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Narayana numbers as products of three repdigits in base g

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Abstract. In this paper, we show that there are only finitely many Narayana's numbers which can be written as a product of three repdigits in base g with $g \geq 2$. Moreover, for $2 \leq g \leq 10$, we determine all these numbers.

1. Introduction

The problems of the terms of linear recurrence sequences written as a product of repdigits in any base have been intensely studied by several researchers specialized in Number Theory. In this article, we consider the linear recurrent sequence of the third order – the Narayana's cows numbers defined as follows:

$$\mathcal{N}_n = \mathcal{N}_{n-1} + \mathcal{N}_{n-3} \quad \text{for} \quad n \geq 3 \quad \text{with} \quad \mathcal{N}_0 = 0, \quad \mathcal{N}_1 = \mathcal{N}_2 = 1.$$

For more details on the work related to the determination of the terms of linear recurrent sequences which are repdigits in any base, we refer the reader to the following recent results [1]–[3], [6], [10].

The concept of Narayana's cows numbers, derived from Indian mythology and Hinduism, holds a significant place in mathematics. These numbers have been extensively studied due to their properties and relationships with other mathematical sequences, and their important applications in other various fields such as cryptography, coding theory, and graph theory. In this paper, we delve into a fascinating aspect of Narayana numbers by examining their representation as a product of three repdigits in base g with $g \geq 2$.

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Repdigits, which consist of repeated digits, have garnered attention for their mathematical properties and patterns. In a fixed base $g \geq 2$, a repdigit has the following form:

$$\sum_{i=0}^{n-1} d \times g^i = d \times \frac{g^n - 1}{g - 1},$$

where $1 \leq d \leq g-1$ and n is a positive integer.

The proofs of our main results are based on a double application of Baker's method and on a reduction algorithm using computations based on continued fractions. The method used to determine Narayana numbers, which are products of three repdigits is similar to that used by Adédji [2] and by Adédji et al. [1].

The present paper is organized as follows: in Section 2, we present our main results, Section 3 is devoted to reminding necessary results for the proofs of our results, and in Section 4, we prove our results.

2. Statement of main results

In this section, we state all the main results obtained in this paper.

Theorem 1. *Let $g \geq 2$ be an integer. Then the Diophantine equation*

$$\mathcal{N}_k = d_1 \frac{g^\ell - 1}{g - 1} \cdot d_2 \frac{g^m - 1}{g - 1} \cdot d_3 \frac{g^n - 1}{g - 1} \tag{1}$$

has only finitely many solutions in integers $k, d_1, d_2, d_3, \ell, m, n$ such that $1 \leq d_i \leq g-1$ for $i = 1, 2, 3$ and $n \geq m \geq \ell \geq 1$. Further, we have

$$n < 5.91 \times 10^{49} \log^9 g \quad \text{and} \quad k < 4.73 \times 10^{50} \log^{10} g.$$

Under the notation and assumptions of Theorem 1, if (1) holds for $(k, d_1, d_2, d_3, \ell, m, n)$, then we write

$$\mathcal{N}_k = [a, b, c]_g = a \times b \times c,$$

where

$$a = d_1 \times \frac{g^\ell - 1}{g - 1} = \underbrace{\overline{d_1 \cdots d_1}_g}_{\ell-1 \text{ times}}, \quad b = d_2 \times \frac{g^m - 1}{g - 1} = \underbrace{\overline{d_2 \cdots d_2}_g}_{m-1 \text{ times}},$$

$$\text{and} \quad c = d_3 \times \frac{g^n - 1}{g - 1} = \underbrace{\overline{d_3 \cdots d_3}_g}_{n-1 \text{ times}}.$$

In the following theorem, we completely and explicitly give all solutions of the equation (1) corresponding to $2 \leq g \leq 10$.

Theorem 2. *The only Narayana numbers which are products of three repdigits in base g with* $2 \le g \le 10$ *are*

$$\{1, 2, 3, 4, 6, 9, 13, 28, 60, 88, 129, 189\}.$$

More precisely, we have

TABLE 1. Narayana numbers which are products of three repdigits in base g, $2 \le g \le 10$.

k	$\mathcal{N}_k$	$[a, b, c]_g$
$1, 2, 3$	1	$[1, 1, 1]_g$ *for* $g = 2, \dots, 10$.
4	2	$[1, 1, 2]_g$ *for* $g = 3, \dots, 10$.
5	3	$[1, 1, 11]_2$, $[1, 1, 3]_g$ *for* $g = 4, \dots, 10$.
6	4	$[1, 1, 11]_3$, $[1, 1, 4]_g$ *for* $g = 5, \dots, 10$, $[1, 2, 2]_g$ *for* $g = 3, \dots, 10$.
7	6	$[1, 1, 11]_5$, $[1, 2, 3]_g$ *forg* $= 4, \dots, 10$, $[1, 1, 6]_g$ *for* $g = 7, \dots, 10$.
8	9	$[1, 11, 11]_2$, $[1, 1, 111]_3$, $[1, 1, 11]_8$, $[1, 1, 9]_{10}$, $[1, 3, 3]_g$ *for* $g = 4, \dots, 10$.
9	13	$[1, 1, 111]_3$
11	28	$[1, 1, 44]_6$, $[1, 2, 22]_6$, $[1, 4, 11]_6$, $[2, 2, 11]_6$, $[1, 4, 7]_g$ *for* $g = 8, \dots, 10$, $[2, 2, 7]_g$ *for* $g = 8, \dots, 10$.
13	60	$[2, 2, 33]_4$, $[2, 3, 22]_4$, $[1, 1, 66]_9$, $[1, 2, 33]_9$, $[1, 3, 22]_9$,$[1, 6, 11]_9$, $[2, 3, 11]_9$, $[2, 5, 6]_g$ *for* $g = 7, \dots, 10$, $[3, 4, 5]_g$ *for* $g = 6, \dots, 10$.
14	88	$[1, 1, 88]_{10}$, $[1, 2, 44]_{10}$, $[1, 4, 22]_{10}$, $[1, 8, 11]_{10}$, $[2, 2, 22]_{10}$, $[2, 4, 11]_{10}$.
15	129	$[1, 1, 333]_6$, $[1, 3, 111]_6$.
16	189	$[1, 11, 111111]_2$, $[1, 3, 333]_4$, $[3, 3, 111]_4$, $[3, 3, 33]_6$, $[1, 3, 77]_8$, $[1, 7, 33]_8$, $[3, 7, 11]_8$, $[3, 7, 9]_{10}$.

3. Preliminary results

In this section, we give some notations and recall certain definitions and results required for the proofs of our main results.

3.1. Some properties of Narayana sequence. Narayana's cows sequence comes from a problem with cows proposed by an Indian mathematician Narayana in the 14th century. In this problem, we assume that there is a cow at the beginning and each cow produces a calf every year from the 4th year. Narayana's cow problem counts the number of calves produced each year [4].

The characteristic polynomial of Narayana's cows sequence $\{\mathcal{N}_n\}_{n\geq 0}$ is

$$\varphi(x) = x^3 - x^2 - 1.$$

Furthermore, the zeros of $\varphi(x)$ are

$$\alpha_{\mathcal{N}} = \frac{1}{3}\left(\sqrt[3]{\frac{1}{2}(29-3\sqrt{93})} + \sqrt[3]{\frac{1}{2}(3\sqrt{93}+29)} + 1\right),$$

$$\beta_{\mathcal{N}} = \frac{1}{3} - \frac{1}{6}\left(1 - i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}(29-3\sqrt{93})} - \frac{1}{6}\left(1 + i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}(3\sqrt{93}+29)},$$

$$\gamma_{\mathcal{N}} = \frac{1}{3} - \frac{1}{6}\left(1 + i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}(29-3\sqrt{93})} - \frac{1}{6}\left(1 - i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}(3\sqrt{93}+29)}.$$

Then, the Narayana sequence can be obtained by Binet formula

$$\mathcal{N}_n = a_{\mathcal{N}}\alpha_{\mathcal{N}}^n + b_{\mathcal{N}}\beta_{\mathcal{N}}^n + c_{\mathcal{N}}\gamma_{\mathcal{N}}^n. \tag{2}$$

From the three initial values of Nayarana sequence, and using Vieta's theorem, one has

$$a_{\mathcal{N}} = \frac{\alpha_{\mathcal{N}}^2}{\alpha_{\mathcal{N}}^3+2}, \quad b_{\mathcal{N}} = \frac{\beta_{\mathcal{N}}^2}{\beta_{\mathcal{N}}^3+2}, \quad \text{and} \quad c_{\mathcal{N}} = \frac{\gamma_{\mathcal{N}}^2}{\gamma_{\mathcal{N}}^3+2}.$$

The minimal polynomial of $a_{\mathcal{N}}$ over $\mathbb{Z}$ is $31x^3 - 3x - 1$.

Setting $\Pi(n) = \mathcal{N}_n - a_{\mathcal{N}}\alpha_{\mathcal{N}}^n = b_{\mathcal{N}}\beta_{\mathcal{N}}^n + c_{\mathcal{N}}\gamma_{\mathcal{N}}^n$, we notice that

$$|\Pi(n)| < \frac{1}{\alpha_{\mathcal{N}}^{n/2}} \quad \text{for all } n \geq 1. \tag{3}$$

We note that the characteristic polynomial has a real zero $\alpha_{\mathcal{N}}(> 1)$ and two complex conjugate zeros $\beta_{\mathcal{N}}$ and $\gamma_{\mathcal{N}}$ with $|\beta_{\mathcal{N}}| = |\gamma_{\mathcal{N}}| < 1$. In fact, $\alpha_{\mathcal{N}} \approx 1.46557$. We also have the following property of $(\mathcal{N}_n)_{n\geq 0}$.

Lemma 1. *For the sequence $(\mathcal{N}_n)_{n\geq 0}$, we have*

$$\alpha_{\mathcal{N}}^{n-2} \leq \mathcal{N}_n \leq \alpha_{\mathcal{N}}^{n-1} \quad \textit{for} \quad n \geq 1.$$

Proof. One can easily prove this lemma using induction on n. □

Let $\mathbb{K}_\varphi := \mathbb{Q}(\alpha_{\mathcal{N}}, \beta_{\mathcal{N}})$ be the splitting field of the polynomial φ over $\mathbb{Q}$. Then $[\mathbb{K}_\varphi, \mathbb{Q}] = 6$. Furthermore, $[\mathbb{Q}(\alpha_{\mathcal{N}}) : \mathbb{Q}] = 3$. The Galois group of $\mathbb{K}_\varphi$ over $\mathbb{Q}$ is given by

$$\begin{aligned}\mathcal{G}_\varphi := \mathrm{Gal}(\mathbb{K}/\mathbb{Q}) &\cong \{(1), (\alpha_{\mathcal{N}}\beta_{\mathcal{N}}), (\alpha_{\mathcal{N}}\gamma_{\mathcal{N}}), (\beta_{\mathcal{N}}\gamma_{\mathcal{N}}), (\alpha_{\mathcal{N}}\beta_{\mathcal{N}}\gamma_{\mathcal{N}}), (\alpha_{\mathcal{N}}\gamma_{\mathcal{N}}\beta_{\mathcal{N}})\}\\ &\cong S_3.\end{aligned}$$

Thus, we identify the automorphisms of $\mathcal{G}_\varphi$ with the permutations of the zeros of the polynomial φ. For example, the permutation $(\alpha_\mathcal{N}\beta_\mathcal{N})$ corresponds to the automorphisms $\sigma_\varphi : \alpha_\mathcal{N} \to \beta_\mathcal{N},\ \beta_\mathcal{N} \to \alpha_\mathcal{N},\ \gamma_\mathcal{N} \to \gamma_\mathcal{N}$.

3.2. Linear forms in logarithms. We begin this subsection with a few reminders about the logarithmic height of an algebraic number. Let η be an algebraic number of degree d, $a_0 > 0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$ and let $\eta = \eta^{(1)}, \dots, \eta^{(d)}$ denote its conjugates. The quantity defined by

$$h(\eta) = \frac{1}{d}\left(\log|a_0| + \sum_{j=1}^{d} \log\max(1, |\eta^{(j)}|)\right)$$

is called the logarithmic height of η. Some properties of height are as follows. For η_1, η_2 algebraic numbers and $m \in \mathbb{Z}$, we have

$$\begin{aligned} h(\eta_1 \pm \eta_2) &\le h(\eta_1) + h(\eta_2) + \log 2, \\ h(\eta_1\eta_2^{\pm 1}) &\le h(\eta_1) + h(\eta_2), \\ h(\eta_1^m) &= |m| h(\eta_1). \end{aligned}$$

In particular, if $\eta = p/q \in \mathbb{Q}$ is a rational number in its reduced form with $q > 0$, then $h(\eta) = \log(\max\{|p|, q\})$.

We can now present the famous Matveev's result used in this study. Let $\mathbb{L}$ be a real number field of degree $d_\mathbb{L}$, $\eta_1, \dots, \eta_s \in \mathbb{L}$ and $b_1, \dots, b_s \in \mathbb{Z} \setminus \{0\}$. Let $B \ge \max\{|b_1|, \dots, |b_s|\}$ and

$$\Lambda = \eta_1^{b_1} \cdots \eta_s^{b_s} - 1.$$

Let $A_1, \dots, A_s$ be real numbers such that

$$A_i \ge \max\{d_\mathbb{L} h(\eta_i), |\log\eta_i|, 0.16\}, \quad i = 1, \dots, s.$$

With the above notation, Matveev [8] proved the following result.

Theorem 3. *Assume that $\Lambda \ne 0$. Then*

$$\log|\Lambda| > -1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot d_\mathbb{L}^2 \cdot (1 + \log d_\mathbb{L}) \cdot (1 + \log B) \cdot A_1 \cdots A_s.$$

We also need the following result from Sanchez and Luca [9].

Lemma 2. *Let $r \ge 1$ and $H > 0$ be such that $H > (4r^2)^r$ and $H > L/(\log L)^r$. Then*

$$L < 2^r H (\log H)^r.$$

3.3. Reduction method. The bounds on the variables obtained via Baker's theory [5] are too large for any computational purposes. To reduce the bounds, we use the reduction method due to Dujella and Pethő [7, Lemma 5a]. For a real number X, let $\|X\| := \min\{|X - n| : n \in \mathbb{Z}\}$ stand for the distance of X to the nearest integer.

Lemma 3. *Let M be a positive integer, p/q be a convergent of the continued fraction expansion of an irrational number τ such that $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Furthermore, let*

$$\varepsilon := \|\mu q\| - M \cdot \|\tau q\|.$$

If $\varepsilon > 0$, then there is no solution to the inequality

$$0 < |u\tau - v + \mu| < AB^{-w}$$

in positive integers u, v and w with

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

4. Proofs of main results

4.1. Proof of Theorem 1. To prove Theorem 1, we will use the following lemma which provides a relation on the size of k versus n and g.

Lemma 4. *All solutions of the Diophantine equation* (1) *satisfy*

$$k < 8n \log g.$$

Proof. From (1), we have

$$\alpha_{\mathcal{N}}^{k-2} \leq \mathcal{N}_k = d_1 \frac{g^\ell - 1}{g - 1} \cdot d_2 \frac{g^m - 1}{g - 1} \cdot d_3 \frac{g^n - 1}{g - 1} \leq (g^n - 1)^3 < g^{3n}.$$

Taking logarithm on both sides, we get $(k-2)\log \alpha_{\mathcal{N}} < 3n \log g$. Since $n \geq 2$ and $g \geq 2$, we obtain the desired inequality. This ends the proof. □

Proof of Theorem 1. If $n = 1$, then $\ell = m = 1$. So, the equation (1) becomes

$$\mathcal{N}_k = d_1 d_2 d_3,$$

which implies

$$\alpha_{\mathcal{N}}^{k-2} \leq (g-1)^3,$$

which leads to

$$k < 2 + 3\frac{\log g}{\log \alpha_{\mathcal{N}}}.$$

Now, suppose $n \geq 2$. From (1) and (2), we have

$$\mathcal{N}_k = a_{\mathcal{N}} \alpha_{\mathcal{N}}^k + b_{\mathcal{N}} \beta_{\mathcal{N}}^k + c_{\mathcal{N}} \gamma_{\mathcal{N}}^k = d_1 \frac{g^\ell - 1}{g - 1} \cdot d_2 \frac{g^m - 1}{g - 1} \cdot d_3 \frac{g^n - 1}{g - 1},$$

which implies

$$a_{\mathcal{N}}\alpha_{\mathcal{N}}^{k}-\frac{d_1d_2d_3g^{\ell+m+n}}{(g-1)^3}=-\frac{d_1d_2d_3\left(g^{\ell+m}+g^{\ell+n}+g^{m+n}\right)}{(g-1)^3}+\frac{d_1d_2d_3\left(g^{l}+g^{m}+g^{n}\right)}{(g-1)^3}-\frac{d_1d_2d_3}{(g-1)^3}-\Pi(k). \tag{4}$$

Taking the absolute values of both sides of (4) and using (3), we get

$$\left|a_{\mathcal{N}}\alpha_{\mathcal{N}}^{k}-\frac{d_1d_2d_3g^{\ell+m+n}}{(g-1)^3}\right|<\frac{d_1d_2d_3\left(g^{\ell+m}+g^{\ell+n}+g^{m+n}\right)}{(g-1)^3}+\frac{d_1d_2d_3\left(g^{l}+g^{m}+g^{n}\right)}{(g-1)^3}+\frac{d_1d_2d_3}{(g-1)^3}+\frac{1}{\alpha_{\mathcal{N}}^{k/2}}. \tag{5}$$

Multiplying both sides of (5) by $\dfrac{(g-1)^3}{d_1d_2d_3g^{\ell+n+m}}$ and noticing the fact that $\ell\le m\le n$, we get the inequality

$$\begin{aligned}\left|\frac{(g-1)^3\cdot a_{\mathcal{N}}\alpha_{\mathcal{N}}^{k}\cdot g^{-(\ell+n+m)}}{d_1d_2d_3}-1\right| &< \frac{1}{g^{\ell}}+\frac{1}{g^{m}}+\frac{1}{g^{n}}+\frac{1}{g^{\ell+m}}+\frac{1}{g^{\ell+n}}\\ &\quad+\frac{1}{g^{m+n}}+\frac{1}{g^{\ell+m+n}}+\frac{(g-1)^3}{\alpha_{\mathcal{N}}^{k/2}d_1d_2d_3g^{\ell+n+m}}\\ &< 8\cdot g^{-\ell}.\end{aligned}$$

So, we get

$$\left|\frac{a_{\mathcal{N}}(g-1)^3}{d_1d_2d_3}\cdot\alpha_{\mathcal{N}}^{k}\cdot g^{-(\ell+n+m)}-1\right|<8\cdot g^{-\ell}. \tag{6}$$

We put

$$\Gamma_1:=\frac{a_{\mathcal{N}}(g-1)^3}{d_1d_2d_3}\cdot\alpha_{\mathcal{N}}^{k}\cdot g^{-(\ell+n+m)}-1.$$

Let us show $\Gamma_1\ne 0$. We proceed by the contrary. Assume that $\Gamma_1=0$. Then

$$a_{\mathcal{N}}\alpha_{\mathcal{N}}^{k}=\frac{d_1d_2d_3}{(g-1)^3}\cdot g^{\ell+m+n},$$

which implies

$$\sigma_{\varphi}\left(a_{\mathcal{N}}\alpha_{\mathcal{N}}^{k}\right)=b_{\mathcal{N}}\beta_{\mathcal{N}}^{k}=\frac{d_1d_2d_3}{(g-1)^3}\cdot g^{\ell+m+n}.$$

Taking the absolute value, we get

$$\left|b_{\mathcal{N}}\beta_{\mathcal{N}}^{k}\right|=\left|\frac{d_1d_2d_3}{(g-1)^3}\cdot g^{\ell+m+n}\right|.$$

We have $\left|b_{\mathcal{N}}\beta_{\mathcal{N}}^k\right| < 1$ instead $\left|\dfrac{d_1d_2d_3}{(g-1)^3}\cdot g^{\ell+m+n}\right| > 1$ since $1 \leq \ell \leq m \leq n$, which leads to a contradiction. Hence $\Gamma_1 \neq 0$.

In order to apply Matveev's result to Γ_1, set

$$t := 3, \quad \eta_1 := \frac{a_{\mathcal{N}}(g-1)^3}{d_1d_2d_3}, \quad \eta_2 := \alpha_{\mathcal{N}}, \quad \eta_3 := g,$$
$$b_1 := 1, \quad b_2 := k, \quad b_3 := -(\ell+m+n),$$

and $\mathbb{K} := \mathbb{Q}(\eta_1,\eta_2,\eta_3) = \mathbb{Q}(\alpha_{\mathcal{N}})$, which is a real number field of degree $d_{\mathbb{K}} = 3$.
Using properties of the logarithmic height, we get

$$h(\eta_2) = h(\alpha_{\mathcal{N}}) = \frac{\log \alpha_{\mathcal{N}}}{3}, \quad h(\eta_3) = h(g) = \log g$$

and

$$\begin{aligned} h(\eta_1) &= h\left(\frac{a_{\mathcal{N}}(g-1)^3}{d_1d_2d_3}\right) \\ &\leq h(a_{\mathcal{N}}) + h\left(\frac{(g-1)^3}{d_1d_2d_3}\right) \\ &\leq \frac{1}{3}\log 23 + \log\left(\max\left\{(g-1)^3, d_1d_2d_3\right\}\right) \\ &< 3\log(g) + 2 < 6\log g \quad \text{since } g \geq 2. \end{aligned}$$

Thus, we can take

$$A_1 = 18\log(g), \quad A_2 = \log\alpha_{\mathcal{N}} \quad \text{and } A_3 = 3\log g.$$

By Lemma 4, we have $k < 8n\log g$, so we put $B = 8n\log g$. Using Theorem 3, we see that

$$\begin{aligned} \log|\Gamma_1| &> -1.4\times 30^6\times 3^{4.5}\times 3^2(1+\log 3)(1+\log(8n\log g)) \\ &\quad\times(18\log(g))(3\log g\log\alpha_{\mathcal{N}}) \\ &> -5.6\times 10^{13}(1+\log(8n\log g))(\log^2 g). \end{aligned}$$

Comparing the above inequality with (6), we obtain that

$$\ell\log g - \log 8 < 5.6\times 10^{13}(1+\log(8n\log g))(\log^2 g).$$

Since $g \geq 2$ and $n \geq 2$, we have

$$1+\log(8n\log g) < 8\log(n\log g).$$

So we get

$$\ell < 4.5\times 10^{14}\log n\log^2 g.$$

Rewriting (1), we get

$$\frac{a_{\mathcal{N}}\alpha_{\mathcal{N}}^k(g-1)}{d_1(g^\ell-1)}+\frac{\Pi(k)(g-1)}{d_1(g^\ell-1)}=\frac{d_2d_3}{(g-1)^2}\left(g^{n+m}-g^m-g^n+1\right),$$

which implies

$$\frac{a_{\mathcal{N}}\alpha_{\mathcal{N}}^k(g-1)}{d_1(g^\ell-1)}-\frac{d_2d_3g^{n+m}}{(g-1)^2}=-\frac{\Pi(k)(g-1)}{d_1(g^\ell-1)}-\frac{d_2d_3g^m}{(g-1)^2}-\frac{d_2d_3g^n}{(g-1)^2}+\frac{d_2d_3}{(g-1)^2}. \tag{7}$$

Taking the absolute values of both sides of (7), we have

$$\left|\frac{a_{\mathcal{N}}\alpha_{\mathcal{N}}^k(g-1)}{d_1(g^\ell-1)}-\frac{d_2d_3g^{n+m}}{(g-1)^2}\right|<\frac{(g-1)}{d_1(g^\ell-1)\alpha_{\mathcal{N}}^{k/2}}+\frac{d_2d_3g^m}{(g-1)^2}+\frac{d_2d_3g^n}{(g-1)^2}+\frac{d_2d_3}{(g-1)^2}.$$

Dividing both sides of the inequality above by $\dfrac{d_2d_3g^{n+m}}{(g-1)^2}$ and using the fact that $n\geq 2$, we see that

$$\begin{aligned}\left|\frac{(g-1)^3}{d_1d_2d_3(g^\ell-1)}\cdot a_{\mathcal{N}}\alpha_{\mathcal{N}}^k\cdot g^{-(n+m)}-1\right| &\leq\frac{(g-1)^3}{d_1d_2d_3(g^\ell-1)\alpha_{\mathcal{N}}^{k/2}g^{n+m}}+\frac{1}{g^n}\\ &\quad+\frac{1}{g^m}+\frac{1}{g^{n+m}}<4\cdot g^{-m}.\end{aligned}$$

Then we have

$$\left|\frac{(g-1)^3}{d_1d_2d_3(g^\ell-1)}\cdot a_{\mathcal{N}}\alpha_{\mathcal{N}}^k\cdot g^{-(n+m)}-1\right|<\frac{4}{g^m}. \tag{8}$$

We put

$$\Gamma_2=\frac{(g-1)^3}{d_1d_2d_3(g^\ell-1)}\cdot a_{\mathcal{N}}\alpha_{\mathcal{N}}^k\cdot g^{-(n+m)}-1.$$

One can check that $\Gamma_2\neq 0$, proceeding as we did for Γ_1. Let us apply Matveev's result for Γ_2. Let

$$\begin{aligned}&t:=3,\quad \eta_1:=\frac{(g-1)^3}{d_1d_2d_3(g^\ell-1)}\cdot a_{\mathcal{N}},\quad \eta_2:=\alpha_{\mathcal{N}},\quad \eta_3:=g,\\ &b_1:=1,\quad b_2:=k,\quad b_3:=-(m+n),\end{aligned}$$

and $\mathbb{K}:=\mathbb{Q}(\eta_1,\eta_2,\eta_3)=\mathbb{Q}(\alpha_{\mathcal{N}})$ of degree $d_{\mathbb{K}}=3$. By Lemma 4, we have $k<8n\log g$, so we put $B=8n\log g$. We have

$$h(\eta_2)=h(\alpha_{\mathcal{N}})=\frac{\log\alpha_{\mathcal{N}}}{3},\quad h(\eta_3)=h(g)=\log g,$$

and

$$h(\eta_1)=h\left(\frac{(g-1)^3}{d_1d_2d_3(g^\ell-1)}\cdot a_{\mathcal{N}}\right)\leq h(a_{\mathcal{N}})+h\left(\frac{(g-1)^3}{d_1d_2d_3(g^\ell-1)}\right)$$

$$
\begin{aligned}
&\leq \frac{1}{3}\log 23 + \log\left(\max\left\{(g-1)^3, d_1d_2d_3\right\}\right) + h\left(\frac{1}{g^\ell - 1}\right)\\
&< 2 + 3\log(g-1) + \log(g^\ell - 1)\\
&< (3+\ell)\log(g) + 2\\
&< (6+\ell)\log g \quad \text{since } g \geq 2.
\end{aligned}
$$

Thus, we can take

$$A_1 = (18+3\ell)\log(g), \quad A_2 = \log\alpha_{\mathcal{N}} \quad \text{and } A_3 = 3\log g.$$

Using Theorem 3, we see that

$$
\begin{aligned}
\log|\Gamma_2| &> -1.4\times 30^6\times 3^{4.5}\times 3^2(1+\log 3)(1+\log(8n\log g))\\
&\quad \times((18+3\ell)\log(g))(3\log g\log\alpha_{\mathcal{N}})\\
&> -3.1\times 10^{12}(1+\log(8n\log g))(\log^2 g)(18+3\ell).
\end{aligned}
$$

Comparing with (8), we get

$$m\log g - \log 4 < 3.1\times 10^{12}(1+\log(8n\log g))(\log^2 g)(18+3\ell).$$

We have

$$1+\log(8n\log g) < 8\log n\log g \quad \text{and} \quad \ell < 4.5\times 10^{14}\log n\log^2 g.$$

So

$$m < 3.8\times 10^{28}\log^2 n\log^4 g.$$

Reorganizing (1), we get

$$\frac{d_3g^n}{g-1} - \frac{(g-1)^2\cdot a_{\mathcal{N}}\alpha_{\mathcal{N}}^k}{d_1d_2(g^\ell-1)(g^m-1)} = \frac{d_3}{g-1} + \frac{\Pi(k)(g-1)^2}{d_1d_2(g^\ell-1)(g^m-1)}.$$

We have

$$\left|\frac{d_3g^n}{g-1} - \frac{(g-1)^2\cdot a_{\mathcal{N}}\alpha_{\mathcal{N}}^k}{d_1d_2(g^\ell-1)(g^m-1)}\right| < \frac{d_3}{g-1} + \frac{(g-1)^2}{\alpha_{\mathcal{N}}^{k/2}d_1d_2(g^\ell-1)(g^m-1)}$$

by taking the absolute values of both sides of (7). Dividing both sides of the above inequality by $(d_3g^n)/(g-1)$ and using the fact that $n \geq 2$, we see that

$$\left|1 - \frac{a_{\mathcal{N}}(g-1)^3}{d_1d_2d_3(g^\ell-1)(g^m-1)}\cdot g^{-n}\cdot\alpha_{\mathcal{N}}^k\right| < \frac{1}{g^n} + \frac{1}{g^{n-1}} < \frac{2}{g^{n-1}} = 2\cdot g^{1-n}.$$

Then, we have

$$\left|\frac{a_{\mathcal{N}}(g-1)^3}{d_1d_2d_3(g^\ell-1)(g^m-1)}\cdot g^{-n}\cdot\alpha_{\mathcal{N}}^k - 1\right| < 2\cdot g^{1-n}. \tag{9}$$

We put

$$\Gamma_3 = \frac{a_{\mathcal{N}}(g-1)^3}{d_1d_2d_3(g^\ell-1)(g^m-1)}\cdot g^{-n}\cdot\alpha_{\mathcal{N}}^k - 1.$$

One can verify that $\Gamma_3 \neq 0$. Let us analyze Matveev's result for Γ_3. Let

$$t := 3, \quad \eta_1 := \frac{a_{\mathcal{N}}(g-1)^3}{d_1 d_2 d_3 (g^\ell - 1)(g^m - 1)}, \quad \eta_2 := \alpha_{\mathcal{N}}, \quad \eta_3 := g,$$
$$b_1 := 1, \quad b_2 := k, \quad b_3 := -n,$$

and $\mathbb{K} := \mathbb{Q}(\eta_1, \eta_2, \eta_3) = \mathbb{Q}(\alpha_{\mathcal{N}})$ of degree $d_{\mathbb{K}} = 3$. By Lemma 4, we have $k < 8n \log g$, so we put $B = 8n \log g$. We have

$$h(\eta_2) = h(\alpha_{\mathcal{N}}) = \frac{\log \alpha_{\mathcal{N}}}{3}, \quad h(\eta_3) = h(g) = \log g,$$

and

$$\begin{aligned} h(\eta_1) &= h\left(\frac{a_{\mathcal{N}}(g-1)^3}{d_1 d_2 d_3 (g^\ell - 1)(g^m - 1)}\right) \leq h(a_{\mathcal{N}}) + h\left(\frac{(g-1)^3}{d_1 d_2 d_3 (g^\ell - 1)(g^m - 1)}\right) \\ &\leq \frac{1}{3} \log 23 + \log\left(\max\{(g-1)^3, d_1 d_2 d_3\}\right) + h\left(\frac{1}{g^\ell - 1}\right) + h\left(\frac{1}{g^m - 1}\right) \\ &< 2 + 3 \log(g-1) + \log\left(g^\ell - 1\right) + \log(g^m - 1) \\ &< (3 + \ell + m) \log(g) + 2 \\ &< (6 + \ell + m) \log g \quad \text{since } g \geq 2. \end{aligned}$$

Thus, we can take

$$A_1 = 3(6 + \ell + m) \log(g), \quad A_2 = \log \alpha_{\mathcal{N}} \quad \text{and } A_3 = 3 \log g.$$

Using Theorem 3, we see that

$$\begin{aligned} \log|\Gamma_3| &> -1.4 \times 30^6 \times 3^{4.5} \times 3^3 (1 + \log 3)(1 + \log(8n \log g)) \\ &\quad \times ((6 + \ell + m) \log(g))(3 \log g \log \alpha_{\mathcal{N}}) \\ &> -9.31 \times 10^{12} (1 + \log(8n \log g))(\log^2 g)(6 + \ell + m). \end{aligned}$$

Comparing with (9), we get

$$(n-1) \log g - \log 2 < 9.31 \times 10^{12} (1 + \log(8n \log g))(\log^2 g)(6 + \ell + m).$$

We have

$$1 + \log(8n \log g) < 8 \log n \log g, \quad m < 3.8 \times 10^{28} \times \log^2 n \log^4 g,$$
$$\ell < 4.5 \times 10^{14} \log n \log^2 g.$$

Thus

$$\begin{aligned} 6 + \ell + m &< 4.51 \times 10^{14} \log n \log^2 g + 3.8 \times 10^{28} \times \log^2 n \log^4 g \\ &< 4 \times 10^{28} \times \log^2 n \log^4 g. \end{aligned}$$

So we have

$$n < 3 \times 10^{42} \log^3 n \log^6 g.$$

Now we apply Lemma 2, by setting

$$r := 3, \quad L := n \quad \text{and} \quad H := 3 \times 10^{42} \cdot \log^6 g,$$

we get

$$\begin{aligned} n &< 2^3 \cdot 3 \times 10^{42} \cdot \log^6 g \times \log^3 \left(3 \times 10^{42} \cdot \log^6 g\right) \\ &< 2.4 \times 10^{43} \cdot \log^6 g \cdot (95.6 + 6 \log\log g)^3 \\ &< 5.91 \times 10^{49} \log^9 g. \end{aligned}$$

Notice that we have used the inequality $95.6 + 6\log\log g < 135 \log g$ which holds since $g \geq 2$. □

4.2. Proof of Theorem 2. Since $2 \leq g \leq 10$, according to Theorem 1, we have

$$\ell \leq m \leq n < 1.08 \times 10^{53} \quad \text{and} \quad k < 1.99 \times 10^{54}.$$

Consequently, the next step is to reduce the upper bounds above in order to identify the set of the interval in which the possible solutions of (1) lie. To do this, we proceed in three steps.

Step 1. Using (6), let

$$\Lambda_1 := -\log(\Gamma_1 + 1) = (\ell + m + n)\log g - k \log \alpha_{\mathcal{N}} - \log\left(\frac{(g-1)^3 a_{\mathcal{N}}}{d_1 d_2 d_3}\right).$$

Notice that (6) can be rewritten as

$$\left|e^{-\Lambda_1} - 1\right| < \frac{8}{g^\ell}.$$

Observe that $\Lambda_1 \neq 0$, since $e^{-\Lambda_1} - 1 = \Gamma_1 \neq 0$. Assume that $\ell \geq 5$. Then

$$\left|e^{-\Lambda_1} - 1\right| < \frac{8}{g^\ell} < \frac{1}{2}.$$

Since $|x| < 2\left|\mathrm{e}^x - 1\right|$, if $|x| < \frac{1}{2}$ holds, then

$$|\Lambda_1| < \frac{16}{g^\ell}.$$

Substituting Λ_1 in the above inequality with its value and dividing through by $\log \alpha_{\mathcal{N}}$, we get

$$\left|(\ell + m + n)\left(\frac{\log g}{\log \alpha_{\mathcal{N}}}\right) - k + \frac{\log\left(\frac{(g-1)^3 a_{\mathcal{N}}}{d_1 d_2 d_3}\right)}{\log \alpha_{\mathcal{N}}}\right| < \frac{16}{\log \alpha_{\mathcal{N}} g^\ell}.$$

Then, we can apply Lemma 3 with the data

$$\tau := \frac{\log g}{\log \alpha_{\mathcal{N}}}, \quad \mu := \frac{\log\left(\dfrac{(g-1)^3 a_{\mathcal{N}}}{d_1 d_2 d_3}\right)}{\log \alpha_{\mathcal{N}}}, \quad A := \frac{16}{\log \alpha_{\mathcal{N}}},$$
$$B := g, \quad \text{and} \quad w := \ell \quad \text{with} \quad 1 \le d_1 \le d_2 \le d_3 \le g-1.$$

We can take $M := 1.99 \times 10^{54}$, since $k < 8n \log g < 1.99 \times 10^{54}$. So, for the remaining proof, we use *Mathematica* to apply Lemma 3. For the computations, if the first convergent q_t is such that $q_t > 6M$ does not satisfy the condition $\varepsilon > 0$, then we use the next convergent until we find the one that satisfies the conditions. Thus, we have the results given in Table 2.

TABLE 2. Upper bound on ℓ.

g	2	3	4	5	6	7	8	9	10
q_t	q_{118}	q_{100}	q_{110}	q_{115}	q_{90}	q_{106}	q_{112}	q_{102}	q_{96}
$\varepsilon \ge$	0.36	0.26	0.03	0.01	0.06	0.001	0.0019	0.005	0.01
$\ell \le$	194	121	99	87	76	72	67	62	59

Therefore

$$1 \le \ell \le \frac{\log((16/\log \alpha_{\mathcal{N}}) \cdot q_{118}/0.36)}{\log 2} \le 194.$$

Step 2. In this step, we have to reduce the upper bound on m. To do this, let us consider

$$\Lambda_2 := -\log(\Gamma_2 + 1) = (m+n)\log g - k \log \alpha_{\mathcal{N}} + \log\left(\frac{(g-1)^3 a_{\mathcal{N}}}{d_1 d_2 d_3 \cdot (g^\ell - 1)}\right).$$

Thus inequality (8) becomes

$$\left|e^{-\Lambda_2} - 1\right| < \frac{4}{g^m} < \frac{1}{2},$$

which holds for $m \ge 4$. It follows that

$$\left|(m+n)\frac{\log g}{\log \alpha_{\mathcal{N}}} - k + \frac{\log\left(\dfrac{(g-1)^3 a_{\mathcal{N}}}{d_1 d_2 d_3 \cdot (g^\ell - 1)}\right)}{\log \alpha_{\mathcal{N}}}\right| < \frac{8}{g^m \log \alpha_{\mathcal{N}}}. \tag{10}$$

So the conditions of Lemma 3 are satisfied. Applying this lemma to the inequality (10) with the following data

$$\tau := \frac{\log g}{\log \alpha_{\mathcal{N}}}, \quad \mu := \frac{\log\left(\dfrac{(g-1)^3 a_{\mathcal{N}}}{d_1 d_2 d_3 \cdot (g^{\ell}-1)}\right)}{\log \alpha_{\mathcal{N}}},$$
$$A := \frac{8}{\log \alpha_{\mathcal{N}}}, \quad B := g, \quad \text{and} \quad w := m$$

with $1 \le d_1 \le d_2 \le d_3 \le g-1$ and $1 \le \ell \le 194$.

As $k < 8n \log g < 1.99 \times 10^{54}$, we can take $M := 1.99 \times 10^{54}$. With *Mathematica* we get the results given in Table 3.

TABLE 3. Upper bound on m.

g	2	3	4	5	6	7	8	9	10
q_t	q_{118}	q_{100}	q_{110}	q_{115}	q_{90}	q_{106}	q_{112}	q_{102}	q_{96}
$\varepsilon \ge$	0.004	0.0007	0.0003	0.001	0.0002	0.0005	0.0001	0.005	0.001
$m \le$	199	125	102	88	78	72	68	62	60

In all cases, we can conclude that

$$1 \le m \le \frac{\log((8/\log \alpha_{\mathcal{N}}) \cdot q_{115}/0.0009)}{\log 2} \le 200.$$

Step 3. Finally, to reduce the bound on n we have to choose

$$\Lambda_3 := \log(\Gamma_3 + 1) = (n)\log g - k \log \alpha_{\mathcal{N}} + \log\left(\frac{(g-1)^3 a_{\mathcal{N}}}{d_1 d_2 d_3 \cdot (g^{\ell}-1)(g^m-1)}\right).$$

We have,

$$\left|e^{-\Lambda_3} - 1\right| < \frac{2}{g^{n-1}} < \frac{1}{2},$$

which is valid for $n \ge 4$ and $g \ge 2$. It follows that

$$\left| m\frac{\log g}{\log \alpha_{\mathcal{N}}} - k + \frac{\log\left(\dfrac{(g-1)^3 a_{\mathcal{N}}}{d_1 d_2 d_3 \cdot (g^{\ell}-1)(g^m-1)}\right)}{\log \alpha_{\mathcal{N}}} \right| < \frac{4}{g^{n-1}\alpha_{\mathcal{N}}}. \tag{11}$$

Now we have to apply Lemma 3 to (11) by taking the following parameters

$$\tau := \frac{\log g}{\log \alpha_{\mathcal{N}}}, \quad \mu := \frac{\log\left(\dfrac{(g-1)^3 a_{\mathcal{N}}}{d_1 d_2 d_3 \cdot (g^{\ell}-1)(g^m-1)}\right)}{\log \alpha_{\mathcal{N}}}, \quad A := \frac{8}{\log \alpha_{\mathcal{N}}},$$
$$B := g, \quad \text{and} \quad w := n-1$$

with $\quad 1 \le d_1 \le d_2 \le d_3 \le g-1, \quad 1 \le \ell \le 194$ and $1 \le m \le 183$.

Using the fact that $k < 8n \log g < 1.99 \times 10^{54}$, we can take $M := 1.99 \times 10^{54}$, and we get the following reduced bounds of n for $2 \le g \le 10$.

Table 4. Upper bound on n.

g	2	3	4	5	6	7	8	9	10
q_t	q_{118}	q_{99}	q_{110}	q_{115}	q_{90}	q_{106}	q_{112}	q_{102}	q_{96}
$\varepsilon \ge$	0.00006	0.002	0.007	0.0002	0.0008	0.002	0.0005	0.02	0.009
$n \le$	204	124	99	89	77	71	67	61	59

It follows from the above table that

$$1 \le n \le \frac{\log((4/\log \alpha_{\mathcal{N}}) \cdot q_{118}/10^{-6})}{\log 2} \le 205,$$

which is valid for all g such that $2 \le g \le 10$. In light of the above results, we need to check the equation (1) in the cases $2 \le g \le 10$ for $1 \le d_1, d_2, d_3 \le 9$, $1 \le n \le 205$, $1 \le m \le 200$, $1 \le \ell \le 194$ and $1 \le k \le 11500$. A quick inspection using *Sagemath* reveals that the Diophantine equation (1) in the range $2 \le g \le 10$ has only the solutions listed in the statement of Theorem 2. This completes the proof of Theorem 2.

5. Discussions

In addition to Baker's method and linear forms in logarithms, there are other so-called "classical" methods and techniques for solving exponential Diophantine equations. These include the modular arithmetic method, p-adic analysis, Fermat's method of infinite descent, the factorization method, solving using inequalities, the mathematical induction method, the parametric method, and so on. It would be interesting to treat the same problems approached in this article with other methods than those of the linear forms in logarithms. The modular arithmetic method could be used to determine Narayana numbers, which are products of three repdigits in base g with $g \ge 2$ due to the interesting divisibility properties possessed by the repdigits.

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