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Quasi-Laplacian energy of fractal graphs

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Abstract. Graph energy is a measurement of determining the structural information content of graphs. The first Zagreb index can be handled with its connection to graph energy. In this paper, a novel and significant application of the first Zagreb index to composite graphs based on fractal graphs is revealed, and by the relation between quasi-Laplacian energy and the vertex degrees of a graph, we derive closed-form formulas for the quasi-Laplacian energy of fractal graphs or namely R-graphs, R-vertex and edge join, R-vertex and edge corona, R-vertex and edge neighborhood graphs in terms of the corresponding energy, the first Zagreb indices, number of vertices and edges of the underlying graphs.

1. Introduction

All graphs considered in this paper are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. If $|V(G)| = n$ and $|E(G)| = m$, we say that G is an (n, m)-graph. The degree of a vertex $v \in V(G)$, denoted by $\deg(v)$, is the number of edges incident to v in G. For a graph G, the sum of the degrees of vertices is equal to the twice the number of edges, that is $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$, a well-known result called the handshaking lemma. Let $D(G) = diag(\deg(v_1), \deg(v_2), \ldots, \deg(v_{|V(G)|}))$ be the diagonal matrix with all vertex degrees of G as its diagonal entries. Let $A(G)$ denote the adjacency matrix of G. The quasi-Laplacian matrix of G is defined as $Q(G) = D(G) + A(G)$ [2]. The quasi-Laplacian matrix $Q(G)$ of G is a real symmetric positive semi-definite matrix [3], and we use $\mu_1(G) \geq \mu_2(G) \geq ... \geq \mu_{|V(G)|}(G) \geq 0$ to denote the

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eigenvalues of $Q(G)$. The quasi-Laplacian energy of G is defined in [8] as

$$E_Q(G) = \sum_{i=1}^{|V(G)|} \mu_i^2,$$

where it is pointed out that the quasi-Laplacian energy of G can also be expressed as

$$E_Q(G) = \sum_{i=1}^{|V(G)|} \mu_i^2 = \sum_{i=1}^{|V(G)|} \deg^2(v_i) + \sum_{i=1}^{|V(G)|} \deg(v_i). \tag{1}$$

The first Zagreb index of a graph was first introduced by Gutman and Trinajstić in [5], defined as

$$M_1(G) = \sum\nolimits_{v \in V(G)} \deg^2(v). \tag{2}$$

The first Zagreb index can be also expressed as a sum over edges of G,

$$M_1(G) = \sum\nolimits_{uv \in E(G)} (\deg_G(u) + \deg_G(v)).$$

We will omit the subscript G when the graph is clear from the context. We encourage the reader to consult [1, 4, 9, 10, 11] for historical background, computational techniques and mathematical properties of the Zagreb indices.

It is well known that composite graph classes arise from simpler graphs via various graph operations. Therefore, it is important and of interest to understand how certain invariants of those composite graphs are related to the corresponding invariants of the simpler graphs.

The *fractal graph* of a graph G, denoted by $R(G)$, is the graph obtained from G by adding a vertex v_e and then joining v_e to the end vertices of e for each $e \in E(G)$. Let $I(G)$ be the set of newly added vertices, that is $I(G) = V(R(G)) \setminus V(G)$. The fractal graph of a graph G is also known as the *R-graph* of a graph G [6].

Let G_1 and G_2 be two vertex-disjoint graphs. In [7] and [6], new graph operations based on fractal graphs are defined as follows.

- The *R-vertex join* of G_1 and G_2, denoted by $G_1 \langle v \rangle G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and G_2 by joining every vertex of $V(G_1)$ with every vertex of $V(G_2)$.
- The *R-edge join* of G_1 and G_2, denoted by $G_1 \langle e \rangle G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and G_2 by joining every vertex of $I(G_1)$ with every vertex of $V(G_2)$.
- The *R-vertex corona* of G_1 and G_2, denoted by $G_1 \cdot G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|V(G_1)|$ copies of G_2 by joining the ith vertex of $V(G_1)$ to every vertex in the ith copy of G_2.

- The *R-edge corona* of G_1 and G_2, denoted by $G_1 \times G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|I(G_1)|$ copies of G_2 by joining the ith vertex of $I(G_1)$ to every vertex in the ith copy of G_2.
- The *R-vertex neighborhood corona* of G_1 and G_2, denoted by $G_1 \odot G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|V(G_1)|$ copies of G_2 by joining the neighbors of the ith vertex of G_1 in $R(G_1)$ to every vertex in the ith copy of G_2.
- The *R-edge neighborhood corona* of G_1 and G_2, denoted by $G_1 \otimes G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|I(G_1)|$ copies of G_2 by joining the neighbors of the ith vertex of $|I(G_1)|$ in $R(G_1)$ to every vertex in the ith copy of G_2.

In this paper we derive closed-form formulas for the quasi-Laplacian energy of fractal graphs or namely R-graphs, R-vertex and edge join, R-vertex and edge corona, R-vertex and edge neighborhood graphs in terms of the corresponding energy (and some other quantities) of G_1 and G_2. The paper ends with a short summary and conclusion.

2. Quasi-Laplacian energy of graphs based on fractal graphs

In this chapter, we consider an (n, m)-graph G and two vertex-disjoint graphs, i.e., an (n_1, m_1)-graph G_1 and an (n_2, m_2)-graph G_2.

2.1. Quasi-Laplacian energy of fractal graphs. The degree of a vertex $v \in V(R(G))$ is given by

$$\deg_{R(G)}(v) = \begin{cases} 2\deg_G(v), & \text{if } v \in V(G); \\ 2, & \text{if } v \in I(G). \end{cases} \tag{3}$$

Theorem 1. *The quasi-Laplacian energy of $R(G)$ is*

$$E_Q(R(G)) = 4(E_Q(G)) + 2m.$$

Proof. By definition (1) we have

$$E_Q(R(G)) = \sum_{i=1}^{|V(R(G))|} \deg^2_{R(G)}(v_i) + \sum_{i=1}^{|V(R(G))|} \deg_{R(G)}(v_i).$$

Splitting the vertex set $V(R(G))$ into disjoint sets $V(G)$ and $I(G)$, we get

$$\begin{aligned} E_Q(R(G)) &= \sum_{v\in V(G)} \deg^2_{R(G)}(v) + \sum_{v\in I(G)} \deg^2_{R(G)}(v) \\ &\quad + \sum_{v\in V(G)} \deg_{R(G)}(v) + \sum_{v\in I(G)} \deg_{R(G)}(v). \end{aligned}$$

Substituting degrees of the vertices of $R(G)$ from (3), we compute

$$E_Q(R(G)) = \left(\sum_{i=1}^{n}(2\deg_G(v_i))^2 + \sum_{i=1}^{m}(2)^2\right) + \left(\sum_{i=1}^{n}(2\deg_G(v_i)) + \sum_{i=1}^{m}(2)\right)$$
$$= \left(4\sum_{i=1}^{n}\deg_G^2(v_i) + 4m\right) + \left(2\sum_{i=1}^{n}\deg_G(v_i) + 2m\right).$$

By handshaking lemma, we compute each summation as follows:

$$\begin{aligned} E_Q(R(G)) &= 4\sum_{i=1}^{n}\deg_G^2(v_i) + 4m + 2(2m) + 2m \\ &= 4\sum_{i=1}^{n}\deg_G^2(v_i) + 4\sum_{i=1}^{n}\deg_G(v_i) + 2m. \end{aligned}$$

Comparing with definition (1) of the quasi-Laplacian energy of G, we receive the result. □

2.2. Quasi-Laplacian energy of R-vertex join graphs. The degree of a vertex $v \in V(G_1 \langle v \rangle G_2)$ is given by

$$\deg_{G_1\langle v\rangle G_2}(v) = \begin{cases} 2\deg_{G_1}(v) + n_2, & \text{if } v \in V(G_1); \\ 2, & \text{if } v \in I(G_1); \\ \deg_{G_2}(v) + n_1, & \text{if } v \in V(G_2). \end{cases} \tag{4}$$

Theorem 2. *The quasi-Laplacian energy of a graph $G_1 \langle v \rangle G_2$ is*

$$\begin{aligned} E_Q(G_1\langle v\rangle G_2) = {} & 4E_Q(G_1) + E_Q(G_2) \\ & + n_1 n_2(n_1 + n_2 + 2) + 2m_1(4n_2 + 1) + 4n_1 m_2. \end{aligned}$$

Proof. By definition (1) we start with

$$E_Q(G_1\langle v\rangle G_2) = \sum_{i=1}^{|V(G_1\langle v\rangle G_2)|}\deg^2_{G_1\langle v\rangle G_2}(v_i) + \sum_{i=1}^{|V(G_1\langle v\rangle G_2)|}\deg_{G_1\langle v\rangle G_2}(v_i).$$

Splitting the vertex set $V(G_1\langle v\rangle G_2)$ into disjoint sets $V(G_1)$, $V(G_2)$ and $I(G_1)$, we get

$$\begin{aligned} E_Q(G_1\langle v\rangle G_2) = {} & \sum_{v\in V(G_1)}\deg^2_{G_1\langle v\rangle G_2}(v) + \sum_{v\in I(G_1)}\deg^2_{G_1\langle v\rangle G_2}(v) \\ & + \sum_{v\in V(G_2)}\deg^2_{G_1\langle v\rangle G_2}(v) + \sum_{v\in V(G_1)}\deg_{G_1\langle v\rangle G_2}(v) \\ & + \sum_{v\in I(G_1)}\deg_{G_1\langle v\rangle G_2}(v) + \sum_{v\in V(G_2)}\deg_{G_1\langle v\rangle G_2}(v). \end{aligned}$$

Substituting the degrees of the vertices of $G_1 \langle v \rangle G_2$ from (4), we compute

$$\begin{aligned}
E_Q(G_1 \langle v \rangle G_2) &= \sum_{i=1}^{n_1} \left(2\deg_{G_1}(v_i) + n_2\right)^2 + \sum_{i=1}^{m_1} (2)^2 + \sum_{i=1}^{n_2} \left(\deg_{G_2}(v_i) + n_1\right)^2 \\
&\quad + \sum_{i=1}^{n_1} \left(2\deg_{G_1}(v_i) + n_2\right) + \sum_{i=1}^{m_1} (2) + \sum_{i=1}^{n_2} \left(\deg_{G_2}(v_i) + n_1\right) \\
&= 4\sum_{i=1}^{n_1} \deg^2_{G_1}(v_i) + 4n_2 \sum_{i=1}^{n_1} \deg_{G_1}(v_i) + n_1 n_2^2 + 4m_1 \\
&\quad + \sum_{i=1}^{n_2} \deg^2_{G_2}(v_i) + 2n_1 \sum_{i=1}^{n_2} \deg_{G_2}(v_i) + n_2 n_1^2 \\
&\quad + 2\sum_{i=1}^{n_1} \deg_{G_1}(v_i) + n_1 n_2 + 2m_1 + \sum_{i=1}^{n_2} \deg_{G_2}(v_i) + n_1 n_2.
\end{aligned}$$

By handshaking lemma, we compute each summation as follows:

$$\begin{aligned}
E_Q(G_1 \langle v \rangle G_2) &= 4\sum_{i=1}^{n_1} \deg^2_{G_1}(v_i) + 4n_2(2m_1) + n_1 n_2^2 + 4m_1 + \sum_{i=1}^{n_2} \deg^2_{G_2}(v_i) \\
&\quad + 2n_1(2m_2) + n_2 n_1^2 + 2(2m_1) + n_1 n_2 + 2m_1 + 2m_2 + n_1 n_2.
\end{aligned}$$

After rearranging the terms, we get

$$\begin{aligned}
E_Q(G_1 \langle v \rangle G_2) &= 4\left(\sum_{i=1}^{n_1} \deg^2_{G_1}(v_i) + 2m_1\right) + \left(\sum_{i=1}^{n_2} \deg^2_{G_2}(v_i) + 2m_2\right) \\
&\quad + 4n_1 m_2 + 2m_1(4n_2 + 1) + n_1 n_2 (n_1 + n_2 + 2).
\end{aligned}$$

Recognizing here the quasi-Laplacian energies for G_1 and G_2 (see (1)), we receive the result. □

2.3. Quasi-Laplacian energy of R-edge join graphs. The degree of a vertex $v \in V(G_1 \langle e \rangle G_2)$ is given by

$$\deg_{G_1 \langle e \rangle G_2}(v) = \begin{cases} 2\deg_{G_1}(v), & \text{if } v \in V(G_1); \\ n_2 + 2, & \text{if } v \in I(G_1); \\ \deg_{G_2}(v) + m_1, & \text{if } v \in V(G_2). \end{cases} \tag{5}$$

Theorem 3. *The quasi-Laplacian energy of a graph $G_1 \langle e \rangle G_2$ is*

$$E_Q(G_1 \langle e \rangle G_2) = 4(E_Q(G_1)) + E_Q(G_2) + m_1(n_2(n_2 + m_1 + 6) + 4m_2).$$

Proof. By definition (1), we have that

$$E_Q(G_1 \langle e \rangle G_2) = \sum_{i=1}^{|V(G_1\langle e \rangle G_2)|} \deg^2_{G_1 \langle e \rangle G_2}(v_i) + \sum_{i=1}^{|V(G_1\langle e \rangle G_2)|} \deg_{G_1 \langle e \rangle G_2}(v_i).$$

Splitting the vertex set $V\left(G_1\left\langle e\right\rangle G_2\right)$ into disjoint sets $V\left(G_1\right)$, $V\left(G_2\right)$ and $I\left(G_1\right)$, we get

$$\begin{aligned}E_Q\left(G_1\left\langle e\right\rangle G_2\right)=&\sum_{v\in V(G_1)}\deg^2_{G_1\langle e\rangle G_2}(v)+\sum_{v\in I(G_1)}\deg^2_{G_1\langle e\rangle G_2}(v)\\&+\sum_{v\in V(G_2)}\deg^2_{G_1\langle e\rangle G_2}(v)+\sum_{v\in V(G_1)}\deg_{G_1\langle e\rangle G_2}(v)\\&+\sum_{v\in I(G_1)}\deg_{G_1\langle e\rangle G_2}(v)+\sum_{v\in V(G_2)}\deg_{G_1\langle e\rangle G_2}(v).\end{aligned}$$

Substituting the degrees of the vertices of $G_1\left\langle e\right\rangle G_2$ from (5), we compute

$$\begin{aligned}E_Q\left(G_1\left\langle e\right\rangle G_2\right)&=\sum_{i=1}^{n_1}\left(2\deg_{G_1}(v_i)\right)^2+\sum_{i=1}^{m_1}(n_2+2)^2+\sum_{i=1}^{n_2}\left(\deg_{G_2}(v_i)+m_1\right)^2\\&+\sum_{i=1}^{n_1}\left(2\deg_{G_1}(v_i)\right)+\sum_{i=1}^{m_1}(n_2+2)+\sum_{i=1}^{n_2}\left(\deg_{G_2}(v_i)+m_1\right)\\&=4\sum_{i=1}^{n_1}\deg^2_{G_1}(v_i)+m_1(n_2+2)^2+\sum_{i=1}^{n_2}\deg^2_{G_2}(v_i)+2m_1\sum_{i=1}^{n_2}\deg_{G_2}(v_i)\\&+n_2m_1^2+2\sum_{i=1}^{n_1}\deg_{G_1}(v_i)+m_1(n_2+2)+\sum_{i=1}^{n_2}\deg_{G_2}(v_i)+n_2m_1.\end{aligned}$$

By handshaking lemma, we compute each summation as follows:

$$\begin{aligned}E_Q\left(G_1\left\langle e\right\rangle G_2\right)=&\,4\sum_{i=1}^{n_1}\deg^2_{G_1}(v_i)+m_1(n_2+2)^2+\sum_{i=1}^{n_2}\deg^2_{G_2}(v_i)+2m_1(2m_2)\\&+n_2m_1^2+2(m_1)+m_1(n_2+2)+2m_2+n_2m_1.\end{aligned}$$

Rearranging terms, we receive

$$\begin{aligned}E_Q\left(G_1\left\langle e\right\rangle G_2\right)=&\,4\left(\sum_{i=1}^{n_1}\deg^2_{G_1}(v_i)+2m_1\right)+\left(\sum_{i=1}^{n_2}\deg^2_{G_2}(v_i)+2m_2\right)\\&+m_1\left(n_2\left(n_2+m_1+6\right)+4m_2\right).\end{aligned}$$

Using definition (1), we see the desired result for $E_Q\left(G_1\left\langle e\right\rangle G_2\right)$. □

2.4. Quasi-Laplacian energy of R-vertex corona graphs. The degree of a vertex $v\in V\left(G_1\cdot G_2\right)$ is given by

$$\deg_{G_1\cdot G_2}(v)=\begin{cases}2\deg_{G_1}(v)+n_2, & \text{if } v\in V(G_1);\\ 2, & \text{if } v\in I(G_1);\\ \deg_{G_2}(v)+1, & \text{if } v\in V(G_2).\end{cases}\tag{6}$$

Theorem 4. *The quasi-Laplacian energy of a graph $G_1 \cdot G_2$ is*

$$E_Q(G_1 \cdot G_2) = 4E_Q(G_1) + n_1 E_Q(G_2) + n_1(4m_2 + n_2(n_2+3)) + 2m_1(4n_2+1).$$

Proof. By definition (1), we have that

$$E_Q(G_1 \cdot G_2) = \sum_{i=1}^{|V(G_1 \cdot G_2)|} \deg^2_{G_1 \cdot G_2}(v_i) + \sum_{i=1}^{|V(G_1 \cdot G_2)|} \deg_{G_1 \cdot G_2}(v_i).$$

Splitting the vertex set $V(G_1 \cdot G_2)$ into disjoint sets $V(G_1)$, $V(G_2)$ and $I(G_1)$, we get

$$\begin{aligned} E_Q(G_1 \cdot G_2) = &\sum_{v \in V(G_1)} \deg^2_{G_1 \cdot G_2}(v) + \sum_{v \in I(G_1)} \deg^2_{G_1 \cdot G_2}(v) \\ &+ \sum_{i=1}^{n_1} \sum_{v \in V(G_2)} \deg^2_{G_1 \cdot G_2}(v) + \sum_{v \in V(G_1)} \deg_{G_1 \cdot G_2}(v) \\ &+ \sum_{v \in I(G_1)} \deg_{G_1 \cdot G_2}(v) + \sum_{i=1}^{n_1} \sum_{v \in V(G_2)} \deg_{G_1 \cdot G_2}(v). \end{aligned}$$

Substituting the degrees of the vertices of $G_1 \cdot G_2$ from (6), we compute

$$\begin{aligned} E_Q(G_1 \cdot G_2) = &\sum_{i=1}^{n_1} \left(2\deg_{G_1}(v_i) + n_2\right)^2 + \sum_{i=1}^{m_1} (2)^2 + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left(\deg_{G_2}(v_j) + 1\right)^2 \\ &+ \sum_{i=1}^{n_1} \left(2\deg_{G_1}(v_i) + n_2\right) + \sum_{i=1}^{m_1} (2) + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left(\deg_{G_2}(v_j) + 1\right) \\ = &\, 4\sum_{i=1}^{n_1} \deg^2_{G_1}(v_i) + 4n_2 \sum_{i=1}^{n_1} \deg_{G_1}(v_i) + n_1 n_2^2 + 4m_1 \\ &+ \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \deg^2_{G_2}(v_j) + 2\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \deg_{G_2}(v_j) + n_1 n_2 \\ &+ 2\sum_{i=1}^{n_1} \deg_{G_1}(v_i) + n_1 n_2 + 2m_1 + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left(\deg_{G_2}(v_j)\right) + n_1 n_2. \end{aligned}$$

Applying handshaking lemma, we get:

$$\begin{aligned} E_Q\left(G_1 \cdot G_2\right) = & \, 4\sum_{i=1}^{n_1} \deg_{G_1}^2\left(v_i\right) + 4n_2\left(2m_1\right) + n_1 n_2^2 + 4m_1 + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \deg_{G_2}^2\left(v_j\right) \\ & + 2\sum_{i=1}^{n_1}\left(2m_2\right) + n_1 n_2 + 2\left(2m_1\right) + n_1 n_2 + 2m_1 + \sum_{i=1}^{n_1}\left(2m_2\right) + n_1 n_2 \\ = & \, 4\sum_{i=1}^{n_1} \deg_{G_1}^2\left(v_i\right) + 4n_2\left(2m_1\right) + n_1 n_2^2 + 4m_1 + n_1\sum_{j=1}^{n_2} \deg_{G_2}^2\left(v_j\right) \\ & + 4n_1 m_2 + n_1 n_2 + 2\left(2m_1\right) + n_1 n_2 + 2m_1 + 2n_1 m_2 + n_1 n_2. \end{aligned}$$

After rearranging,

$$\begin{aligned} E_Q\left(G_1 \cdot G_2\right) = & \, 4\left(\sum_{i=1}^{n_1} \deg_{G_1}^2\left(v_i\right) + 2m_1\right) + n_1\left(\sum_{j=1}^{n_2} \deg_{G_2}^2\left(v_j\right) + 2m_2\right) \\ & + n_1\left(4m_2 + n_2\left(n_2 + 3\right)\right) + 2m_1\left(4n_2 + 1\right), \end{aligned}$$

from which, by definition (1), we see that the desired result for $E_Q\left(G_1 \cdot G_2\right)$ holds. □

2.5. Quasi-Laplacian energy of R-edge corona graphs. The degree of a vertex $v \in V\left(G_1 \times G_2\right)$ is given by

$$\deg_{G_1 \times G_2}(v) = \begin{cases} 2\deg_{G_1}(v), & \text{if } v \in V\left(G_1\right); \\ n_2 + 2, & \text{if } v \in I\left(G_1\right); \\ \deg_{G_2}(v) + 1, & \text{if } v \in V\left(G_2\right). \end{cases} \tag{7}$$

Theorem 5. *The quasi-Laplacian energy of a graph $G_1 \times G_2$ is*

$$E_Q\left(G_1 \times G_2\right) = 4E_Q\left(G_1\right) + m_1\left(E_Q\left(G_2\right) + n_2\left(n_2 + 6\right) + 4m_2 + 2\right).$$

Proof. By definition (1), we have that

$$E_Q\left(G_1 \times G_2\right) = \sum_{i=1}^{\left|V\left(G_1 \times G_2\right)\right|} \deg_{G_1 \times G_2}^2\left(v_i\right) + \sum_{i=1}^{\left|V\left(G_1 \times G_2\right)\right|} \deg_{G_1 \times G_2}\left(v_i\right).$$

Splitting the vertex set $V(G_1 \times G_2)$ into disjoint sets $V(G_1)$, $V(G_2)$ and $I(G_1)$, we get

$$\begin{aligned}
E_Q(G_1 \times G_2) = &\sum_{v \in V(G_1)} \deg^2_{G_1 \times G_2}(v) + \sum_{v \in I(G_1)} \deg^2_{G_1 \times G_2}(v) \\
&+ \sum_{i=1}^{m_1} \sum_{v \in V(G_2)} \deg^2_{G_1 \times G_2}(v) \sum_{v \in V(G_1)} \deg_{G_1 \times G_2}(v) \\
&+ \sum_{v \in I(G_1)} \deg_{G_1 \times G_2}(v) + \sum_{i=1}^{m_1} \sum_{v \in V(G_2)} \deg_{G_1 \times G_2}(v).
\end{aligned}$$

Substituting the degrees of the vertices of $G_1 \times G_2$ from (7), we compute

$$\begin{aligned}
E_Q(G_1 \times G_2) = &\sum_{i=1}^{n_1} \big(2\deg_{G_1}(v_i)\big)^2 + \sum_{i=1}^{m_1} (n_2+2)^2 + \sum_{i=1}^{m_1}\sum_{j=1}^{n_2} \big(\deg_{G_2}(v_j)+1\big)^2 \\
&+ \sum_{i=1}^{n_1} \big(2\deg_{G_1}(v_i)\big) + \sum_{i=1}^{m_1} (n_2+2) + \sum_{i=1}^{m_1}\sum_{j=1}^{n_2} \big(\deg_{G_2}(v_j)+1\big)
\end{aligned}$$

$$\begin{aligned}
= &\,4\sum_{i=1}^{n_1} \deg^2_{G_1}(v_i) + m_1(n_2+2)^2 + \sum_{i=1}^{m_1}\sum_{j=1}^{n_2} \deg^2_{G_2}(v_j) + 2\sum_{i=1}^{m_1}\sum_{j=1}^{n_2} \deg_{G_2}(v_j) \\
&+ m_1 n_2 + 2\sum_{i=1}^{n_1} \deg_{G_1}(v_i) + m_1(n_2+2) + \sum_{i=1}^{m_1}\sum_{j=1}^{n_2} \deg_{G_2}(v_j) + m_1 n_2.
\end{aligned}$$

By handshaking lemma, we have:

$$\begin{aligned}
E_Q(G_1 \times G_2) &= 4\sum_{i=1}^{n_1} \deg^2_{G_1}(v_i) + m_1(n_2+2)^2 + \sum_{i=1}^{m_1}\sum_{j=1}^{n_2} \deg^2_{G_2}(v_j) + 2\sum_{i=1}^{m_1}(2m_2) \\
&\quad + m_1 n_2 + 2(2m_1) + m_1(n_2+2) + \sum_{i=1}^{m_1}(2m_2) + m_1 n_2 \\
&= 4\sum_{i=1}^{n_1} \deg^2_{G_1}(v_i) + m_1(n_2+2)^2 + \sum_{i=1}^{m_1}\sum_{j=1}^{n_2} \deg^2_{G_2}(v_j) \\
&\quad + 4m_1 m_2 + m_1 n_2 + 2(2m_1) + m_1(n_2+2) + 2m_1 m_2 + m_1 n_2.
\end{aligned}$$

Rearranging, we get that

$$\begin{aligned}
E_Q(G_1 \times G_2) = &\,4\left(\sum_{i=1}^{n_1} \deg^2_{G_1}(v_i) + 2m_1\right) + m_1\left(\sum_{j=1}^{n_2} \deg^2_{G_2}(v_j) + 2m_2\right) \\
&+ m_1\left(n_2(n_2+6) + 4m_2 + 2\right),
\end{aligned}$$

from which by definition (1), the desired result for $E_Q\left(G_1 \times G_2\right)$ is received. □

2.6. Quasi-Laplacian energy of R-vertex neighborhood corona graphs. The degree of a vertex $v \in V\left(G_1 \odot G_2\right)$ is given by

$$\deg_{G_1 \odot G_2}(v) = \begin{cases} \left(n_2+2\right)\deg_{G_1}(v), & \text{if } v \in V\left(G_1\right); \\ 2\left(n_2+1\right), & \text{if } v \in I\left(G_1\right); \\ \deg_{G_2}(v) + 2\deg_{G_1}(u), & \text{if } v \in V\left(G_2\right), \text{ where } u \text{ is } i\text{th} \\ & \text{vertex of } G_1 \text{ in } R\left(G_1\right) \text{ and } v \\ & \text{is a vertex in } i\text{th copy of } G_2. \end{cases} \tag{8}$$

Theorem 6. *The quasi-Laplacian energy of a graph $G_1 \odot G_2$ is*

$$\begin{aligned} E_Q\left(G_1 \odot G_2\right) = & \left(\left(n_2+2\right)^2 + 2n_2\right) E_Q\left(G_1\right) + n_1(E_Q(G_2)) \\ & + 2(n_2 M_1\left(G_1\right) + m_1(8m_2+1)), \end{aligned}$$

where $M_1\left(G_1\right)$ is the first Zagreb index of G_1, given in (2).

Proof. By definition (1), we have that

$$E_Q\left(G_1 \odot G_2\right) = \sum_{i=1}^{|V(G_1 \odot G_2)|} \deg^2_{G_1 \odot G_2}\left(v_i\right) + \sum_{i=1}^{|V(G_1 \odot G_2)|} \deg_{G_1 \odot G_2}\left(v_i\right).$$

Splitting the vertex set $V\left(G_1 \odot G_2\right)$ into disjoint sets $V\left(G_1\right)$, $V\left(G_2\right)$ and $I\left(G_1\right)$, we get

$$\begin{aligned} E_Q\left(G_1 \odot G_2\right) = & \sum_{v \in V(G_1)} \deg^2_{G_1 \odot G_2}(v) + \sum_{v \in I(G_1)} \deg^2_{G_1 \odot G_2}(v) \\ & + \sum_{i=1}^{n_1} \sum_{v \in V(G_2)} \deg^2_{G_1 \odot G_2}(v) + \sum_{v \in V(G_1)} \deg_{G_1 \odot G_2}(v) \\ & + \sum_{v \in I(G_1)} \deg_{G_1 \odot G_2}(v) + \sum_{i=1}^{n_1} \sum_{v \in V(G_2)} \deg_{G_1 \odot G_2}(v). \end{aligned}$$

Substituting the degrees of the vertices of $G_1 \odot G_2$ from (8), we get

$$\begin{aligned}
E_Q(G_1 \odot G_2) = &\sum_{i=1}^{n_1}\big((n_2+2)\deg_{G_1}(v_i)\big)^2 + \sum_{i=1}^{m_1}(2(n_2+1))^2 \\
&+ \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\big(\deg_{G_2}(v_j) + 2\deg_{G_1}(u_i)\big)^2 + \sum_{i=1}^{n_1}\big((n_2+2)\deg_{G_1}(v_i)\big) \\
&+ \sum_{i=1}^{m_1}(2(n_2+2)) + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\big(\deg_{G_2}(v_j) + 2\deg_{G_1}(u_i)\big) \\
&= (n_2+2)^2\sum_{i=1}^{n_1}\deg^2_{G_1}(v_i) + m_1(2(n_2+1))^2 + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\deg^2_{G_2}(v_j) \\
&+4\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\deg_{G_2}(v_j)\deg_{G_1}(u_i) + 4\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\deg^2_{G_1}(u_i) \\
&+ (n_2+2)\sum_{i=1}^{n_1}\deg_{G_1}(v_i) + m_1(2(n_2+2)) + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\deg_{G_2}(v_j) \\
&+2\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\deg_{G_1}(u_i).
\end{aligned}$$

By handshaking lemma, we compute each summation as follows:

$$\begin{aligned}
E_Q(G_1 \odot G_2) = &(n_2+2)^2\sum_{i=1}^{n_1}\deg^2_{G_1}(v_i) + m_1(2(n_2+1))^2 + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\deg^2_{G_2}(v_j) \\
&+4\left(\sum_{i=1}^{n_1}\deg_{G_1}(u_i)\right)\left(\sum_{i=1}^{n_2}\deg_{G_2}(v_i)\right) + 4\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\deg^2_{G_1}(u_i) \\
&+ (n_2+2)(2m_1) + m_1(2(n_2+2)) + \sum_{i=1}^{n_1}(2m_2) + 2n_2\sum_{i=1}^{n_1}\deg_{G_1}(u_i) \\
&= (n_2+2)^2\sum_{i=1}^{n_1}\deg^2_{G_1}(v_i) + m_1(2(n_2+1))^2 + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\deg^2_{G_2}(v_j) \\
&+4(2m_1)(2m_2) + 4\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\deg^2_{G_1}(u_i) + (n_2+2)(2m_1) \\
&+m_1(2(n_2+2)) + n_1(2m_2) + 2n_2(2m_1).
\end{aligned}$$

Rearranging, and then using definitions (1) and (2), we have the desired result:

$$E_Q\left(G_1 \odot G_2\right)=\left(\left(n_2+2\right)^2+2 n_2\right)\left(\sum_{i=1}^{n_1} \operatorname{deg}_{G_1}^2\left(v_i\right)+2 m_1\right)$$
$$+2\left(n_2 \sum_{i=1}^{n_1} \operatorname{deg}_{G_1}^2\left(u_i\right)+m_1\left(8 m_2+1\right)\right)+n_1\left(\sum_{j=1}^{n_2} \operatorname{deg}_{G_2}^2\left(v_j\right)+2 m_2\right)$$
$$=\left(\left(n_2+2\right)^2+2 n_2\right) E_Q\left(G_1\right)+n_1\left(E_Q\left(G_2\right)\right)+2\left(n_2 M_1(G_1)+m_1\left(8 m_2+1\right)\right).$$

□

2.7. Quasi-Laplacian energy of R-edge neighborhood corona graphs. The degree of a vertex $v \in V\left(G_1 \otimes G_2\right)$ is given by

$$\operatorname{deg}_{G_1 \otimes G_2}(v)=\begin{cases}\left(n_2+2\right) \operatorname{deg}_{G_1}(v), & \text{if } v \in V\left(G_1\right); \\ \operatorname{deg}_{G_2}(v)+2, & \text{if } v \in V\left(G_2\right); \\ 2, & \text{if } v \in I\left(G_1\right).\end{cases} \tag{9}$$

Theorem 7. *The quasi-Laplacian energy of a graph $G_1 \otimes G_2$ is*

$$E_Q\left(G_1 \otimes G_2\right)=4\left(n_2+1\right) E_Q\left(G_1\right)+n_2^2 M_1\left(G_1\right)+m_1\left(E_Q\left(G_2\right)+2\left(4 m_2+1\right)\right).$$

Proof. By definition (1), we have that

$$E_Q\left(G_1 \otimes G_2\right)=\sum_{i=1}^{\left|V\left(G_1 \otimes G_2\right)\right|} \operatorname{deg}_{G_1 \otimes G_2}^2\left(v_i\right)+\sum_{i=1}^{\left|V\left(G_1 \otimes G_2\right)\right|} \operatorname{deg}_{G_1 \otimes G_2}\left(v_i\right).$$

Splitting the vertex set $V\left(G_1 \otimes G_2\right)$ into disjoint sets $V\left(G_1\right)$, $V\left(G_2\right)$ and $I\left(G_1\right)$, we have

$$\begin{aligned}E_Q\left(G_1 \otimes G_2\right)= & \sum_{v \in V\left(G_1\right)} \operatorname{deg}_{G_1 \otimes G_2}^2(v)+\sum_{v \in I\left(G_1\right)} \operatorname{deg}_{G_1 \otimes G_2}^2(v) \\ & +\sum_{v \in V\left(G_2\right)} \operatorname{deg}_{G_1 \otimes G_2}^2(v)+\sum_{v \in V\left(G_1\right)} \operatorname{deg}_{G_1 \otimes G_2}(v) \\ & +\sum_{v \in I\left(G_1\right)} \operatorname{deg}_{G_1 \otimes G_2}(v)+\sum_{v \in V\left(G_2\right)} \operatorname{deg}_{G_1 \otimes G_2}(v).\end{aligned}$$

Substituting the degrees of the vertices of $G_1 \otimes G_2$ from (9), we get

$$E_Q(G_1 \otimes G_2) = \sum_{i=1}^{n_1} \left((n_2+2)\deg_{G_1}(v_i)\right)^2 + \sum_{i=1}^{m_1}(2)^2 + \sum_{i=1}^{m_1}\sum_{j=1}^{n_2}\left(\deg_{G_2}(v_j)+2\right)^2$$

$$+ \sum_{i=1}^{n_1}\left((n_2+2)\deg_{G_1}(v_i)\right) + \sum_{i=1}^{m_1}(2) + \sum_{i=1}^{m_1}\sum_{j=1}^{n_2}\left(\deg_{G_2}(v_j)+2\right)$$

$$= (n_2+2)^2\sum_{i=1}^{n_1}\deg^2_{G_1}(v_i) + 4m_1 + \sum_{i=1}^{m_1}\sum_{j=1}^{n_2}\deg^2_{G_2}(v_j) + 4\sum_{i=1}^{m_1}\sum_{j=1}^{n_2}\deg_{G_2}(v_j)$$

$$+ 4m_1n_2 + (n_2+2)\sum_{i=1}^{n_1}\deg_{G_1}(v_i) + 2m_1 + \sum_{i=1}^{m_1}\sum_{j=1}^{n_2}\deg_{G_2}(v_j) + 2m_1n_2.$$

By handshaking lemma, we compute each summation as follows:

$$E_Q(G_1 \otimes G_2) = (n_2+2)^2\sum_{i=1}^{n_1}\deg^2_{G_1}(v_i) + 4m_1 + \sum_{i=1}^{m_1}\sum_{j=1}^{n_2}\deg^2_{G_2}(v_j)$$

$$+ 4\sum_{i=1}^{m_1}(2m_2) + 4m_1n_2 + (n_2+2)(2m_1) + 2m_1 + \sum_{i=1}^{m_1}(2m_2) + 2m_1n_2.$$

After rearranging,

$$E_Q(G_1 \otimes G_2) = 4(n_2+1)\left(\sum_{i=1}^{n_1}\deg^2_{G_1}(v_i) + 2m_1\right) + n_2^2\sum_{i=1}^{n_1}\deg^2_{G_1}(v_i)$$

$$+ m_1\left(\sum_{j=1}^{n_2}\deg^2_{G_2}(v_j) + 2m_2\right) + 2m_1(4m_2+1).$$

Comparing with definitions (1) and (2), the desired result for $E_Q(G_1 \otimes G_2)$ follows.

□

3. Summary and conclusion

Graph energy has so many applications in the field of chemistry, physics, biology, mathematics and sociology. By the approach presented in [8], the relation between quasi-Laplacian energy and the vertex degrees of a graph was envisaged. In this paper, it is also observed that the first Zagreb index can be handled with its connection to graph energy. A new and significant application of the first Zagreb index to composite graphs based on fractal graphs is revealed, and exact formulae for quasi-Laplacian energy are derived in terms of the corresponding energies, the first Zagreb indices, number of vertices and edges of the underlying graphs of those composite graph types.

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