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Atoms of the lattices of residuated mappings

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Abstract. Given a lattice L, we denote by $\mathrm{Res}(L)$ the lattice of all residuated maps on L. The main objective of the paper is to study the atoms of $\mathrm{Res}(L)$ where L is a complete lattice. Note that the description of dual atoms of $\mathrm{Res}(L)$ easily follows from earlier results of Shmuely (1974). We first consider lattices L for which all atoms of $\mathrm{Res}(L)$ are mappings with 2-element range and give a sufficient condition for this. Extending this result, we characterize these atoms of $\mathrm{Res}(L)$ which are weakly regular residuated maps in the sense of Blyth and Janowitz (Residuation Theory, 1972). In the rest of the paper we investigate the atoms of $\mathrm{Res}(M)$ where M is the lattice of a finite projective plane, in particular, we describe the atoms of $\mathrm{Res}(F)$, where F is the lattice of the Fano plane.

1. Introduction

Let A and B be (partially) ordered sets. Recall that a mapping $f : A \to B$ is called *residuated* if there exists a mapping $g : B \to A$ such that

$$\forall x \in A, \forall y \in B \quad f(x) \leq y \ \Leftrightarrow\ x \leq g(y)\,.$$

In this case the mapping g is uniquely determined, called the *residual* of f and denoted f^*. The set of all residuated mappings from A to B is denoted by $\mathrm{Res}(A, B)$.

Let now L be a complete lattice. We denote $\mathrm{Res}(L) = \mathrm{Res}(L, L)$ and $\mathrm{Res}^*(L) = \{f^* \mid f \in \mathrm{Res}(L)\}$. It is well known that then $\mathrm{Res}(L)$ and $\mathrm{Res}^*(L)$ consist of complete join endomorphisms and complete meet endomorphisms of L, respectively. Clearly $\mathrm{Res}(L)$ and $\mathrm{Res}^*(L)$ are complete lattices with respect to the pointwise defined order relation and the bijection $f \leftrightarrow f^*$ is

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an anti-isomorphism between them. The study of the lattice $\mathrm{Res}(L)$ goes back to [4], where Grätzer and Schmidt gave a general form of its elements for the case of distributive lattice L. The aim of the present paper is to investigate atoms of the lattice $\mathrm{Res}(L)$ where L is a complete lattice. For motivation, we refer to the paper [6], where the atoms of $\mathrm{Res}(L)$ came up in an attempt to describe locally order affine complete lattices.

We first observe that dual atoms of this lattice can be easily described without any restrictions on the lattice L. In fact this is an easy consequence of results by Shmuely [13].

Unfortunately, the situation with atoms is drastically different. Actually only in very special cases one can completely describe the atoms of $\mathrm{Res}(L)$. We first consider the lattices L for which all atoms of $\mathrm{Res}(L)$ have 2-element range. It is known [11, 12] that completely distributive lattices enjoy such property. Our new sufficient condition (Theorem 7) allows to find more lattices with this property, for example, pseudocomplemented lattices of finite height. Since the atoms with 2-element range are weakly regular in the sense of Blyth and Janowitz [2], we then study when the atoms of $\mathrm{Res}(L)$ are weakly regular. In the rest of the paper we investigate the atoms of $\mathrm{Res}(M)$ where M is the lattice of a finite projective plane. It turns out that all atoms of $\mathrm{Res}(F)$ where F is the lattice of the Fano plane are weakly regular but this is not true for projective planes, in general.

2. Preliminaries

In what follows all lattices are assumed to be complete. Given a lattice L, we call a binary relation $R \subseteq L \times L$ *compatible* if it is a complete sublattice of $L \times L$. Recall that reflexive and symmetric binary relations are called *tolerances.* A *tolerance* on a lattice L is by definition a tolerance relation on L which also is a compatible binary relation on L. The set of all tolerances on a lattice L will be denoted by $\mathrm{Tol}(L)$. Obviously, the set $\mathrm{Tol}(L)$ is a lattice with respect to the set theoretical inclusion. We call a lattice *tolerance simple* if the trivial tolerances $\Delta_L = \{(x,x) \mid x \in L\}$ and $\nabla_L = L \times L$ are the only members of $\mathrm{Tol}(L)$. It is important for us that there is a close relationship between residuated maps and tolerances, established by Janowitz [5], see also [8], Proposition 3.2. Given a tolerance $T \in \mathrm{Tol}(L)$, consider the mapping $f^T : L \to L$ defined by $f^T(x) = \bigwedge\{z \mid (z,x) \in T\}$. Then $f^T \in \mathrm{Res}(L)$ and, moreover, f^T is decreasing, that is, $f^T(x) \le x$ for every $x \in L$. We will denote the set of all decreasing residuated maps on L by $\mathrm{Res}_{\downarrow}(L)$. On the other hand, given a mapping $f \in \mathrm{Res}_{\downarrow}(L)$, the binary relation

$$T = T^f = \{(x,y) \in L^2 \mid f(x \vee y) \le x \wedge y\}$$

is a tolerance on L and $f^T = f$. Thus there is a one-to-one correspondence, actually an anti-isomorphism between lattices $\mathrm{Tol}(L)$ and $\mathrm{Res}_{\downarrow}(L)$. It is easy to see that a tolerance T is a congruence of L if and only if the corresponding residuated mapping f^T is idempotent. For easy reference, we summarize these facts in the following theorem.

Theorem 1. *For every lattice L, the correspondence $T \leftrightarrow f^T$ is an anti-isomorphism between the lattices* $\mathrm{Tol}(L)$ *and* $\mathrm{Res}_{\downarrow}(L)$. *In particular, f^T is a minimal decreasing residuated mapping on L if and only if T is a maximal tolerance of L. The mapping f^T is idempotent if and only if T is a congruence of L.*

Clearly all automorphisms of a lattice L are residuated mappings. In view of our interest in atoms of $\mathrm{Res}(L)$, the following lemma is useful. Its proof is straightforward.

Lemma 2. *Let $f \in \mathrm{Res}(L)$ and $\alpha \in \mathrm{Aut}(L)$. The mapping f is an atom of* $\mathrm{Res}(L)$ *if and only if $f\alpha$ is an atom of* $\mathrm{Res}(L)$ *if and only if αf is an atom of* $\mathrm{Res}(L)$.

3. Dual atoms of $\mathrm{Res}(L)$

Given elements $a, b \in L$, we define a mapping $f_{ab} : L \to L$ by the rule:

$$f_{ab}(x) = \begin{cases} 0, \text{ if } x = 0; \\ b, \text{ if } x \leq a,\ x \neq 0; \\ 1, \text{ otherwise.} \end{cases}$$

It is easy to check that $f_{ab} \in \mathrm{Res}(L)$ for arbitrary $a, b \in L$. Note that if $a = 0$ or $b = 1$, then f_{ab} is the greatest element of $\mathrm{Res}(L)$, that is, $f_{ab}(x) = 1$ whenever $x \neq 0$.

Probably for the first time such mappings appeared in Shmuely's paper [13]. Technically, Shmuely's approach was somewhat different. Given ordered sets A and B, she studied the set of functions $A \otimes B$ which in our notation is precisely $\mathrm{Res}(A, B^d)$ where B^d denotes the dual of the ordered set B. Thus all of her results can be translated into the language of residuated functions. In particular, Shmuely introduced the mappings L_b^a that correspond to our functions f_{ab} and obtained the following results, in different notation, of course.

Lemma 3. ([13], Lemma 2.5) *Let $a, b, c, d \in L$, $c \neq 0$ and $d \neq 1$. Then:*

$$f_{ab} \leq f_{cd} \quad \Leftrightarrow \quad c \leq a \text{ and } b \leq d\,.$$

It follows that if $a, c \in L \setminus \{0\}$ and $b, d \in L \setminus \{1\}$, then

$$f_{ab} = f_{cd} \quad \Leftrightarrow \quad c = a \text{ and } b = d\,.$$

Lemma 4. ([13], Theorem 2.5) *Every $g \in \mathrm{Res}(L)$ can be represented as the meet of some set of mappings f_{ab}, $a, b \in L$. In particular,*

$$g = \bigwedge \{ f_{ag(a)} \mid a \in L \}\,.$$

Proof. Let $a \in L$, we first prove that $g(x) \leq f_{ag(a)}(x)$ for any $x \in L$. This is clearly true if $x = 0$ or $x \nleq a$. If $x \neq 0$ but $x \leq a$, then $g(x) \leq g(a) = f_{ag(a)}(a)$. This proves that $g \leq \bigwedge\{f_{ag(a)} \mid a \in L\}$. Let now $h \in \operatorname{Res}(L)$ be such that $h \leq f_{ag(a)}$ for every $a \in L$. Then, in particular, $h(a) \leq f_{ag(a)}(a) = g(a)$ for every $a \in L$. Thus, $g = \bigwedge\{f_{ag(a)} \mid a \in L\}$. □

Theorem 5. *Every dual atom of* $\operatorname{Res}(L)$ *has the form* f_{ab} *where* a *is a dual atom of* L *and* b *is an atom of* L.

Proof. Let g be a dual atom of $\operatorname{Res}(L)$. By Lemma 4, there exist $a, b \in L$ such that $g = f_{ab}$. Now Lemma 3 yields that a and b must be a dual atom and an atom of L, respectively. □

Corollary 6. *If* L *is atomistic and dually atomistic, then* $\operatorname{Res}(L)$ *is a dually atomistic (complete) lattice.*

4. Atoms with 2-element range

If $f \in \operatorname{Res}(L)$ and f is not the zero map, then $|f(L)| \geq 2$. In fact the residuated maps with $|f(L)| = 2$ always exist. Take arbitrary $a, b \in L$ and define:

$$e_{ab}(x) = \begin{cases} 0, \text{ if } x \leq a; \\ b, \text{ if } x \nleq a\,. \end{cases}$$

It is easy to check that all mappings e_{ab} are residuated. It is also easy to see that every $f \in \operatorname{Res}(L)$ with $|f(L)| = 2$ has the form e_{ab}. Such mappings first appeared almost simultanously in the papers by Schreiner [12] and Shmuely [13]. They both used the same notation E^a_b but Shmuely's E^a_b is the dualized version of Schreiner's E^a_b which is exactly our e_{ab}.

Following Raney's paper [11], Schreiner [12] introduced tight residuated mappings and proved that, for any lattice L, every tight residuated mapping is a join of some set of mappings e_{ab}. In view of [11] and [12], all members of $\operatorname{Res}(L)$ are tight if and only if the lattice L is completely distributive. These facts easily imply that if L is completely distributive (in particular distributive of finite height), then all atoms of $\operatorname{Res}(L)$ have the form e_{ab}. In what follows, we prove a stronger result.

It is easy to see that the mapping e_{ab} is an atom of $\operatorname{Res}(L)$ if and only if a and b are a dual atom and an atom of L, respectively. Similarly, it is an easy exercise to check that the mapping e_{ab} is decreasing if and only if L satisfies the condition: for every $x \in L$, either $x \leq a$ or $b \leq x$. Following [12], we will call such pairs (a, b) *decreasing*. Clearly all the pairs (a, b) with $a = 1$ or $b = 0$ are decreasing. We will call them *trivial decreasing pairs* because the function they yield is the zero function. In view of Theorem 1, the non-trivial decreasing mappings of the form e_{ab} are precisely those induced by tolerances with exactly two blocks.

In what follows, $\uparrow a$ ($\downarrow a$), where $a \in L$, denotes the principal filter (the principal ideal) of the lattice L generated by a.

Theorem 7. *If every non-zero principal ideal of L, as a lattice, has a non-trivial decreasing pair, then all atoms of* $\mathrm{Res}(L)$ *have the form e_{ab}.*

Proof. Let f be an atom of $\mathrm{Res}(L)$, $|f(L)| \geq 3$. Let $a = f(1)$, $L' = \downarrow a$. By assumption L' has a non-trivial decreasing pair (b, c), let $g = e_{bc}f$. Now $g \in \mathrm{Res}(L)$ and $g \leq f$, since e_{bc} is decreasing. Clearly, $|g(L)| \leq 2$, thus $g \neq f$. Since f is an atom, g should be the zero map. However, $g(1) = e_{bc}(f(1)) = e_{bc}(a) = c \neq 0$. This contradiction proves that f must have 2-element range. □

Corollary 8. *If L is completely distributive, then all atoms of* $\mathrm{Res}(L)$ *have the form e_{ab}.*

Proof. Let L be a completely distributive lattice and $0 \neq a \in L$. Then the principal ideal $L_1 = \downarrow a$ is completely distributive, too. Now, the existence of a non-trivial decreasing pair in L_1 is proved in Theorem 5 of [11]. □

Problem 1. Does there exist a distributive lattice L such that some atom of $\mathrm{Res}(L)$ is not of the form e_{ab}?

Corollary 9. *If L has a single atom, then all atoms of* $\mathrm{Res}(L)$ *have the form e_{ab}.*

Proof. Let p be the atom of L and $0 \neq a \in L$. Then $p \leq a$ and p is the only atom of $L_1 = \downarrow a$. Now, (p, p) is a nontrivial decreasing pair for L_1. □

Corollary 10. *Let L be a lattice of finite height such that every principal ideal of L has a non-trivial distributive homomorphic image. Then all atoms of* $\mathrm{Res}(L)$ *have the form e_{ab}.*

Proof. Let $a \in L$ and $L_1 = \downarrow a$. Since L_1 has a non-trivial distributive homomorphic image, it also has a 2-element homomorphic image, hence also a congruence with exactly two blocks. It follows that L_1 has a decreasing pair. □

Corollary 11. *If L is a pseudocomplemented lattice of finite height, then all atoms of* $\mathrm{Res}(L)$ *have the form e_{ab}.*

Proof. Recall that a lattice L is called pseudocomplemented if, for every $a \in L$, the set $\{x \in L \mid x \wedge a = 0\}$ has a largest element, denoted by a^*. It is well known that the mapping $\phi : L \to L$, $\phi(x) = x^{**}$, is a lattice homomorphism and $\phi(L)$ is a Boolean lattice. If $a \in L$ is arbitrary, then obviously $a \leq a^{**}$. Hence $\phi(\downarrow a)$ is a non-trivial distributive homomorphic image of $\downarrow a$. Now Corollary 10 implies that all atoms of $\mathrm{Res}(L)$ have the form e_{ab}. □

Problem 2. Does there exist a finite lattice L such that all atoms of $\mathrm{Res}(L)$ have the form e_{ab} but there is a principal ideal of L with no non-trivial distributive homomorphic image?

A partial answer to this problem is given by the following proposition.

Proposition 12. *If L is a lattice of finite height and all atoms of* $\mathrm{Res}(L)$ *have the form e_{ab}, then L has a non-trivial decreasing pair.*

Proof. Let T be a maximal tolerance of L. Then $f = f^T$ is a non-zero decreasing residuated map of L. Since L is of finite height, there exists an atom g of $\mathrm{Res}(L)$ such that $g \leq f$. By our assumption, g has the form e_{ab} and since f is decreasing, g is decreasing, too. It follows that (a, b) is a decreasing pair. ☐

5. Weakly regular atoms of $\mathrm{Res}(L)$

We start with a proposition that gives necessary conditions for a mapping f to be an atom of $\mathrm{Res}(L)$. Note that if $f \in \mathrm{Res}(L)$, then $f(L)$ is not necessarily a sublattice of L. However, still the ordered set $f(L)$ is a lattice, which we will denote by $\mathcal{L}_f$. Similarly, we denote by $\mathcal{L}_f^*$ the lattice $f^*(L)$.

Proposition 13. *If f is an atom of* $\mathrm{Res}(L)$*, then the lattices $\mathcal{L}_f$ and $\mathcal{L}_f^*$ are tolerance simple.*

Proof. Suppose T is a tolerance of $\mathcal{L}_f$ and $T \neq \nabla_{\mathcal{L}_f}$. Consider the mapping $g = f^T f : L \to L$. Since the join operation is the same in $\mathcal{L}_f$ and L, we have $g \in \mathrm{Res}(L)$ and $g(x) = f^T f(x) \leq f(x)$ for every $x \in L$. This yields $0 \leq g \leq f$ in $\mathrm{Res}(L)$. Since f is an atom, we have either $g = 0$ or $g = f$. Since $T \neq \nabla_{\mathcal{L}_f}$, f^T is not the zero map, hence $g \neq 0$. Thus, $g = f$, i.e. $f^T(f(x)) = f(x)$, for all $x \in L$. We see that f^T is identical on $f(L)$, whence $T = \Delta_{\mathcal{L}_f}$. This proves that the lattice $\mathcal{L}_f$ is tolerance simple.

The second claim is proved using dual arguments because f is an atom of $\mathrm{Res}(L)$ if and only if f^* is a coatom of $\mathrm{Res}^*(L)$. ☐

A disadvantage of the condition in Proposition 13 is that it is given in terms of the lattice $\mathcal{L}_f$, not in terms of L. This suggests to consider atoms f for which $f(L)$ is a sublattice of L. In particular, this condition is satisfied by so-called *weakly regular* residuated mappings introduced by Blyth and Janowitz in [2]. As they claim, such mappings are extremely important, especially because of their relationship with modularity of lattices.

Definition 1. A mapping $f \in \mathrm{Res}(L)$ is called *range closed* if $f(L) = \downarrow f(1)$ and *dually range closed* if $f^*(L) = \uparrow f^*(0)$. A mapping which is both range closed and dually range closed is called *weakly regular*.

In view of [2, Theorem 13.2] and the remark after it, weakly regular residuated mappings have the following characterization in terms of isomorphisms between ideals and filters.

Proposition 14. *Let L be a lattice and $f : L \to L$. Then the following are equivalent:*

(1) f is a weakly regular residuated mapping;
(2) there are $a, b \in L$ and a lattice isomorphism $\alpha : \uparrow a \to \downarrow b$ such that $f(x) = \alpha(x \vee a)$ for every $x \in L$.

Theorem 15. *Let $f \in \mathrm{Res}(L)$ be weakly regular. Then f is an atom of* $\mathrm{Res}(L)$ *if and only if the lattice $f(L) = \mathcal{L}_f$ is tolerance simple.*

Proof. The necessity follows from Proposition 13. For sufficiency, let $\alpha : \uparrow a \to \downarrow b$ be the lattice isomorphism determining f. Then clearly $f(1) = b$ and $f(L) = \downarrow b$. For simplicity, write $\mathcal{L}$ instead of $\mathcal{L}_f$. Assume that the lattice $\mathcal{L}$ is tolerance simple and take $g \in \mathrm{Res}(L)$ such that $g < f$. Then $g(a) \leq f(a) = \alpha(a) = 0$, hence $g(a) = 0$. Next we show that, for any $x, y \in L$,

$$f(x) = f(y) \;\Rightarrow\; g(x) = g(y)\,. \tag{1}$$

Indeed, if $f(x) = f(y)$, then $x \vee a = y \vee a$ and

$$g(x) = g(x) \vee g(a) = g(x \vee a) = g(y \vee a) = g(y) \vee g(a) = g(y)\,.$$

Since $f(L) = \downarrow b$ and $g \leq f$, the formula (1) implies that the mapping $h : f(L) \to f(L)$, $h(f(x)) = g(x)$ is well defined. It is routine to check that h is residuated and $g \leq f$ implies that $h \in \mathrm{Res}_{\mathcal{L}}$.

Therefore, by Theorem 1, there exists a tolerance T of $\mathcal{L}$ such that $h = f^T$. Since $\mathcal{L}$ is tolerance simple, $T \in \{\Delta_{\mathcal{L}}, \nabla_{\mathcal{L}}\}$. If $T = \Delta_{\mathcal{L}}$, then $h = 1_{\mathcal{L}}$, hence $g = hf = f$, a contradiction. Thus, $T = \nabla_{\mathcal{L}}$ implying $h = 0$ and $g = hf = 0$. This proves that f is an atom of $\mathrm{Res}(L)$. □

Now we use Theorem 15 for exhibiting some interesting examples of atoms of $\mathrm{Res}(L)$.

Example 1. Atoms of the form e_{ab} are weakly regular. Recall that a mapping e_{ab} is an atom of $\mathrm{Res}(L)$ if and only if a and b are a dual atom and an atom of L, respectively. It is easy to see that e_{ab} is the weakly regular residuated mapping determined by the lattice isomorphism $\alpha : \uparrow a \to \downarrow b$.

Example 2. Every automorphism of L is a weakly regular residuated mapping. Indeed, an automorphism α of L is the weakly regular residuated mapping of L determined by the isomorphism $\alpha : \uparrow 0 \to \downarrow 1$. It follows from Theorem 15 that an automorphism of L is an atom of $\mathrm{Res}(L)$ if and only if the lattice L is tolerance simple.

Example 3. Let L be a modular lattice and $s, t \in L$ the complements of each other. Define $f : L \to L$ by the formula $f(x) = (x \vee s) \wedge t$. Such mappings are often called *projections*. It is well known that in this situation the mapping $\alpha : \uparrow s \to \downarrow t$, $\alpha(y) = y \wedge t$, is a lattice isomorphism. Thus $f(x) = \alpha(x \vee s)$ for every $x \in L$ which means that f is a weakly regular residuated mapping. It follows from Theorem 15 that the mapping f is an atom of $\mathrm{Res}(L)$ if and only if $\downarrow t$ (and also $\uparrow s$) is a tolerance simple lattice.

Example 4. ([1], Example 4.2) Let V be a vector space over a field K and $\phi \colon V \to V$ a linear mapping. Then ϕ induces the mapping $f : \mathrm{Sub}(V) \to \mathrm{Sub}(V)$, $f(W) = \phi(W) = \{\phi(x) \mid x \in W\}$. According to [1], f is a weakly regular residuated mapping of $\mathrm{Sub}(V)$. It follows from Theorem 15 that the mapping f is an atom of $\mathrm{Res}(\mathrm{Sub}(V))$ if and only if the lattice $\downarrow f(\mathrm{Sub}(V))$ (the lattice of subspaces of $\phi(V)$) is tolerance simple. It is well known that this is always the case when $\phi(V)$ is finite dimensional.

Example 5. Let M_n, $n \geq 3$, be a finite lattice of size $n + 2$ where all members of $L \setminus \{0, 1\}$ are both atoms and coatoms. Note that all these lattices are tolerance simple. In particular, M_3 is the smallest modular but non-distributive lattice. Foreman [3] has described join irreducibles of $\mathrm{Res}(M_n)$ for every $n \geq 3$. It follows that every atom of $\mathrm{Res}(M_n)$ is either an automorphism of M_n or the mapping e_{ab} where a and b are atoms of M_n. Hence, all atoms of $\mathrm{Res}(M_n)$ are weakly regular.

6. Lattices of projective planes

We have seen that atoms of $\mathrm{Res}(L)$ have a nice description if L is a distributive lattice of finite height. Since modularity is a property of lattices close to distributivity, it is natural to try to describe atoms of $\mathrm{Res}(L)$ for finite modular lattices L. Unfortunately, it has turned out that even in the case of very nice modular lattices L this problem can be very hard. About 15 years ago Vladimir Kuchmei and Stefan Schmidt (the former coauthors of the first author) considered atoms of the lattices $\mathrm{Res}(M)$ where M is a finite complemented modular lattice. Surely this is a very important class of lattices. Eventually they restricted to the lattices of projective planes but even then a satisfactory result was obtained only for the Fano plane (the smallest projective plane). For all other projective planes they proved that the corresponding lattice $\mathrm{Res}(M)$ contains certain "exotic" atoms. It has to be mentioned that the idea of construction of those atoms was suggested by Ralph Mckenzie. The proofs were complicated and technical and eventually the manuscript [10] remained unpublished.

Kuchmei and Shmidt kindly allowed us to use their manuscript for which we are most grateful. Our hope was that we can considerably simplify their proofs. To be honest, we were not succesful. Still we think it makes sense to publish these results just in order to show the complexity of the problem. Our proofs only partly follow those given in [10]. They are not shorter but perhaps more transparent and easier to follow than those in [10].

Let M be the lattice of a finite projective plane of order n ($n \geq 2$). Then $M = \{0, 1\} \cup P \cup L$ where P and L are the sets of atoms (called points) and coatoms (called lines) of M, respectively. It is required that:

- $P \cap L = \emptyset$;
- every line covers exactly $n + 1$ points and, dually, every point is covered by exactly $n + 1$ lines;
- two different lines cover exactly one common point and, dually, two different points are covered by exactly one common line.

It is natural to identify every line with the set of points below it, that is, if $l \in L$, then $l = \{x \in P \mid x < l\}$. Hence, if $x \in P$ and $l \in L$, then $x < l$ is equivalent to $x \in l$. Occasionally we will use geometric terminology like two lines intersect in a point or a line goes through a point.

It is well known that $|P| = |L| = n^2 + n + 1$. All known finite projective planes have prime power order and for every prime power n there exists a projective plane of order n.

Since the lattice M is atomistic, every $f \in \mathrm{Res}(M)$ is uniquely determined by its restriction to P. The following lemma describes the functions $P \to M$ that can be extended (uniquely) to residuated mappings on M.

Lemma 16. *If $f \in \mathrm{Res}(M)$ and $x, y, z \in P$, then*

$$x \leq y \vee z \quad \Rightarrow \quad f(x) \leq f(y) \vee f(z)\,. \tag{2}$$

On the other hand, every mapping $f : P \to M$ satisfying (2) can be uniquely extended to a residuated mapping on M.

Proof. The first claim is obvious because f is a join endomorphism of M. In order to prove the second claim, assume that a mapping $f : P \to M$ satisfies the condition (2) and define a mappping $g : M \to M$ by

$$g(x) = \bigvee \{f(p) \mid p \in P, p \leq x\}\,.$$

It is easy to see that $g(0) = 0$, $g|_P = f$, and g is order preserving. Let $x, y \in M$. Since g is order preserving, we have $g(x \vee y) \geq g(x) \vee g(y)$. For the converse, take $p \in P$ such that $p \leq x \vee y$ and let $p_x, p_y \in P$ be such that $p_x \leq x$, $p_y \leq y$ and $p \leq p_x \vee p_y$. The existence of such points can be easily proved by checking cases. For example, consider the case $x, y \in L$ which probably is most complicated. If $x = y$, then take $p_x = p_y = p$. Otherwise, choose any line l such that $x \wedge y \notin l$ and define $p_x = x \wedge l$ and $p_y = y \wedge l$.

Now, using condition (2) and the definition of g, we have:

$$f(p) \leq f(p_x) \vee f(p_y) \leq g(x) \vee g(y)\,.$$

Using again the definition of g, we conclude $g(x \vee y) \leq g(x) \vee g(y)$. $\square$

In view of this lemma, we will use the same notation for $f \in \mathrm{Res}(M)$ and its restriction to P. Moreover, we call the mappings $P \to M$ satisfying (2) the residuated mappings on M. Note that actually condition (2) puts a restriction to the action of f on every single line. This allows us to speak about applying Lemma 16 to some line.

7. The Fano lattice

As mentioned above, there exists a projective plane of order 2. It is unique up to isomorphism and it is called *the Fano plane.* We call the lattice of the Fano plane *the Fano lattice* and denote it by F. The Fano lattice has 7 points and 7 lines, thus $\mid F \mid = 16$, every line of F contains exactly 3 points and there are exactly 3 lines that go through a given point.

It is well known that the size of the automorphism group $\mathrm{Aut}(F)$ is 168 and it is easy to see that the number of non-collinear ordered triples of points of the Fano plane is 168, too. This implies the following lemma.

Lemma 17. *The group* $\mathrm{Aut}(F)$ *acts transitively on the set of non-collinear ordered triples of points of* F. *Consequently, it also acts transitively on the set of ordered pairs* (a, b), $a \neq b$, *of points of* F.

Our aim is to prove the following theorem.

Theorem 18. *Every atom of* $\mathrm{Res}(F)$ *is weakly regular.*

As the first step, we handle the case $f(1) \neq 1$.

Lemma 19. *Let* f *be an atom of* $\mathrm{Res}(F)$. *If* $f(1) \neq 1$, *then* f *is weakly regular.*

Proof. Since f is an atom of $\mathrm{Res}(F)$, it is not the zero map, Thus, there are two possibilities: either $f(1)$ is a point or $f(1)$ is a line. The first case is trivial because then $|f(F)| = 2$, so f is weakly regular. We now focus on the second case when $f(1) = s$ is a line. Assume that $s = \{a_1, a_2, a_3\}$. There are 4 possibilities: 1) $f(F) = \{0, s\}$, 2) $f(F) = \{0, a_1, s\}$, 3) $f(F) = \{0, a_1, a_2, s\}$, 4) $f(F) = \{0, a_1, a_2, a_3, s\} = \downarrow s$. In the first case f is not an atom because $e_{0a_1} < f$. Also in the cases 2) and 3) f cannot be an atom, because then the lattice $\mathcal{L}_f$ is not tolerance simple (see Proposition 13). Remains the case 4) when f is range closed.

We have to prove that f is also dually range closed, that is, $f^*(F) = \uparrow f^*(0)$. Note that just dualizing the beginning of the proof, we get the statement: if f^* is a coatom of $\mathrm{Res}^*(F)$ and $f^*(0) \neq 0$, then $f^*(F) = \uparrow f^*(0)$. Since f is an atom of $\mathrm{Res}(F)$ if and only if f^* is a coatom of $\mathrm{Res}^*(F)$, we only need to prove that, under assumptions of our lemma, $f^*(0) \neq 0$, that is, $f(x) = 0$ for some $x \neq 0$.

Let $b_i = \bigvee f^{-1}(a_i)$, $i = 1, 2, 3$. Clearly, the elements b_i are points or lines. Suppose that two of them, say b_1 and b_2 are lines. Then $c = b_1 \wedge b_2$ is a point and $f(c) \le a_1 \wedge a_2 = 0$. Let now two of the elements b_i be points, say $b_2, b_3 \in P$. Then $x \in P \setminus \{b_2, b_3\}$ implies $f(x) \in \{0, a_1, s\}$. If $f(x) = 0$ for some $x \in P$, then we are done. Otherwise it is easy to see that $e_{la_1} < f$ where $l = b_2 \vee b_3$. This contradicts the assumption that f is an atom of $\mathrm{Res}(F)$. □

Lemma 20. *Let f be an atom of* $\mathrm{Res}(F)$*. If $f^*(0) \neq 0$, then f is weakly regular.*

Proof. This can be proved similarly to Lemma 19, or derived from that lemma using a duality argument. □

Thus, in order to prove Theorem 18, we may restrict ourselves to the case $f(1) = 1$ and $f^*(0) = 0$. An important step of the proof is given by the following lemma.

Lemma 21. *Let f be an atom of* $\mathrm{Res}(F)$ *such that $f(1) = 1$ and $f^*(0) = 0$. Then the restriction of f to P is injective.*

Proof. Assume that there exist $a_1, a_2 \in P$, $a_1 \neq a_2$, such that $f(a_1) = f(a_2)$. Let $l_0 = a_1 \vee a_2$, a_3 be the third point of l_0 and $B = P \setminus l_0 = \{b_1, b_2, b_3, b_4\}$. The numeration can be chosen so that the sets

$$l_1 = \{a_1, b_1, b_4\},\ l_2 = \{a_2, b_2, b_4\},\ l_3 = \{a_3, b_3, b_4\},$$

$$l_4 = \{a_1, b_2, b_3\},\ l_5 = \{a_2, b_1, b_3\},\ l_6 = \{a_3, b_1, b_2\}$$

are the lines. For the rest the following observation is important:

$$a_1 < (l_i \wedge l_j) \vee (l_k \wedge l_m) \tag{3}$$

whenever $\{i, j, k, m\} = \{2, 3, 5, 6\}$ which means that the set $\{a_1, l_i \wedge l_j, l_k \wedge l_m\}$ is a line. Indeed, whatever order of $\{2, 3, 5, 6\}$ we choose, always

$$\{l_i \wedge l_j, l_k \wedge l_m\} \in \{\{a_2, a_3\}, \{b_1, b_4\}, \{b_2, b_3\}\}\,.$$

We must prove that $a_1 = a_2$. We split the proof of this equality into three parts depending on where the element $a = f(a_1) = f(a_2)$ lies. Note that $a \neq 0$ because $f^*(0) = 0$.

Part 1, $a = 1$. Applying Lemma 16 to the lines that contain either a_1 or a_2, we conclude

$$f(b_1) \vee f(b_4) = f(b_2) \vee f(b_4) = f(b_2) \vee f(b_3) = f(b_1) \vee f(b_3) = 1\,. \tag{4}$$

It follows that if $f(b_3)$ or $f(b_4)$ is a point, then $f(b_1), f(b_2) \in L \cup \{1\}$. Thus, either $f(b_1), f(b_2) \in L \cup \{1\}$ or $f(b_3), f(b_4) \in L \cup \{1\}$. We handle the first case, the second one is similar. Since $f(b_1), f(b_2) \in L \cup \{1\}$, there exists $b \in P$ so that $b \le f(b_1) \wedge f(b_2)$. Let $g = e_{l_3,b}$. It is easy to see that $g \le f$ but $g \neq f$ because $g(a_1) = b < 1 = f(a_1)$. This contradicts the minimality of f.

Part 2, $a \in P$. Due to Lemmas 17 and 2, without loss of generality, $a = a_1$. It follows from Lemma 16 that $f(a_3) = a_1$, too. For every $i \in \{1, 2, 3, 4\}$, we have $1 = a_1 \vee a_2 \vee b_i$ which implies

$$1 = f(1) = f(a_1) \vee f(a_2) \vee f(b_i) = a_1 \vee f(b_i)\,.$$

Thus, $f(b_i) = 1$ or $f(b_i) \in L$ and in the latter case $a_1 \not\leq f(b_i)$. Note that in view of Part 1, $f(b_i) = 1$ can hold only for one value of i. Observe that f acts injectively on B. Suppose, on the contrary, that there are i and j such that $i \neq j$ but $f(b_i) = f(b_j) = l$. We know that $l = 1$ is impossible, so $l \in L$. The line $b_i \vee b_j$ intersects the line l_0 in one of the points a_k, $k \in \{1, 2, 3\}$. Hence $a_k \leq b_i \vee b_j$ implying $a_1 = f(a_k) \leq f(b_i) \vee f(b_j) = f(b_i)$, a contradiction.

Thus, $f(B) = M = \{l_2, l_3, l_5, l_6\}$ or there is $i \in \{2, 3, 5, 6\}$ such that $f(B) = (M \setminus \{l_i\}) \cup \{1\}$. Assume first that $f(B) = M$ and define a mapping $g : P \to F$ by

$$g(a_1) = 0, g(a_2) = g(a_3) = a_1, g(b_1) = g(b_4) = f(b_1) \wedge f(b_4),$$

$$g(b_2) = g(b_3) = f(b_2) \wedge f(b_3)\,.$$

Since $f(B) = M$, formula (3) implies that the set

$$\{a_1, f(b_1) \wedge f(b_4), f(b_2) \wedge f(b_3)\} \tag{5}$$

is a line. It is easy to see, using Lemma 16, that g is residuated. Indeed, g maps the lines l_0, l_1 and l_4 to two-element sets containing 0 and the other four lines to the line (5). Since obviously $g < f$, we have a contradiction with minimality of f.

It remains to consider the case when there is $i \in \{2, 3, 5, 6\}$ such that $f(b_i) = 1$. Then there is exactly one mapping $h : P \to F$ such that $h(x) = f(x)$ if $x \neq b_i$ and $h(B) = M$. Clearly, $h < f$ and using Lemma 16 it is easy to check that h is residuated. Hence, in view of the previous step of the proof, h is not an atom of $\mathrm{Res}(F)$ and hence also f cannot be an atom of $\mathrm{Res}(F)$.

Part 3, $a \in L$. In view of Lemmas 17 and 2, without loss of generality, $a = l_0$. First note that

$$f(b_i) \not\leq l_0 \text{ for every } i \in \{1, 2, 3, 4\}\,. \tag{6}$$

Indeed, otherwise $f(a_1), f(a_2), f(b_i) \leq l_0$ and $a_1 \vee a_2 \vee b_i = 1$ would imply $1 = f(1) \leq l_0$, a contradiction. Next observe that the equalities (4) hold in the present case, too. Indeed, applying Lemma 16 to the line l_5, we get $l_0 \leq f(b_1) \vee f(b_3)$. Hence $f(b_1) \vee f(b_3) \in \{l_0, 1\}$ but $f(b_1) \vee f(b_3) = l_0$ is impossible because $f(b_i) \not\leq l_0$. The other three equalities are proved similarly.

The equalities (4) imply that either $f(b_1), f(b_2) \in L \cup \{1\}$ or $f(b_3), f(b_4) \in L \cup \{1\}$. Assume, without loss of generality, that $f(b_3), f(b_4) \in L \cup \{1\}$ and suppose that there exists $c \in P$ such that $f(b_3) \wedge f(b_4) \wedge l_0 \geq c \in P$. Then

$g = e_{l_6,c} \leq f$. Since $g(a_1) = c < a = f(a_1)$, f is not minimal, a contradiction. Thus,

$$f(b_3) \wedge f(b_4) \wedge l_0 = 0 \tag{7}$$

which means that $f(b_3)$ and $f(b_4)$ are different lines, moreover, the triple of lines $f(b_3), f(b_4), l_0$ is not confluent (has no common point). Let d be the only point that does not belong to any of these three lines. Also, there must exist one more line h such that $d \not< h$.

Suppose now that $f(b_1) \in P$. We know that $f(b_1) \notin l_0$ and in view of (4) also $f(b_1) \notin f(b_j)$, $j = 3, 4$. Hence, $f(b_1) = d$. Similarly, if $f(b_2) \in P$, then $f(b_2) = d$. Since by Part 2 of our proof $f(b_1) = f(b_2) \in P$ is impossible, we conclude that only one of the elements $f(b_1)$ and $f(b_2)$ can be a point.

Next we show that $f(b_1) \neq 1$, the proof that $f(b_2) \neq 1$ is similar. First suppose that $f(b_1) = 1$, $f(b_2) \in P$. By the previous paragraph, we have $f(b_2) = d$ and applying Lemma 16 to the line l_6 we get $1 \leq f(a_3) \vee d$. Hence, $f(a_3) \in L$ and $f(a_1) = f(a_2) = l_0$ implies $f(a_3) = l_0$.

Now define a mapping $g : P \to F$ as follows: $g(b_1) = h$, $g(x) = f(x)$ if $x \neq b_1$. We use again Lemma 16 to show that g is residuated. Since f and g differ only on the point b_1, we have to check only the lines l_1, l_5 and l_6. The points of the first two lines are mapped by g to three different lines, so the condition of Lemma 16 is satisfied. The points of l_6 are mapped to the set $\{l_0, h, d\}$. Since

$$l_0 \vee h = l_0 \vee d = h \vee d = 1\,,$$

we are done. Thus, g is a non-zero residuated mapping strictly less than f. This contradiction with minimality of f proves that $f(b_1) \neq 1$.

It remains to handle two cases: (i) $f(b_1) \in P$, $f(b_2), f(b_3), f(b_4) \in L$ and (ii) $f(b_i) \in L$, $i \in \{1, 2, 3, 4\}$.

(i) We denote $p_i = l_0 \wedge f(b_i)$, $i = 2, 3, 4$, and show that these three points are different, that is, $l_0 = \{p_2, p_3, p_4\}$. Assume first that $f(a_3) \in L$, hence $f(a_3) = l_0$. Note that the point $f(b_1)$ does not belong to any of the four lines $l_0, f(b_2), f(b_3), f(b_4)$. For $l_0, f(b_3)$ and $f(b_4)$ this follows from formulas (6) and (4) and as for $f(b_2)$, if $f(b_1) \leq f(b_2)$, then applying Lemma 16 to the line l_6, we get $l_0 = f(a_3) \leq f(b_1) \vee f(b_2) = f(b_2)$, a contradiction. Hence no three lines from this set are confluent because otherwise they would cover the whole P, in particular also $f(b_1)$. But then $p_i = p_j$ is impossible for $i \neq j$, as desired.

Assume now that $f(a_3) \in P$. We know that $f(a_3) < l_0$ and applying Lemma 16 to the line l_6, we conclude $f(b_2) \leq f(a_3) \vee f(b_1)$. Since $f(a_3), f(b_1) \in P$, this yields the equality $f(b_2) = f(a_3) \vee f(b_1)$, thus $f(a_3) < f(b_2)$ and consequently $f(a_3) = l_0 \wedge f(b_2) = p_2$. As $f(b_3) \neq f(b_4)$, Lemma 16 applied to the line l_3 implies $f(a_3) \not\leq f(b_3)$ and $f(a_3) \not\leq f(b_4)$, hence $p_i \neq p_2$, $i = 3, 4$. Finally, it follows from formula (7) that $p_3 \neq p_4$.

Now define a mapping $g : P \to F$ as follows:

$$g(b_1) = 0,\ g(a_1) = g(b_4) = p_4,\ g(a_2) = g(b_3) = p_3,\ g(a_3) = g(b_2) = p_2\,.$$

The mapping g maps the lines l_1, l_5 and l_6 to one of the sets $\{0, a_k\}$ where $k = 1, 2, 3$, and the remaining lines bijectively to l_0. Therefore Lemma 16 implies that g is residuated and obviously $g < f$. This is a contradiction with minimality of f.

(ii) Recall that now $f(b_i)$, $i \in \{1, 2, 3, 4\}$, are distinct lines. Let first $f(a_3) \in L$, hence, as above, $f(a_3) = l_0$. Since the four lines $f(b_i)$ are different, at least two of them must intersect l_0 in the same point, let it be a_k. Let l be any line satisfying the following condition:

$$x \in P,\ a_k \not\leq f(x) \ \Rightarrow\ x < l\,.$$

Such a line l exists because there are at least five points x such that $a_k \leq f(x)$. Then clearly $e_{la_k} < f$, a contradiction with minimality of f.

Finally, assume that $f(a_3) \in P$. Consider the mapping $g : P \to F$ defined by:

$$g(x) = \begin{cases} f(a_3), \text{ if } x < l_0; \\ f(x),\ \text{otherwise}, \end{cases}$$

and show that it is residuated. In view of Lemma 16, we have to show that $g(x) \leq g(y) \vee g(z)$ holds for any three collinear points x, y, z. Clearly, this is the case if these points belong to l_0. Otherwise $\{g(x), g(y), g(z)\} = \{f(a_3), f(b_i), f(b_j)\}$ where $i \neq j$. Since $f(b_i) \vee f(b_j) = 1$, we will be done if we show that $f(a_3) \not\leq f(b_i)$, that is, $f(a_3) \vee f(b_i) = 1$ for every i. We prove this for $i = 1$, the other cases are similar. If $f(a_3) \leq f(b_1)$, then, applying Lemma 16 to the line l_6, we have $f(b_2) \leq f(b_1) \vee f(a_3) = f(b_1)$, i.e. $f(b_2) = f(b_1)$, a contradiction. It remains to notice that $g(a_1) = f(a_3) < l_0 = f(a_1)$, hence $g < f$. □

We continue the proof of Theorem 18. It seems to us that a different, "symmetric" notation of points is convenient now. We choose a non-collinear triple of points a_1, a_2, a_3 and denote by a_{ij} the third point of the line $a_i \vee a_j$, $i < j$. The seventh point will be denoted by b. As it was mentioned already, we may assume that $f(1) = 1$ and $f^*(0) = 0$, that is, $f(x) = 0$ holds only for $x = 0$. In view of Lemma 21, we know that f is injective. We split the remaining part of the proof into cases depending on properties of the set $X = P \cap f^{-1}(P)$. It will turn out that in Case 1 the mapping f is an automorphism of F while the other cases do not occur at all.

Case 1: $|X| = 7$. Clearly, in this case $X = P = f(P)$. It is easy to see that then also $f(L) = L$, hence f is bijective. Thus, f is an automorphism of F, in particular, it is a weakly regular residuated mapping.

Case 2: $|X| < 7$ and the set of points $f(X)$ is not collinear, that is, $f(X)$ contains a non-collinear triple of points. Note that by Lemma 21 this is certainly the case when $|X| \geq 4$. Indeed, if f is injective, then $|X| = |f(X)|$ but every line contains only three points.

Without loss of generality, assume that $a_1, a_2, a_3 \in X$ and the triple $f(a_1), f(a_2), f(a_3)$ is non-collinear. Again without loss of generality, in view of Lemma 17, assume that $f(a_i) = a_i$, $i = 1, 2, 3$. Then Lemma 16 implies that $f(a_{ij}) \leq a_i \vee a_j$ and applying Lemma 21, we conclude that $f(a_{ij}) \in \{a_{ij}, a_i \vee a_j\}$. We see that $x \leq f(x)$ holds for all points $x = a_i$ and $x = a_{ij}$. Now, if also $b \leq f(b)$, then $\mathrm{id}_F \leq f$ and since f is an atom of $\mathrm{Res}(F)$, we have $\mathrm{id}_F = f$. But then $f(P) = P$, a contradiction.

Consequently, we must handle the possibilities when $b \not\leq f(b)$. There are essentially two different cases: (i) $f(b) \leq a_i \vee a_j$ for some $i \neq j$ and (ii) $f(b) = \{a_{12}, a_{23}, a_{13}\}$.

(i) Without loss of generality, assume that $f(b) \leq a_1 \vee a_2$. Then, applying Lemma 16 first to the line $\{a_1, b, a_{23}\}$ and then to the line $\{a_3, a_{23}, a_2\}$, we obtain

$$f(a_{23}) \leq f(a_1) \vee f(b) \leq a_1 \vee a_1 \vee a_2 = a_1 \vee a_2$$

and

$$a_3 = f(a_3) \leq f(a_{23}) \vee f(a_2) \leq a_1 \vee a_2 \vee a_2 = a_1 \vee a_2\,,$$

a contradiction, because the points a_1, a_2 and a_3 are not collinear.

(ii) Let now $f(b) = \{a_{12}, a_{23}, a_{13}\}$ and consider the projection mapping $g(x) = (a_3 \vee x) \wedge (a_1 \vee a_2)$. Then easy straightforward calculations show that $g(x) \leq f(x)$ for every $x \in P$. Since $g(a_3) = 0 < a_3 = f(a_3)$, we have a contradiction with minimality of f.

Case 3: $|X| = 3$. The subcase with $f(X)$ non-collinear was actually handled in Case 2, thus we may assume that the triple $f(X)$ is collinear.

First suppose that the triple X is not collinear. As above, we may assume that $X = \{a_1, a_2, a_3\}$. Then

$$1 = f(1) = f(a_1 \vee a_2 \vee a_3) = f(a_1) \vee f(a_2) \vee f(a_3) < 1\,,$$

a contradiction. Thus, it remains to consider the case when the triple X is collinear. Without loss of generality, $X = \{a_1, a_{12}, a_2\}$ and $f(a_i) = a_i$, $i = 1, 2$. Then $f(a_{12}) \leq a_1 \vee a_2$ and, by Lemma 21, $f(a_{12}) = a_{12}$. Since $|X| = 3$, the points that do not belong to the line $a_1 \vee a_2$, are mapped by f to lines or to 1 and by Lemma 21 there is $x \in P$ such that $f(x) \in L$. Let y be a common point of lines $f(x)$ and $a_1 \vee a_2$ and z be the third point of the line $x \vee y$. Then $f(y) = y \leq f(x)$ and $z \notin X$, thus $f(z) \notin P$. Consequently,

$$f(z) \leq f(y) \vee f(x) = y \vee f(x) = f(x)$$

implying $f(z) = f(x)$, a contradiction with Lemma 21.

Case 4: $|X| = 2$. As above, we may assume that $X = \{a_1, a_2\}$ and $f(a_i) = a_i$, $i = 1, 2$. Further, since $f(a_{12}) \leq f(a_1) \vee f(a_2) = a_1 \vee a_2$ and

$a_{12} \notin X$, we have $f(a_{12}) = a_1 \vee a_2$. Let now $x \in P$ be such that $x \not\leq a_1 \vee a_2$ and $f(x) \in L$. Note that in view of Lemma 21, there are at least three such points and $f(x) \neq a_1 \vee a_2$. Let $y = f(x) \wedge (a_1 \vee a_2)$ and let z be the third point of the line $x \vee y$. Then $f(z) \leq f(x) \vee f(y)$. We see that if $y \in \{a_1, a_2\}$, then $f(z) \leq f(x) \vee y = f(x)$ and since $f(z) \notin P$, we have a contradiction with Lemma 21. This means that the line $f(x)$ intersects $a_1 \vee a_2$ at a_{12}. Since there are only two lines through a_{12}, different from $a_1 \vee a_2$, we have a contradiction again.

Case 5: $|X| = 1$. As above, we may assume that $X = \{a_1\}$ and $f(a_1) = a_1$. Since $|P \setminus X| = 6$ and there are only 4 lines not containing the point a_1, there exist two different points x_1, x_2 such that $a_1 \leq f(x_1), f(x_2)$. If the triple $\{a_1, x_1, x_2\}$ is collinear, then $f(x_1) \leq f(a_1) \vee f(x_2) = a_1 \vee f(x_2) = f(x_2)$ and similarly $f(x_2) \leq f(x_1)$. Thus we get $f(x_1) = f(x_2)$, a contradiction with Lemma 21.

It remains to consider the case when the triple $\{a_1, x_1, x_2\}$ is not collinear. Let y_i be the third point of the line $= a_1 \vee x_i$, $i = 1, 2$. Then

$$f(y_i) \leq f(a_1) \vee f(x_i) = a_1 \vee f(x_i) = f(x_i), \ i = 1, 2.$$

Because of Lemma 21, $f(x_i) = f(y_i)$ is impossible, thus $f(y_1) = 1 = f(y_2)$ which again contradicts Lemma 21.

Case 6: $|X| = 0$. Now f maps P injectively into $L \cup \{1\}$. Suppose $f(P) \not\subseteq L$, that is, there exists $a \in P$ such that $f(a) = 1$. Then clearly there exists a unique injective mapping $g : P \to L$ such that $g < f$ and in view of Lemma 16, g is residuated meaning that f is not minimal. Thus, we may restrict ourselves to the case $f(P) \subseteq L$. Actually, because of injectivity of f, we have $f(P) = L$.

Suppose first that there is a line l such that the lines $f(x)$, $x < l$, are confluent and let c be the common point of these three lines. Then define the mapping $g : P \to F$ as follows:

$$g(x) = \begin{cases} c, \text{ if } x < l; \\ f(x), \text{ otherwise.} \end{cases}$$

It is easy to check, using Lemma 16, that g is residuated, and obviously $g < f$.

Finally, assume that for any line l the three lines $f(x)$, $x < l$, are not confluent. Since the number of non-confluent triples of lines (what is 28) is larger than the number of lines (what is 7), there also exist three non-collinear points b_1, b_2, b_3 such that the lines $l_i = f(b_i)$, $i = 1, 2, 3$, are non-confluent. Then clearly the triple of points $l_1 \wedge l_2$, $l_2 \wedge l_3$, $l_3 \wedge l_1$ is non-collinear, too. Without loss of generality, assume that

$$a_1 = l_1 \wedge l_3 \,, a_2 = l_1 \wedge l_2 \,, a_3 = l_2 \wedge l_3$$

and note that then

$$l_1 = a_1 \vee a_2 \ , l_2 = a_2 \vee a_3 \ , l_3 = a_1 \vee a_3 \,.$$

In view of Lemma 17 there exists an automorphism α of F such that $\alpha(a_i) = b_i$, $i = 1, 2, 3$. Due to Lemma 2, the mapping $g = f\alpha$ is an atom of $\mathrm{Res}(F)$ and it is easy to see that g inherits the properties of f that are important for us. Namely, $g(P) = L$ and the triple of lines $g(x)$, $x < l$, is non-confluent for every line l. Now we have $g(a_i) = l_i$, $i = 1, 2, 3$. Hence,

$$g(a_1) = a_1 \vee a_2,\ g(a_2) = a_2 \vee a_3,\ g(a_3) = a_1 \vee a_3 \,.$$

Since the collinear points a_{12}, a_{23}, a_{13} cannot be mapped to confluent lines, one of these points is mapped to the line $a_{12} \vee a_{23}$. Assume, without loss of generality, that $g(a_{12}) = a_{12} \vee a_{23}$.

Now, the remaining three points a_{13}, a_{23} and b have to be mapped by g bijectively to the lines $a_1 \vee a_{23}$, $a_2 \vee a_{13}$ and $a_3 \vee a_{12}$. Since the triple $\{b, a_3, a_{12}\}$ is collinear and $g(a_3) \wedge g(a_{12}) = a_{13}$, there are two possibilities: $g(b) = a_1 \vee a_{23}$ or $g(b) = a_3 \vee a_{12}$. Indeed, otherwise the triple of lines $\{g(b), g(a_3), g(a_{12})\}$ would be confluent.

Assume that $g(b) = a_1 \vee a_{23}$, the other case is similar. Further, for $g(a_{23})$ there are again two possibilities, either $a_2 \vee a_{13}$ or $a_3 \vee a_{12}$. However, since $g(a_2) \wedge g(a_3) = a_3$, the second case is impossible. Thus, $g(a_{23}) = a_2 \vee a_{13}$ and consequently $g(a_{13}) = a_3 \vee a_{12}$.

Now define a mapping $h : P \to P$ as follows:

$$h(a_1) = a_1,\ h(a_2) = a_2,\ h(a_{12}) = a_{12},$$
$$h(a_3) = a_{13},\ h(a_{13}) = a_3,\ h(a_{23}) = b,\ h(b) = a_{23} \,.$$

Obviously, h is a bijection and Lemma 16 shows that h is residuated. It is easy to check that $h < g$, a contradiction with minimality of g.

The proof of Theorem 18 is complete now. □

8. Example

The proof of Theorem 18 significantly depends on the specific structure of F, especially on the facts that every line contains exactly three points and every point belongs to exactly three lines. Therefore it is not surprising that it cannot be generalized to larger projective planes. This is what the following example exhibits.

Example. Let M be the lattice of a projective plane of order $n > 2$. Then the lattice $\mathrm{Res}(M)$ has atoms that are not weakly regular.

We fix a point a and a line s such that $a \nleq s$. Let

$$P^s = \{x \in P \mid x \not< s\}, \quad L_a = \{x \in L \mid a \not< x\} \,.$$

Since $|P^s| = n^2 = |L_a|$, there exists a bijection $\alpha : P^s \to L_a$. We first observe that α can be chosen so that $\alpha(a) = s$ and

$$\bigwedge\{\alpha(y) \mid y \leq a \vee x, y \neq x\} = 0 \tag{8}$$

holds for every point $x < s$.

Let ρ be the partition of the set $P^s \setminus \{a\}$ whose blocks are the lines through a (without a). Obviously, each block of ρ can be written as $(x \vee a) \setminus \{a\}$ for some $x < s$. Let τ be the partition of the set $L_a \setminus \{s\}$ whose blocks consist of lines that intersect s in the same point. Clearly both ρ and τ have $n+1$ blocks, all of size $n-1$. Let $\beta : P^s \setminus \{a\} \to L_a \setminus \{s\}$ be a bijection that maps blocks of ρ to blocks of τ.

Let Z be a transversal of the partition ρ, that is, a set of points that contains exactly one element from each ρ-block and let γ be a cyclic permutation on Z. Now define $\alpha : P^s \to L_a$ as follows:

$$\alpha(x) = \begin{cases} s, \text{ if } x = a; \\ \beta(x), \text{ if } x \neq a \text{ and } x \notin Z; \\ \beta(\gamma(x)), \text{ if } x \in Z\,; \end{cases}$$

and check that it satisfies the condition (8). Let x be an arbitrary point of s and let the points $y, z < a \vee x$ be such that $z \in Z$ and $y \notin Z$. It follows from the definition of α that the lines $\alpha(y)$ and $\alpha(z)$ intersect s in different points. Since $\alpha(a) = s$, we have $\alpha(a) \wedge \alpha(y) \wedge \alpha(z) = 0$, thus (8) is satisfied.

Now define $f : P \to M$ as follows:

$$f(x) = \begin{cases} a, \text{ if } x < s; \\ \alpha(x), \text{ if } x \in P^s\,. \end{cases}$$

Lemma 16 easily implies that $f \in \mathrm{Res}(M)$. Indeed, we have to check that whenever $x \leq y \vee z$ where $x, y, z \in P$, also $f(x) \leq f(y) \vee f(z)$. Now, if $y, z < s$, then also $x < s$, hence $f(x) = a \leq a = f(y) \vee f(z)$. Otherwise, it follows from the definition of α that $f(y) \vee f(z) = 1$ and we are done. Furthermore, obviously $f(1) = 1$ but $f(M) \neq M$, thus f is not weakly regular.

Now we are going to prove that f is an atom of $\mathrm{Res}(M)$. Suppose there is $g \in \mathrm{Res}(M)$ such that $g < f$, that is, $g(x) \leq f(x)$ for all points $x \in P$ and $g(b) < f(b)$ for some $b \in P$. We first show that such point b must exist in the line s and then of course $g(b) = 0$ because $g(b) < f(b) = a \in P$.

Suppose $b \not< s$. Since $g(b) < f(b) \in L$, we have $b \in P \cup \{0\}$. Thus there are at least $n+1$ lines h such that $g(b) \leq h$. We show that among them there is a line h such that $h \neq f(b)$ and $a \not< h$. If $g(b) = 0$, then this follows from the inequality

$$|\{h \in L \mid a \not< h, h \neq f(b)\}| \geq n^2 - 1 \geq 3\,.$$

If $g(b) \in P$, then note that $g(b) \neq a$. This is because $b \not< s$ implies $a \not< f(b)$. Now, among $n+1$ lines through $g(b)$ only $g(b) \vee a$ goes through a. It follows that

$$|\{h \in L \mid a \not\leq h, h \neq f(b)\}| \geq (n+1) - 2 \geq 1\,.$$

By the definition of f there exists $y \in P \setminus s$ such that $f(y) = h$. Denote $c = (y \vee b) \wedge s$. Since $b \not\leq s$, we have $c \in P$ and clearly $c < s$ which implies $g(c) \leq f(c) = a$. On the other hand,

$$g(c) \leq g(y \vee b) = g(y) \vee g(b) \leq f(y) \vee h = h\,.$$

Thus, $g(c) \leq h \wedge a = 0$. This says that, without loss of generality, the point b can be chosen in the line s.

It follows from Lemma 16 that if $b < l \in L$, then g is constant on the set $l \setminus \{b\}$, in particular, g is constant on the set $(a \vee b) \setminus \{b\}$. Thus, $g(a) = g(y) \leq f(y) = \alpha(y)$ holds for every point $y \leq a \vee b$, $y \neq b$. But then, using formula (8), we have

$$g(a) \leq \bigwedge\{\alpha(y) \mid y \leq a \vee b, y \neq b\} = 0\,.$$

Since $g(b) = 0$, also $g(a \vee b) = 0$. Now Lemma 16 easily implies that g is constant on $P \setminus (a \vee b)$, let $g(x) = d \in M$ for all points $x \not\leq a \vee b$. Hence $d < f(x)$ holds for all points x that do not belong to lines s and $a \vee b$. Since f is one-to-one on the set $P \setminus (s \cup (a \vee b))$ this would imply that

$$|P \setminus (s \cup (a \vee b))| \leq |\{l \in L \mid d < l\}| = n + 1.$$

However, this is not true because for $n \geq 3$ we have:

$$\begin{aligned}|P \setminus (s \cup (a \vee b))| - (n+1) &= (n^2 + n + 1) - (2n + 1) - (n + 1)\\ &= (n-1)^2 - 2 > 0\,.\end{aligned}$$

This contradiction proves that actually there is no $g \in \mathrm{Res}(M)$ such that $0 < g < f$. In other words, f is an atom of $\mathrm{Res}(M)$.

Remark. We suggest the reader to find out why our argument would not work when $n = 2$, that is, in the case of the Fano plane.

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