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On a generalization of the Nagumo–Brezis theorem

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Abstract. We generalize the Nagumo–Brezis theorem to the category of MC^k-Fréchet manifolds. Then we will apply the obtained result to locate a critical value of a real-valued mapping over these manifolds.

1. Introduction

The problem of verifying the (positive) invariance of a set with respect to the flow generated by a vector field (or a dynamical system) has a long and rich history. Some early results were obtained by Bouligand in 1932 [2]. Later, Nagumo in 1942 provided the first necessary and sufficient conditions on set invariance [11], which state that a set is invariant if and only if a vector field lies in the tangent cone. In the late 1960's and early 1970's, Nagumo's theorem was independently rediscovered by Bony [1] and Brezis [3]. The results of Brezis used the tangent cone to give conditions on set invariance, while Bony used a comparison theorem type result. In the infinite dimensional case, the ideas of Nagumo and Brezis were utilized for Banach manifolds, and are well-documented in [12].

If we want to develop these ideas further to Fréchet manifolds, we will encounter some obstacles. It is well-known that, in general, a vector field on a Fréchet manifold M has no integral curve, or even if it does, the integral curve may not be unique. Also, in general, a vector field may not generate a (differentiable) flow, and the domain of a flow may not be open in $M \times \mathbb{R}$. Therefore, we need to work on concrete manifolds.

The above issues were treated for a generalized category of Fréchet manifolds, known as MC^k(or bounded)-Fréchet manifolds in [5, 6, 8]. Thus,

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it would be intriguing to investigate conditions for set invariance in this category of manifolds too.

In this paper, we extend the ideas of Nagumo and Brezis to the MC^k-Fréchet manifolds. We give a criterion for a closed subset of an MC^k-Fréchet nuclear manifold to be invariant under the flow defined by an MC^k-vector field in Theorem 2. Then, we will apply the latter theorem to locate critical points of real-valued mappings (which is a very challenging problem for Fréchet manifolds) in Theorem 3.

A related point to consider is that for Fréchet manifolds only the Palais–Smale condition has been used to locate critical points so far, see [4, 7]. However, due to the difficulty of verifying the Palais-Smale condition, and the existence of important functions that do not satisfy the condition it would be a gain to develop other methods such as the Nagumo–Brezis theorem to locate critical points.

2. Prerequisites

In this section, we briefly recall the basic knowledge about MC^k-Fréchet manifolds that we will need, we use the notations and the definitions of the papers [5, 6, 8].

We will use the notation $U \Subset T$ to indicate that a set U is open in a topological space T. Throughout the paper, we assume that E, F are Fréchet spaces, and $CL(E, F)$ is the space of all continuous linear mappings from E to F, endowed with the compact-open topology.

Let $\varphi : U \Subset E \to F$ be a map. If for all $x \in U$ and all $h \in E$ the directional derivatives

$$\mathrm{D}\,\varphi(x)h = \lim_{t\to 0} \frac{\varphi(x+th) - \varphi(x)}{t}$$

exist and the induced map $\mathrm{D}\,\varphi(x) : U \to CL(E, F)$ is continuous for all $x \in U$, then we say that φ is a Keller's differentiable map of class C_c^1. The higher directional derivatives and C_c^k-mappings, $k \geq 2$, are defined in the obvious inductive fashion, see [10].

To define MC^k-differentiability (or bounded differentiability), we define the topology of a Fréchet space F with a translation invariant metric ϱ, and then introduce the metric concepts which strongly depend on the choice of ϱ. We consider only metrics of the following form

$$\varrho(x, y) = \sup_{n\in\mathbb{N}} \frac{1}{2^n} \frac{\|x-y\|_{F,n}}{1+\|x-y\|_{F,n}}, \tag{2.1}$$

where $\|\cdot\|_{F,n}$ is a collection of seminorms generating the topology of F. The distance between two subsets A, B (a point to sets) is defined by

$$\mathtt{Dist}(A, B) = \inf\{\, \varrho(x, y) \mid x \in A,\ y \in B \,\}.$$

We denote by $\|x\|_{F,\varrho} := \varrho(x, 0_F)$ the distance of a point x form the origin 0_F, and by $\mathbb{B}_\varrho(x, r)$ the open ϱ-ball with center x and radius $r > 0$. Since ϱ is translation-invariant,

$$\varrho(x, y) = \|x - y\|_{F,\varrho} .$$

Let σ be a metric that defines the topology of a Fréchet space E. Let $\mathbb{L}_{\sigma,\varrho}(E, F)$ be the set of all linear mappings $L : E \to F$ which are (globally) Lipschitz continuous as mappings between metric spaces E and F, that is

$$\operatorname{Lip}(L) := \sup_{x \in E \setminus \{0_F\}} \frac{\varrho(L(x), 0_F)}{\sigma(x, 0_F)} < \infty,$$

where $\operatorname{Lip}(L)$ is the (minimal) Lipschitz constant of L.

The translation invariant metric

$$\mathbf{d}_{\sigma,\varrho} : \mathbb{L}_{\sigma,\varrho}(E, F) \times \mathbb{L}_{\sigma,\varrho}(E, F) \longrightarrow [0, \infty), \ (L, H) \mapsto \operatorname{Lip}(L - H)_{\sigma,\varrho} , \quad (2.2)$$

on $\mathbb{L}_{\sigma,\varrho}(E, F)$ turns it into an Abelian topological group. We always topologize the space $\mathbb{L}_{\sigma,\varrho}(E, F)$ by the metric (2.2).

Let $\varphi : U \Subset E \to F$ be a map. If φ is Keller's differentiable, $\mathrm{D}\,\varphi(x) \in \mathbb{L}_{\sigma,\varrho}(E, F)$ for all $x \in U$ and the induced map $\mathrm{D}\,\varphi(x) : U \to \mathbb{L}_{\sigma,\varrho}(E, F)$ is continuous, then φ is called bounded differentiable or MC^1, and we write $\varphi^{(1)} = \varphi'$. We define for $k > 1$ mappings of the class MC^k recursively. An MC^k-Fréchet manifold is a Fréchet manifold whose coordinate transition functions are all MC^k-mappings.

We recall the definition of nuclear Fréchet manifolds as we mainly work with these manifolds. Let $(B_1, |\cdot|_1)$ and $(B_2, |\cdot|_2)$ be Banach spaces. A linear operator $T : B_1 \to B_2$ is called nuclear or trace class if it can be written in the form

$$T(x) = \sum_{j=1}^{\infty} \lambda_j \langle x, x_j \rangle y_j,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between B_1 and its dual $(B_1', |\cdot|_1')$, $x_j \in B_1'$ with $| x_j |_1' \leq 1$, $y_j \in B_2$ with $| y_1 |_2 \leq 1$, and λ_j are complex numbers such that $\sum_j | \lambda_j | < \infty$.

If $\|\cdot\|_{F,i}$ is a seminorm on a Fréchet space F, we denote by F_i the Banach space given by completing F using the seminorm $\|\cdot\|_{F,i}$, there is a natural map from F to F_i whose kernel is $\ker \|\cdot\|_{F,i}$. A Fréchet space F is called nuclear if for any seminorm $\|\cdot\|_{F,i}$, we can find a larger seminorm $\|\cdot\|_{F,j}$ so that the natural induced map from F_j to F_i is nuclear. A nuclear Fréchet manifold is a manifold modeled on a nuclear Fréchet space.

Since nuclear operators factor over Hilbert spaces, each nuclear Fréchet space admits a fundamental system of Hilbertian seminorms. Another key feature of Fréchet nuclear spaces is that they have the Heine–Borel property, that is, a closed bounded subset of such a space is compact. We have this advantage over Banach manifolds, since there is no infinite dimensional Banach space which is nuclear.

A key example of an MC^∞-Fréchet nuclear manifold is the manifold of all smooth sections of a fiber bundle (such as the manifold of Riemannian metrics) on a closed manifold.

3. Flow-invariant sets

Henceforth, we assume that $k \geq 2$, and M is an MC^k-Fréchet manifold modeled on a Fréchet space F whose topology is defined by the metric (2.1).

We denote by $\Pi : TM \to M$ the tangent bundle, and by $T\varphi$ (or φ_*) the tangent map (the differential) of a map φ defined on M. For an MC^r-vector field $\mathsf{X} : U \Subset F \to F$, $r \geq 1$, let $\mathcal{I}_t(x_0)$ denote its integral curve passing through x_0, i.e., $\mathcal{I}'(t) = \mathsf{X}(\mathcal{I}(t))$ with $\mathcal{I}(0) = x_0$.

We will need the following results.

Proposition 1 ([6], Proposition 5.1)**.** *Let $\xi : U \Subset F \to F$ be an $MC^r, r \geq 1$, vector field. Then, for $p_0 \in U$, there is an MC^1-integral curve $\ell : I \to F$ at p_0. Furthermore, any two such curves are equal on the intersection of their domains.*

Corollary 1 ([6], Corollary 5.1)**.** *Suppose the hypotheses of the previous proposition hold. Let $\mathcal{I}_t(p_0)$ be the solution of $\ell'(t) = \xi(\ell(t))$, $\ell(t_0) = p_0$. Then there is an open neighborhood U_0 of p_0 and a positive real number α such that for every $q \in U_0$ there exists a unique integral curve $\ell(t) = \mathcal{I}_t(q)$ satisfying $\ell(0) = q$ and $\ell'(t) = \xi(\ell(t))$ for all $t \in (-\alpha, \alpha)$.*

Theorem 1 ([9], Gronwall's inequality)**.** *Let $\varphi, \psi : [a, b) \to \mathbb{R}^+ \cup \{0\}$ be continuous. If for a constant R and all $t \in [a, b)$ we have the inequality*

$$\varphi(t) \leq R + \int_a^t \varphi(s)\psi(s)\mathtt{d}s,$$

then $\varphi(t) \leq R \exp\big(\int_0^t \psi(s)\mathtt{d}s\big)$ for all $t \in [a, b)$.

Lemma 1. *Let $\mathsf{X} : U \Subset F \to F$ be an MC^1-vector field. Then, there exists a neighborhood V of $x_0 \in U$, and $r > 0$ such that, for every $y \in V$, there is a unique integral curve $\mathcal{I}$ satisfying*

$$I(0) = y; \;\; \mathcal{I}'(t) = \mathsf{X}(\mathcal{I}(t)), \quad -r \leq t \leq r.$$

Moreover, for some $R > 0$ we have

$$\|\mathcal{I}_t(x) - \mathcal{I}_t(y)\|_{F,\varrho} = e^{R|t|} \|x - y\|_{F,\varrho}.$$

Proof. Since X is MC^1, it is bounded, say by R. Thus, for any $x, y \in U$ we have

$$\|\mathsf{X}(x) - \mathsf{X}(y)\|_{F,\varrho} \leq R \|x - y\|_{F,\varrho}.$$

Let $m > 0$ be such that the closed ϱ-ball $\overline{\mathbb{B}_\varrho(x_0, m)}$ lies in U, and $\|\mathsf{X}(x)\|_{F,\varrho} \leq \mathbf{n}$ for all x in $\overline{\mathbb{B}_\varrho(x_0, m)}$. Now, let $V = \overline{\mathbb{B}_\varrho(x_0, m/2)}$ and $r = \min\left\{\frac{1}{R}, \frac{m}{2\mathbf{n}}\right\}$. Then, for a fixed $y \in V$ we have $\overline{\mathbb{B}_\varrho(p, m/2)} \subset \overline{\mathbb{B}_\varrho(x_0, m)}$, and therefore

$$\|\mathsf{X}(z)\|_{F,\varrho} \leq \mathbf{n}, \quad \forall z \in \overline{\mathbb{B}_\varrho(y, m/2)}.$$

By Proposition 1 and Corollary 1, with x_0 replaced by y and t_0 by 0, there exists an integral curve $\mathcal{I}(t)$ of X for $t \in [-r, r]$ with $\mathcal{I}(0) = y$.

Now, define the mapping $\Psi(t) \coloneqq \|\mathcal{I}_t(x) - \mathcal{I}_t(y)\|_{F,\varrho}$. Then,

$$\begin{aligned}\Psi(t) &= \left\| \int_0^t \big(\mathsf{X}(\mathcal{I}_s(x)) - \mathsf{X}(\mathcal{I}_s(y)\big)\mathtt{d}s + x + y \right\|_{F,\varrho} \\ &\leq \|x - y\|_{F,\varrho} + R\int_0^t \Psi(s)\mathtt{d}s.\end{aligned}$$

Therefore, by Gronwall's inequality we obtain

$$\|\mathcal{I}_t(x) - \mathcal{I}_t(y)\|_{F,\varrho} = e^{R|t|}\,\|x - y\|_{F,\varrho}\,.$$

□

Definition 1. Let $A \subset M$ and let $\mathtt{V}$ be an MC^1-vector field on M. The set A is called flow-invariant with respect to $\mathtt{V}$, if whenever $\mathcal{I}(t)$ is the integral curve of $\mathtt{V}$ with $\mathcal{I}(0) \in A$ (starting from A), then $\mathcal{I}(t) \in A$ for all $t \geq 0$ in the domain of $\mathcal{I}(\cdot)$.

Theorem 2. *Let M be a nuclear MC^k-Fréchet manifold, and $\mathsf{X} : M \to TM$ an MC^1-vector field. Let $A \subset M$ be closed. Then A is flow-invariant with respect to X if and only if for each $x \in M$ there is a chart $(x \in U, \phi)$ such that*

$$\lim_{s\to 0} t^{-1}\varrho\big(\phi(x) + s\,\mathrm{D}\,\phi(x)\mathsf{X}(x), \phi(U \cap A)\big) = 0. \tag{3.3}$$

Proof. Suppose A is a flow-invariant set with respect to X. Let $\mathcal{I}_t(x)$ denote the integral curve of X passing through $x \in A$, i.e., $\mathcal{I}'(t) = \mathsf{X}(\mathcal{I}(t))$ with $\mathcal{I}(0) = x$. Let (U, ϕ) be a chart at x. For small s we have

$$\begin{aligned}&\varrho\big(\phi(x) + s\,\mathrm{D}\,\phi(x)\mathsf{X}(x), \phi(U \cap A)\big) \\ &\qquad \leq \varrho\big(\phi(x) + s\,\mathrm{D}\,\phi(x)\mathsf{X}(x), \phi(\mathcal{I}(h))\big) \\ &\qquad = \sup_{n\in\mathbb{N}} \frac{1}{2^n} \frac{\|\phi(\mathcal{I}(h)) - \phi(x) - s\,\mathrm{D}\,\phi(x)\mathsf{X}(x)\|_{F,\varrho}}{1 + \|\phi(\mathcal{I}(h)) - \phi(x) - s\,\mathrm{D}\,\phi(x)\mathsf{X}(x)\|_{F,\varrho}} \\ &\qquad = \sup_{n\in\mathbb{N}} \frac{1}{2^n} \frac{|s| \left\| \frac{\phi(\mathcal{I}(h)) - \phi(x)}{s} - \mathrm{D}\,\phi(x)\mathsf{X}(x) \right\|_{F,\varrho}}{1 + |s| \left\| \frac{\phi(\mathcal{I}(h)) - \phi(x)}{s} - \mathrm{D}\,\phi(x)\mathsf{X}(x) \right\|_{F,\varrho}}\end{aligned}$$

$$\leq |s|\mathbf{N} \sup_{n\in\mathbb{N}} \frac{1}{2^n} \frac{\left\| \dfrac{\phi(\mathcal{I}(h)) - \phi(x)}{s} - \mathrm{D}\,\phi(x)\mathsf{X}(x) \right\|_{F,\varrho}}{1 + \left\| \dfrac{\phi(\mathcal{I}(h)) - \phi(x)}{s} - \mathrm{D}\,\phi(x)\mathsf{X}(x) \right\|_{F,\varrho}},$$

where $\mathbf{N} = \lceil 1/s \rceil + 1$ in the last inequality, whence we obtain Equation (3.3).

Now, suppose that Equation (3.3) holds. It is sufficient to prove $\mathcal{I}(t) \in A$ for small t. Let $x = \mathcal{I}(0) \in A$, and let (U, ϕ) be a chart at x. Since X is MC^1, it is bounded by some constant $R > 0$. Near x, say $\mathbb{B}_\varrho(x, r)$ (for some $r > 0$), we have

$$\|\phi(\mathsf{X}(x_1)) - \phi(\mathsf{X}(x_2))\|_{F,\varrho} \leq R\, \|\phi(x_1) - \phi(x_2)\|_{F,\varrho},$$

and therefore by Lemma 1 we get

$$\|\phi(\mathcal{I}_t(x_1)) - \phi(\mathcal{I}_t(x_2))\|_{F,\varrho} \leq e^{tR}\, \|\phi(x_1) - \phi(x_2)\|_{F,\varrho}.$$

We may assume $\|\phi(\mathcal{I}_t(x)) - \phi(x)\|_{F,\varrho} \leq \dfrac{r}{2}$. Define the mapping

$$\Psi(t) \coloneqq \varrho\big(\phi(\mathcal{I}_t(x) \cap U), \phi(A \cap U)\big).$$

We have $\Psi(0) = 0$, so for small t, $\Psi(t) < r/2$. Since F is a nuclear Fréchet space, and so the Heine–Borel Theorem is available for F, by closedness of A we obtain

$$\varrho\big(\phi(\mathcal{I}_t(x) \cap U), \phi(A \cap U)\big) = \|\phi(\mathcal{I}_t(x)) - \phi(y_t)\|_{F,\varrho}, \quad \text{for some } y_t \in A.$$

Therefore, $\|\phi(y_t) - \phi(x)\|_{F,\varrho} < r$.

For small s, we have $\|\phi(\mathcal{I}_s(y_t)) - \phi(x)\|_{F,\varrho} < r$, so that

$$\begin{aligned}
\Psi(t+s) &= \inf_{z\in A} \Big\{ \|\phi(\mathcal{I}_{t+s}(x)) - \phi(z)\|_{F,\varrho} \Big\} \\
&\leq \inf_{z\in A} \Big\{ \|\phi(\mathcal{I}_{t+s}(x)) - \phi(\mathcal{I}_s(y_t))\|_{F,\varrho} + \\
&\qquad + \|\phi(\mathcal{I}_s(y_t)) - \phi(y_t) - s\,\mathrm{D}\,\phi(y_t)\mathsf{X}(y_t)\|_{F,\varrho} + \\
&\qquad + \|\phi(y_t) + s\,\mathrm{D}\,\phi(y_t)\mathsf{X}(y_t) - \phi(z)\|_{F,\varrho} \Big\} \\
&= \|\phi(\mathcal{I}_{t+s}(x)) - \phi(\mathcal{I}_s(y_t))\|_{F,\varrho} + \\
&\qquad + \|\phi(\mathcal{I}_s(y_t)) - \phi(y_t) - s\,\mathrm{D}\,\phi(y_t)\mathsf{X}(y_t)\|_{F,\varrho} + \\
&\qquad + \varrho\big(\phi(y_t) + s\,\mathrm{D}\,\phi(y_t)\mathsf{X}(y_t), \phi(A \cap U)\big) \\
&\leq e^{Rt}\, \|\phi(y_t) - \phi(\mathcal{I}_t(x))\|_{F,\varrho} + \|\phi(\mathcal{I}_s(y_t)) - \phi(y_t) - s\,\mathrm{D}\,\phi(y_t)\mathsf{X}(y_t)\|_{F,\varrho} \\
&\qquad + \varrho\big(\phi(y_t) + s\,\mathrm{D}\,\phi(y_t)\mathsf{X}(y_t), \phi(A \cap U)\big).
\end{aligned}$$

Consequently,

$$\frac{\Psi(t+s) - \Psi(t)}{s} \leq \big(\frac{e^{Rs} - 1}{s}\big)\Psi(t) + \left\| \frac{\phi(\mathcal{I}_s(y_t)) - \phi(y_t)}{s} - \mathrm{D}\,\phi(y_t)\mathsf{X}(y_t) \right\|_{F,\varrho} +$$

$$+\frac{1}{s}\varrho\big(\phi(y_t) + s\,\mathrm{D}\,\phi(y_t)\mathsf{X}(y_t), \phi(A \cap U)\big).$$

Thus

$$\limsup_{s\to 0} \frac{\Psi(t+s) - \Psi(t)}{s} \leq R\Psi(t).$$

Hence, like in Gronwall's inequality we obtain

$$\Psi(t) \leq e^{Rt}\Psi(0),$$

whence $\Psi(t) = 0$, which concludes the proof. □

Note that in the proof of the theorem we used the nuclear property of a Fréchet space F to find the distance minimizer point y_t. However, there may be other ways to prove this theorem by not assuming the nuclearness of spaces. But this does not restrict us that much, because the most important Fréchet manifolds, manifolds of mappings, are nuclear.

4. Applications to critical point theory

In this section, we follow the ideas in [5] and [12]. First, we recall the definition of a Finsler metric for MC^k-Fréchet manifolds.

Definition 2. Let F be a Fréchet space, T a topological space and $V = T \times F$ the trivial bundle with fiber F over T. A Finsler structure for V is a collection of continuous functions $\|\cdot\|_{V,n} : V \to \mathbb{R}^+$, $n \in \mathbb{N}$, such that the following conditions hold.

(F1): For $b \in T$ fixed, $\|x\|^b_{F,n} := \|(b,x)\|_{V,n}$ is a collection of seminorms on F which gives the topology of F.

(F2): Given $k > 1$ and $x_0 \in T$, there exists a neighborhood W of x_0 such that

$$\frac{1}{k}\,\|x\|^{x_0}_{F,n} \leq \|x\|^{w}_{F,n} \leq k\,\|x\|^{x_0}_{F,n} \quad \text{for all} \quad w \in W, n \in \mathbb{N}, x \in F.$$

Let $\|\cdot\|_{M,n} : TM \to \mathbb{R}^+$ be a collection of continuous functions, $n \in \mathbb{N}$. We say that $\{\|\cdot\|_{M,n}\}_{n\in\mathbb{N}}$ is a Finsler structure for TM, if for a given $x \in M$ there exists a bundle chart $\psi : U \times F \simeq TM\mid_U$ with $x \in U$ such that

$$\{\|\cdot\|_{V,n} \circ \psi^{-1}\}_{n\in\mathbb{N}}$$

is a Finsler structure for $V = U \times F$.

An MC^k-Fréchet Finsler manifold is a Fréchet manifold together with a Finsler structure on its tangent bundle. Regular (in particular nuclear) manifolds admit Finsler structures.

If $\{\|\cdot\|_{M,n}\}_{n\in\mathbb{N}}$ is a Finsler structure for M, then we can obtain a graded Finsler structure, denoted by $(\|\cdot\|_{M,n})_{n\in\mathbb{N}}$, that is $\|\cdot\|_{M,i} \leq \|\cdot\|_{M,i+1}$ for all $i \in \mathbb{N}$.

The length of an MC^1-curve $\gamma : [a, b] \to M$ for the n-th component is defined by

$$L_n(\gamma) = \int_a^b \left\|\gamma'(t)\right\|_{M,n}^{\gamma(t)} \,\mathrm{d}t.$$

The length of a piecewise path with respect to the n-th component is the sum over the curves constituting the path. On each connected component of M, the distance is defined by

$$\rho_n(x, y) = \inf_{\gamma} L_n(\gamma),$$

where infimum is taken over all piecewise MC^1-curves connecting x to y. Thus, we obtain an increasing sequence of metrics $\rho_n(x, y)$. Define the distance ρ by

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\rho_n(x, y)}{1 + \rho_n(x, y)}. \tag{4.4}$$

The distance ρ defined in (4.4) is a metric for M which is bounded by 1. Also, the topology induced by this metric coincides with the original topology of M by [5, Theorem 4.6].

An MC^k-vector field $\mathsf{X} : M \to TM$ has a unique MC^k-local flow, see [8, Theorem 4]. It follows from the theorem that the union of the domains of all integral curves of an MC^k-vector field $\mathsf{X} : M \to TM, (k \geq 1)$ through $x \in M$ is an open interval which we denote by $I_x = (T_x^-, T_x^+)$, where T_x^- (resp. T_x^+) are the sup (resp., inf) of the times of existence of the integral curves. Let $\mathcal{D}_{\mathsf{X}} := \bigcup_{x\in M}(\{x\} \times I_x)$, then we have the map $\mathbf{F} : \mathcal{D}_{\mathsf{X}} \to M$ defined on the entire $\mathcal{D}_{\mathsf{X}}$ by letting $\mathbf{F}(x, t)$ be the local flow of X at x. We call this map the flow determined by X, and we call $\mathcal{D}_{\mathsf{X}}$ the domain of the flow which is open in $M \times \mathbb{R}$, see [8, Lemma 1].

Let (M, ρ) be an $MC^k (k \geq 2)$ Fréchet Finsler manifold, c a real number, and ϵ a positive real number. Let $\varphi : M \to \mathbb{R}$ be an MC^1-mapping. Define the following sets:

(1) $\varphi_c := \{ x \in M \mid \varphi(x) \leq c \}$,
(2) $\mathsf{C}_\varphi(c) := \{ x \in M \mid \varphi(x) = c \text{ and } \mathrm{D}\,\varphi(x) = 0 \}$,
(3) $\mathcal{N}(\mathsf{C}_\varphi(c), \epsilon) := \{ x \in M \mid \rho(x, \mathsf{C}_\varphi(c)) < \epsilon \}$.

Proposition 2. *Let $\varphi : M \to \mathbb{R}$ be an MC^1-mapping, $\mathsf{X} : M \to TM$ an MC^1-vector field, and $\mathbf{F} : \mathcal{D}_{\mathsf{X}} \to M$ the flow determined by X. Suppose $c \in \mathbb{R}$, and the following holds.*

(**C.1**) *If $(x_n) \subset M$ is a sequence such that $\varphi(x_n) \to c$ and $\mathrm{D}\,\varphi(x_n)(\mathsf{X}(x_n)) \to 0$, then (x_n) has a convergent subsequence.*
(**C.2**) *There is ϵ_0 such that for $x \in \varphi_{c+\epsilon_0}$, $\mathbf{F}(x, t)$ is defined, and $\varphi(\mathbf{F}(x, t))$ is non-increasing for $t \in [0, 1]$.*

Then we obtain:

(D.1) *$\mathbf{F}(x,t)=x$, for all $x\notin\varphi^{-1}([c-\epsilon_0,c+\epsilon_0])$ and $t\in[0,1]$,*
(D.2) *$\mathbf{F}\big(\varphi_{c+\delta}\setminus\mathcal{U},1\big)\subset\varphi_{c-\delta}$ for some $\delta\in(0,\epsilon_0)$, and any critical neighborhood $\mathcal{U}$ of $\mathsf{C}_\varphi(c)$.*

Proof. It follows from (C.1) that $\mathsf{C}_\varphi(c)$ is compact. Hence there are positive constants $\overline{\delta}<\overline{\delta}_0$ and m such that for all $n\in\mathbb{N}$

(E.1) $\|\mathsf{X}(x)\|^x_{F,n}\le m,\quad\forall x\in\mathcal{N}_{\overline{\delta}_0}(\mathsf{C}_\varphi(c))$,
(E.2) $\mathsf{C}_\varphi(c)\subset\mathcal{N}_{\overline{\delta}(\mathsf{C}_\varphi(c))}\subset\mathcal{N}_{\overline{\delta}_0(\mathsf{C}_\varphi(c))}\subset\mathcal{U}$.

It follows from (C.1) and (C.2) that there exist positive numbers $\mathbf{c}$ and ϵ_1,ϵ_0 such that

$$\mathrm{D}\,\varphi(x)(\mathsf{X}(x))\le-\mathbf{c},\quad\forall x\in\varphi^{-1}[c-\epsilon_1,c+\epsilon_1]\setminus\mathcal{N}_{\overline{\delta}}(\mathsf{C}_\varphi(c)).\tag{4.5}$$

Now, pick a fixed $x\in\varphi_{c+\delta}\setminus\mathcal{U}$. Since otherwise (C.2) implies (D.1), we may assume that

$$\mathbf{F}(x,t)\in\varphi^{-1}(c-\delta,c+\delta],\quad\forall t\in[0,1)\tag{4.6}$$

Now, suppose that

$$\mathbf{F}(x,t)\notin\mathcal{N}_{\overline{\delta}}(\mathsf{C}_\varphi(c)),\quad\forall t\in[0,1],\tag{4.7}$$

and

$$\delta<\frac{1}{2}\min\left\{2\epsilon_0,\mathbf{c},\frac{\mathbf{c}(\overline{\delta}-\overline{\delta}_0)}{m}\right\}.\tag{4.8}$$

Therefore, from (4.5)–(4.8) it follows that

$$\begin{aligned}\varphi(\mathbf{F}(x,1))&=\varphi(x)+\int_0^1\frac{\mathtt{d}}{\mathtt{d}t}\big(\varphi(\mathbf{F}(x,t))\big)\mathtt{d}t\\&\le c+\epsilon+\int_0^1\mathrm{D}\,\varphi(\mathbf{F}(x,t))\big(\mathsf{X}(\mathbf{F}(x,t))\big)\mathtt{d}t\\&\le c-\epsilon.\end{aligned}$$

Thus, for any δ which satisfies (4.8) the statement (D.2) is valid. If the assumption (4.7) is false, by (E.2) there are $s_1\in(0,1)$ and $s_2\in(0,s_1)$ such that

$$\rho(\mathbf{F}(x,s_1),\mathsf{C}_\varphi(c))=\overline{\delta},\quad\rho(\mathbf{F}(x,s_2),\mathsf{C}_\varphi(c))=\overline{\delta}_0.$$

Thus, by (E.1) and (E.2) for all $n\in\mathbb{N}$ we obtain that

$$\begin{aligned}\overline{\delta}_0-\overline{\delta}&\le\int_{s_2}^{s_1}\left\|\frac{\mathtt{d}\mathbf{F}(x,t)}{\mathtt{d}t}\right\|^{\mathbf{F}(x,t)}_{F,n}\mathtt{d}t\\&\le m(s_1-s_2).\end{aligned}\tag{4.9}$$

It follows from (C.2), (4.9), (4.8) and (4.5) that

$$\varphi(\mathbf{F}(x,t))=\varphi(x)+\int_0^1\frac{\mathtt{d}}{\mathtt{d}t}\big(\mathrm{D}\,\varphi(\mathbf{F}(x,t))(\mathsf{X}(\mathbf{F}(x,t)))\big)\mathtt{d}t$$

$$\begin{aligned}
&\leq c + \epsilon + \int_0^1 \frac{\mathtt{d}}{\mathtt{d}t}\big(\varphi(\mathbf{F}(x,t))\big)\mathtt{d}t \\
&\leq c + \epsilon - m(s_1 - s_2) \\
&\leq c - \epsilon.
\end{aligned}$$

Thus, even if (4.7) is false, the statement (D.2) is verified.

Now, define an MC^1-mapping $\mathbf{h}$ over M by

$$\mathbf{h}(x) = \begin{cases} 1, & x \in \varphi^{-1}[c - \epsilon_1, c + \epsilon_1], \\ 0, & \text{otherwise}. \end{cases}$$

Also, define

$$\overline{\mathsf{X}(x)} = \mathsf{X}(x)\mathbf{h}(x).$$

Then, by repeating the above arguments for $\overline{\mathsf{X}(x)}$ we can easily verify (C.1), and (C.2) is also true if we exchange $\mathbf{F}$ by the flow determined by $\overline{\mathsf{X}(x)}$. $\square$

Theorem 3. *Suppose M is a nuclear MC^k-Fréchet manifold, $k \geq 2$. Let $\varphi : M \to \mathbb{R}$ be an MC^1-mapping, $A \subset M$ a closed subset, and $\varphi \mid_A$ bounded from below. Let $\mathsf{X} : M \to TM$ be an MC^1-vector field such that for each $x \in M$ there is a chart $(x \in U, \phi)$ such that*

$$\lim_{s \to 0} t^{-1} \varrho\big(\phi(x) + s \,\mathrm{D}\,\phi(x)\mathsf{X}(x), \phi(U \cap A)\big) = 0.$$

Also, suppose we have (C.1) *and* (C.2) *with $c = \inf_A \varphi(a)$. Then c is a critical value of φ.*

Proof. Define the set

$$\mathcal{A} := \{\, \mathsf{S} \subset A \cap \varphi_{c+\epsilon_0} \mid \mathsf{S} \text{ is compact subset of } M \,\}.$$

It follows from Theorem 2 that A is flow-invariant with respect to X. Therefore, the integral curve $\mathbf{F}(x,\cdot)$ of X remains in A if $x \in A$. Now, by (C.2) we obtain that if $\mathsf{S} \in \mathcal{A}$, then

$$\mathbf{F}(\mathsf{S}, 1) \text{ exists and } \mathbf{F}(\mathsf{S}, 1) \in \mathcal{A}.$$

From the definition of $\mathcal{A}$, we see that

$$c = \inf_{\mathsf{S} \in \mathcal{A}} \max_{x \in \mathsf{A}} \varphi(x). \tag{4.10}$$

Since φ is bounded from below on A, the number c cannot be $-\infty$. We claim that c is a critical value. Because otherwise, if we employ $\mathcal{U} = \emptyset$ in Proposition 2, for some positive δ we will obtain

$$\mathbf{F}(\varphi_{c+\delta}) \subset \varphi_{c-\delta}.$$

Hence, there exists $\mathsf{S} \in \mathcal{A}$ such that $\mathbf{F}(\mathsf{S}, 1) \subset \varphi_{c-\delta}$, which contradicts (4.10). $\square$

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