta(x).$$

Let $\Phi(v) = g(v)+h(v)+\eta(v)$, where $g\colon \mathfrak{a}\to J$, $h\colon\mathfrak{a}\to\mathfrak{a}$ and $\eta\colon\mathfrak{a}\to J^*$. Using $\Phi'\circ i = i$, we obtain $g(v)=0$ and $h(v)=v$. Then $\Phi(v)=v+\eta(v)$. Using $B(v,x)=B(\Phi(v),\Phi(x))$, we obtain $\eta(v)(x) = -B_{\mathfrak{a}}(v,\tau(x))$. Since Φ is an isometry and $\Phi(J^*)\subset J^*$, we obtain $\Phi(Z)=Z$.

Using $B(\Phi(x),\Phi(y)) = B(x,y)$, we obtain $B_{\mathfrak{a}}(\tau(x),\tau(y)) = -\zeta(x)(y)-\zeta(y)(x)$. Since $\zeta(x)(y)=\zeta(y)(x)$, we obtain $\zeta(x,y) = -\frac{1}{2}B_{\mathfrak{a}}(\tau(x),\tau(y))$. By $\Phi(d(x,y)) = d'(\Phi(x),\Phi(y))$, we obtain

$$\theta(x,y) = \theta'(x,y)-\tau([(x,y])+\rho(x)\tau(y)+\rho(y)\tau(x) = \theta'(x,y)-d^1\tau(x,y)$$

and

$$\gamma(x,y,\cdot) = \gamma'(x,y,\cdot)-B_{\mathfrak{a}}\left((\theta'+\frac{1}{2}d(-\tau))\wedge(-\tau)\right)(x,y,\cdot).$$

Hence

$$\begin{cases}\theta = \theta'+d^1\left(-\tau\right),\\ \gamma = \gamma'(x,y,\cdot)-B_{\mathfrak{a}}\left((\alpha'+\frac{1}{2}d(-\tau))\wedge(-\tau)\right)(x,y,\cdot).\end{cases}$$

Using Proposition 2.2, we have $d_c^2\theta=0$. Therefore, using Proposition 4.9, we have $d_Q^2(\theta,\gamma) = d_Q^2(\theta',\gamma')$. □

Bringing the previous results together, we have the following result.

Theorem 5.6. *The set* $Ext(J, \mathfrak{a})$ *of equivalence classes of quadratic extensions* $(\mathfrak{J}, I, i, \pi)$ *of* J *by* $\mathfrak{a}$ *is in a one-to-one correspondence with* $Z^2_Q(J, \mathfrak{a})/G$*, that is,*

$$Ext(J, \mathfrak{a}) \cong H^2_Q(J, \mathfrak{a}).$$

References

[1] A. L. Agore and G. Militaru, *On a type of commutative algebras*, Linear Algebra Appl. **485** (2015), 222–249.
[2] F. Ammar, Z. Ejbehi, and A. Makhlouf, *Cohomology and deformations of Hom-algebras*, J. Lie Theory **21** (2011), 813–836.
[3] S. Benayadi and A. Makhlouf, *Hom-Lie algebras with symmetric invariant nondegenerate bilinear forms*, J. Geom. Phys. **76** (2014), 38–60.
[4] D. Burde and A. Fialowski, *Jacobi–Jordan algebras*, Linear Algebra Appl. **459** (2014), 586–594.
[5] J. M. Casas, M. A. Insua, and N. Pacheco Rego, *On universal central extensions of Hom-Lie algebras*, Hacet. J. Math. Stat. **44** (2015), 277–288.
[6] C. E. Haliya and G. D. Houndedji, *Hom-Jacobi–Jordan and Hom-antiassociative algebras with symmetric invariant nondegenerate bilinear forms*, Quasigroups Related Systems **29** (2021), 61–88.
[7] J. T. Hartwig, D. Larsson, and S. D. Silvestrov, *Deformations of Lie algebras using* σ*-derivations*, J. Algebra **295** (2006), 314–361.
[8] Q. Jin and X. Li, *Hom-Lie algebra structures on semi-simple Lie algebras*, J. Algebra **319** (2008), 1398–408.
[9] I. Kath and M. Olibrich, *Metric Lie algebras and quadratic extensions*, Transform. Groups **11** (2006), 87–131.
[10] I. Kath and M. Olbrich, *Metric Lie algebras with maximal isotropic centre*, Math. Z. **246** (2004), 23–53.
[11] D. Larsson and S. D. Silvestrov, *Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities.* J. Algebra **288** (2005), 321–344.
[12] A. Makhlouf and S. D. Silvestrov, *Hom-algebra structures*, J. Gen. Lie Theory Appl. **2** (2008), 51–64.
[13] A. Makhlouf and S. D. Silvestrov, *Notes on 1-parameter formal deformations of Hom-associative and Hom-Lie algebras*, Forum Math. **22** (2010), 715–739.
[14] A. Medina and P. Revoy, *Algèbres de Lie et produit scalaire invariant*, Ann. Sci. l'É.N.S. **18** (1985), 553–561.
[15] J. V. Neumann, P. Jordan, and E. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, Ann. Math. **35** (1934), 29–64.
[16] N. Saadaoui, *Extensions of Hom-Jacobi–Jordan algebras*, 2022, 21 pp. URL
[17] Y. Sheng, *Representations of Hom-Lie algebras*, Algebr. Represent. Theory **15** (2012), 1081–1098.
[18] S. Okubo and K. Noriaki, *Jordan-Lie super algebra and Jordan-Lie triple system*, J. Algebra **198** (1997), 388–411.
[19] A. Wörz-Busekroz, *Bernstein algebras*, Arch. Math. **48** (1987), 388–398.
[20] P. Zusmanovich, *Special and exceptional mock-Lie algebras*, Linear Algebra Appl. **518** (2017), 79–96.

University of Gabes, Higher Institute of Computer Science and Multimedia Gabes, City Elamal 4, Gabes, Tunisia
E-mail address: `najib.saadaoui@isimg.tn`