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Construction of continuous controlled K-g-fusion frames in Hilbert spaces

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Abstract. We present the notion of continuous controlled K-g-fusion frame in a Hilbert space which is generalization of discrete controlled K-g-fusion frame. We discuss some characterizations of a continuous controlled K-g-fusion frame. A relationship between a continuous controlled K-g-fusion frame and a quotient operator has been studied. Finally, stability of a continuous controlled g-fusion frame has been described.

1. Introduction

In 1952, Duffin and Schaeffer [10] introduced frame for a Hilbert space to study some fundamental problems in non-harmonic Fourier series. Later on, after some decades, frame theory was popularized by Daubechies et al. [8].

A frame for a Hilbert space was defined as a sequence of basis-like elements in that Hilbert space. A sequence $\{f_i\}_{i=1}^{\infty} \subseteq H$ is called a frame for a separable Hilbert space $(H, \langle \cdot, \cdot \rangle)$, if there exist positive constants $0 < A \leq B < \infty$ such that

$$A \| f \|^2 \leq \sum_{i=1}^{\infty} | \langle f, f_i \rangle |^2 \leq B \| f \|^2 \text{ for all } f \in H.$$

For the past few years many other types of frames were proposed such as K-frame [13], fusion frame [5], g-frame [26], g-fusion frame [16, 24] and K-g-fusion frame [1] etc. Ghosh and Samanta [15] have discussed generalized atomic subspaces for operators in Hilbert spaces.

Controlled frame is one of the newest generalizations of frame. Balaz et al. [4] introduced controlled frame to improve the numerical efficiency

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of interactive algorithms for inverting the frame operator. In recent times, several generalizations of controlled frame namely, controlled K-frame [21], controlled g-frame [22], controlled fusion frame [19], controlled g-fusion frame [25], controlled K-g-fusion frame [23] etc. have appeared. Continuous frames were proposed by Kaiser [18] and these were independently studied by Ali et al. [2]. At present, frame theory has been widely used in signal and image processing, filter bank theory, coding and communications, system modeling and so on.

In this paper, continuous controlled K-g-fusion frames in Hilbert spaces are studied and some of their properties are going to be established. Under some sufficient conditions, we will see that any continuous controlled K-g-fusion frame is equivalent to a continuous K-g-fusion frame. A necessary and sufficient condition for a continuous controlled g-fusion Bessel family to be a continuous controlled K-g-fusion frame with the help of a quotient operator is established. At the end, we study some stability results of continuous controlled g-fusion frames.

Throughout this paper, H is considered to be a separable Hilbert space with associated inner product $\langle \cdot, \cdot \rangle$ and $\mathbb{H}$ is the collection of all closed subspaces of H, I_H is the identity operator on H, $\mathcal{B}(H_1, H_2)$ is a collection of all bounded linear operators from H_1 to H_2. In particular $\mathcal{B}(H)$ denotes the space of all bounded linear operators on H. For $S \in \mathcal{B}(H)$, we write $\mathcal{N}(S)$ and $\mathcal{R}(S)$ for the null space and the range of S, respectively. Also, $P_M \in \mathcal{B}(H)$ is the orthonormal projection onto a closed subspace $M \subset H$. $\mathcal{GB}(H)$ denotes the set of all bounded linear operators which have bounded inverse. If $S, R \in \mathcal{GB}(H)$, then R^*, R^{-1} and SR also belong to $\mathcal{GB}(H)$. Finally, $\mathcal{GB}^+(H)$ is the set of all positive operators in $\mathcal{GB}(H)$ and T, U are invertible operators in $\mathcal{GB}(H)$.

2. Preliminaries

In this section, we recall some necessary definitions and theorems.

Theorem 1 ([7])**.** *Let H_1, H_2 be two Hilbert spaces and $U : H_1 \to H_2$ be a bounded linear operator with closed range $\mathcal{R}(U)$. Then there exists a bounded linear operator $U^\dagger : H_2 \to H_1$ such that $U U^\dagger x = x$ for all $x \in \mathcal{R}(U)$.*

The operator $U^\dagger$ defined in Theorem 1 is called the pseudo-inverse of U.

Theorem 2 (Douglas' factorization theorem, [9])**.** *Let $S, V \in \mathcal{B}(H)$. Then the following conditions are equivalent.*

(i) $\mathcal{R}(S) \subseteq \mathcal{R}(V)$.
(ii) $S S^* \leq \lambda^2 V V^*$ *for some* $\lambda > 0$.
(iii) $S = V W$ *for some bounded linear operator* W *on* H.

Theorem 3 ([7])**.** *The set $\mathcal{S}(H)$ of all self-adjoint operators on H is a partially ordered set with respect to the partial order $\leq$ which is defined for $R, S \in \mathcal{S}(H)$ by*

$$R \leq S \Leftrightarrow \langle R f, f \rangle \leq \langle S f, f \rangle \;\; \forall f \in H.$$

Definition 1 ([20])**.** A self-adjoint operator $U : H_1 \to H_1$ is called positive if $\langle U x, x \rangle \geq 0$ for all $x \in H_1$. In notation, we can write $U \geq 0$. A self-adjoint operator $V : H_1 \to H_1$ is called a square root of U if $V^2 = U$. If, in addition, $V \geq 0$, then V is called a positive square root of U and is denoted by $V = U^{1/2}$.

Theorem 4 ([20])**.** *The positive square root $V : H_1 \to H_1$ of an arbitrary positive self-adjoint operator $U : H_1 \to H_1$ exists and is unique. Further, the operator V commutes with every bounded linear operator on H_1 which commutes with U.*

In a complex Hilbert space, every bounded positive operator is self-adjoint and any two bounded positive operators commute with each other.

Theorem 5 ([12])**.** *Let $M \subset H$ be a closed subspace and $T \in \mathcal{B}(H)$. Then $P_M T^* = P_M T^* P_{\overline{TM}}$. If T is a unitary operator (i.e $T^* T = I_H$), then $P_{\overline{TM}} T = T P_M$.*

Definition 2. [24] Let $\{W_j\}_{j \in J}$ be a collection of closed subspaces of H and $\{v_j\}_{j \in J}$ be a collection of positive weights, $\{H_j\}_{j \in J}$ be a sequence of Hilbert spaces and let $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in J$. Then $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ is called a generalized fusion frame or a g-fusion frame for H with respect to $\{H_j\}_{j \in J}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A \|f\|^2 \leq \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j}(f) \right\|^2 \leq B \|f\|^2 \;\; \forall f \in H. \tag{1}$$

The constants A and B are called the lower and upper bounds of the g-fusion frame, respectively. If $A = B$, then Λ is called a tight g-fusion frame and if $A = B = 1$, then we say Λ is a Parseval g-fusion frame. If Λ satisfies only the right inequality of (3) it is called a g-fusion Bessel sequence in H with a bound B.

Define the space

$$l^2\left(\{H_j\}_{j \in J}\right) = \left\{ \{f_j\}_{j \in J} : f_j \in H_j, \sum_{j \in J} \|f_j\|^2 < \infty \right\}$$

with inner product given by

$$\langle \{f_j\}_{j \in J}, \{g_j\}_{j \in J} \rangle = \sum_{j \in J} \langle f_j, g_j \rangle_{H_j}.$$

Clearly $l^2\left(\{H_j\}_{j \in J}\right)$ is a Hilbert space with the pointwise operations [1].

Definition 3 ([25])**.** Let $\{W_j\}_{j \in J}$ be a collection of closed subspaces of H and $\{v_j\}_{j \in J}$ be a collection of positive weights. Let $\{H_j\}_{j \in J}$ be a sequence of Hilbert spaces, $T, U \in \mathcal{G}\mathcal{B}(H)$ and $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in J$. Then the family $\Lambda_{TU} = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ is a (T, U)-controlled g-fusion frame for H if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \leq B\|f\|^2 \;\; \forall f \in H. \quad (2)$$

If $A = B$, then Λ_{TU} is called a (T, U)-controlled tight g-fusion frame and if $A = B = 1$, then we say that Λ_{TU} is a (T, U)-controlled Parseval g-fusion frame. If Λ_{TU} satisfies only the right inequality of (2), then it is called a (T, U)-controlled g-fusion Bessel sequence in H.

Definition 4. [25] Let Λ_{TU} be a (T, U)-controlled g-fusion Bessel sequence in H with a bound B. The synthesis operator $T_C : \mathcal{K}_{\Lambda_j} \to H$ is defined as

$$T_C\left(\left\{v_j\left(T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U\right)^{1/2} f\right\}_{j \in J}\right) = \sum_{j \in J} v_j^2\, T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U f$$

for all $f \in H$ and the analysis operator $T_C^* : H \to \mathcal{K}_{\Lambda_j}$ is given by

$$T_C^* f = \left\{v_j\left(T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U\right)^{1/2} f\right\}_{j \in J} \;\; \forall f \in H,$$

where

$$\mathcal{K}_{\Lambda_j} = \left\{\left\{v_j\left(T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U\right)^{1/2} f\right\}_{j \in J} : f \in H\right\} \subset l^2\left(\{H_j\}_{j \in J}\right).$$

The frame operator $S_C : H \to H$ is defined as follows:

$$S_C f = T_C T_C^* f = \sum_{j \in J} v_j^2\, T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U f \;\; \forall f \in H,$$

and it is easy to verify that

$$\langle S_C f, f\rangle = \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \;\; \forall f \in H.$$

Furthermore, if Λ_{TU} is a (T, U)-controlled g-fusion frame with bounds A and B, then $A I_H \leq S_C \leq B I_H$. Hence, S_C is a bounded, invertible, self-adjoint and positive linear operator. It is easy to verify that

$$B^{-1} I_H \leq S_C^{-1} \leq A^{-1} I_H.$$

Definition 5 ([23]). Let $K \in \mathcal{B}(H)$ and $\{W_j\}_{j \in J}$ be a collection of closed subspaces of H and $\{v_j\}_{j \in J}$ be a collection of positive weights. Let $\{H_j\}_{j \in J}$ be a sequence of Hilbert spaces, $T, U \in \mathcal{G}\mathcal{B}(H)$ and $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in J$. Then the family $\Lambda_{TU} = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ is a (T, U)-controlled K-g-fusion frame for H if there exist constants $0 < A \leq B < \infty$ such that

$$A \| K^* f \|^2 \leq \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle \leq B \| f \|^2 \;\; \forall f \in H.$$

Definition 6 ([11]). Let $F : X \to \mathbb{H}$ be such that, for each $h \in H$, the mapping $x \to P_{F(x)}(h)$ is measurable (i.e. is weakly measurable) and $v : X \to \mathbb{R}^+$ be a measurable function and let $\{K_x\}_{x \in X}$ be a collection of Hilbert spaces. For each $x \in X$, suppose that $\Lambda_x \in \mathcal{B}(F(x), K_x)$. Then $\Lambda_F = \{(F(x), \Lambda_x, v(x))\}_{x \in X}$ is called a generalized continuous fusion frame or a gc-fusion frame for H with respect to (X, μ) and v, if there exists $0 < A \leq B < \infty$ such that

$$A \| h \|^2 \leq \int_X v^2(x) \left\| \Lambda_x P_{F(x)}(h) \right\|^2 d\mu \leq B \| h \|^2 \quad \forall h \in H,$$

where $P_{F(x)}$ is the orthogonal projection onto the subspace $F(x)$. Moreover, Λ_F is called a tight gc-fusion frame for H if $A = B$ and Parseval if $A = B = 1$. If we have only the upper bound, we call Λ_F a Bessel gc-fusion mapping for H.

Let $K = \oplus_{x \in X} K_x$ and $L^2(X, K)$ be the collection of all measurable functions $\varphi : X \to K$ such that for each $x \in X$, $\varphi(x) \in K_x$ and $\int_X \| \varphi(x) \|^2 d\mu < \infty$. It can be verified that $L^2(X, K)$ is a Hilbert space with inner product given by

$$\langle \phi, \varphi \rangle = \int_X \langle \phi(x), \varphi(x) \rangle \, d\mu$$

for $\phi, \varphi \in L^2(X, K)$.

Definition 7 ([11]). Let $\Lambda_F = \{(F(x), \Lambda_x, v(x))\}_{x \in X}$ be a Bessel gc-fusion mapping for H. Then the gc-fusion pre-frame operator or synthesis operator $T_{gF} : L^2(X, K) \to H$ is defined by

$$\langle T_{gF}(\varphi), h \rangle = \int_X v(x) \left\langle P_{F(x)} \Lambda_x^* (\varphi(x)), h \right\rangle,$$

where $\varphi \in L^2(X, K)$ and $h \in H$. Then T_{gF} is a bounded linear mapping and its adjoint operator is given by

$$T_{gF}^* : H \to L^2(X, K), \; T_{gF}^*(h) = \left\{ v(x) \Lambda_x P_{F(x)}(h) \right\}_{x \in X}, \; h \in H,$$

and $S_{gF} = T_{gF} T_{gF}^{*}$ is called a gc-fusion frame operator. Thus, for each $f, h \in H$,

$$\langle S_{gF}(f), h \rangle = \int_X v^2(x) \left\langle P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} f, h \right\rangle.$$

The operator S_{gF} is bounded, self-adjoint, positive and invertible on H.

3. Continuous controlled K-g-fusion frame

In this section, a continuous version of controlled K-g-fusion frame for H is presented. We expand some of the recent results on controlled K-g-fusion frames to continuous controlled K-g-fusion frames.

Definition 8. Let $K \in \mathcal{B}(H)$ and $F : X \to \mathbb{H}$ be a mapping, $v : X \to \mathbb{R}^+$ be a measurable function and $\{K_x\}_{x \in X}$ be a collection of Hilbert spaces. For each $x \in X$, suppose that $\Lambda_x \in \mathcal{B}(F(x), K_x)$ and $T, U \in \mathcal{G}\mathcal{B}^+(H)$. Then $\Lambda_{TU} = \{(F(x), \Lambda_x, v(x))\}_{x \in X}$ is called a continuous (T, U)-controlled K-g-fusion frame for H with respect to (X, μ) and v, if

- (i) for each $f \in H$, the mapping $x \to P_{F(x)}(f)$ is measurable (i.e. is weakly measurable),
- (ii) there exist constants $0 < A \leq B < \infty$ such that

$$A \|K^* f\|^2 \leq \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} U f, \Lambda_x P_{F(x)} T f \right\rangle d\mu_x \leq B \|f\|^2 \tag{3}$$

for all $f \in H$, where $P_{F(x)}$ is the orthogonal projection onto the subspace $F(x)$. The constants A, B are called the frame bounds.

Furthermore,

- (i) if only the last inequality of (3) holds, then Λ_{TU} is called a continuous (T, U)-controlled K-g-fusion Bessel family for H,
- (ii) if $T = I_H$, then Λ_{TU} is called a continuous (I_H, U)-controlled K-g-fusion frame for H,
- (iii) if $T = U = I_H$, then Λ_{TU} is called a continuous K-g-fusion frame for H,
- (iv) if $K = I_H$, then Λ_{TU} is called a continuous (T, U)-controlled g-fusion frame for H [17].

Remark 1. If the measure space $X = \mathbb{N}$ and μ is the counting measure then a continuous (T, U)-controlled g-fusion frame will be the discrete (T, U)-controlled g-fusion frame.

Example 1. Let $H = \mathbb{R}^3$ and $\{e_1, e_2, e_3\}$ be the standard orthonormal basis for H. Consider

$$\mathcal{B} = \left\{ x \in \mathbb{R}^3 : \|x\| \leq 1 \right\}.$$

Then it is a measure space equipped with the Lebesgue measure μ. Suppose $\{B_1, B_2, B_3\}$ is a partition of $\mathcal{B}$ where $\mu(B_1) \geq \mu(B_2) \geq \mu(B_3) > 1$. Let $\mathbb{H} = \{W_1, W_2, W_3\}$, where $W_1 = \overline{span}\{e_1, e_2\}$, $W_2 = \overline{span}\{e_2, e_3\}$ and $W_3 = \overline{span}\{e_1, e_3\}$. Define

$$F : \mathcal{B} \to \mathbb{H} \quad \text{by} \quad F(x) = \begin{cases} W_1 & \text{if } x \in B_1, \\ W_2 & \text{if } x \in B_2, \\ W_3 & \text{if } x \in B_3, \end{cases}$$

and

$$v : \mathcal{B} \to [0, \infty) \quad \text{by} \quad v(x) = \begin{cases} 1 & \text{if } x \in B_1, \\ 2 & \text{if } x \in B_2, \\ -1 & \text{if } x \in B_3. \end{cases}$$

It is easy to verify that F and v are measurable functions. For each $x \in \mathcal{B}$, define the operators

$$\Lambda(x)(f) = \frac{1}{\sqrt{\mu(B_k)}} \langle f, e_k \rangle e_k, \quad f \in H,$$

where k is such that $x \in \mathcal{B}_k$ and $K : H \to H$ is defined by

$$K e_1 = e_1, K e_2 = 0, K e_3 = e_3.$$

It is easy to verify that $K^* e_1 = e_1, K^* e_2 = 0, K^* e_3 = e_3.$ Now, for any $f \in H$, we have

$$\|K^* f\|^2 = \left\| \sum_{i=1}^{3} \langle f, e_k \rangle K^* e_k \right\|^2 = |\langle f, e_1 \rangle|^2 + |\langle f, e_3 \rangle|^2 \leq \|f\|^2.$$

Let $T(f_1, f_2, f_3) = (5 f_1, 4 f_2, 5 f_3)$ and $U(f_1, f_2, f_3) = \left(\frac{f_1}{6}, \frac{f_2}{3}, \frac{f_3}{6} \right)$ be two operators on H. Then it is easy to verify that $T, U \in \mathcal{G}\mathcal{B}^+(H)$ and $TU = UT$. Now, for any $f = (f_1, f_2, f_3) \in H$, we have

$$\begin{aligned} &\int_{\mathcal{B}} v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\ &= \sum_{i=1}^{3} \int_{\mathcal{B}_i} v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\ &= \frac{5}{6} f_1^2 + \frac{16}{3} f_2^2 + \frac{5}{6} f_3^2. \end{aligned}$$

This implies that

$$\frac{5}{6}\,\|K^{*}f\,\|^{2} \leq \int\limits_{\mathcal{B}} v^{2}\,(\,x\,)\,\big\langle\,\Lambda\,(\,x\,)P_{F\,(\,x\,)}U\,f,\,\Lambda\,(\,x\,)P_{F\,(\,x\,)}T\,f\,\big\rangle\,d\mu_{x}$$
$$\leq\,\frac{16}{3}\,\|\,f\,\|^{2}.$$

Thus $\Lambda_{T\,U}$ is a continuous $(\,T,\,U\,)$-controlled K-g-fusion frame for $\mathbb{R}^{3}$.

Proposition 1. *Let* $\Lambda_{T\,U}$ *be a continuous* $(\,T,\,U\,)$*-controlled* g*-fusion Bessel family for* H *with bound* B*. Then there exists a unique bounded linear operator* $S_{C}\,:\,H\,\to\,H$ *such that*

$$\langle\,S_{C}\,f,\,g\,\rangle\,=\,\int\limits_{X} v^{2}\,(\,x\,)\,\big\langle\,T^{*}\,P_{F\,(\,x\,)}\,\Lambda_{x}^{*}\,\Lambda_{x}\,P_{F\,(\,x\,)}\,U\,f,\,g\,\big\rangle\,d\mu_{x}$$

for all $f,\,g\,\in\,H$*. Furthermore, if* $\Lambda_{T\,U}$ *is a continuous* $(\,T,\,U\,)$*-controlled* K*-*g*-fusion frame for* H*, then* $A\,K\,K^{*}\,\leq\,S_{C}\,\leq\,B\,I_{H}$*.*

Proof. Proof of this proposition follows directly from Proposition 3.3 of [17].

Furthermore, if $\Lambda_{T\,U}$ is a continuous $(\,T,\,U\,)$-controlled K-g-fusion frame for H then by (3) it is easy to verify that $A\,K\,K^{*}\,\leq\,S_{C}\,\leq\,B\,I_{H}$. □

The operator defined in Proposition 1 is called the frame operator for $\Lambda_{T\,U}$.

Definition 9. Let $\Lambda_{T\,U}$ be a continuous $(\,T,\,U\,)$-controlled g-fusion Bessel family for H. Then the bounded linear operator $T_{C}\,:\,L^{2}\,(\,X,\,K\,)\,\to\,H$ defined by

$$\langle\,T_{C}\,\Phi,\,g\,\rangle\,=\,\int\limits_{X} v^{2}\,(\,x\,)\,\big\langle\,T^{*}\,P_{F\,(\,x\,)}\,\Lambda_{x}^{*}\,\Lambda_{x}\,P_{F\,(\,x\,)}\,U\,f,\,g\,\big\rangle\,d\mu_{x},$$

where for all $f\,\in\,H$,

$$\Phi\,=\,\Big\{\,v\,(\,x\,)\,\big(\,T^{*}\,P_{F\,(\,x\,)}\,\Lambda_{x}^{*}\,\Lambda_{x}\,P_{F\,(\,x\,)}\,U\,\big)^{1\,/\,2}\,f\,\Big\}_{x\,\in\,X}$$

and $g\,\in\,H$, is called the synthesis operator. Its adjoint operator, described by

$$T_{C}^{*}\,g\,=\,\Big\{\,v\,(\,x\,)\,\big(\,T^{*}\,P_{F\,(\,x\,)}\,\Lambda_{x}^{*}\,\Lambda_{x}\,P_{F\,(\,x\,)}\,U\,\big)^{1\,/\,2}\,g\,\Big\}_{x\,\in\,X},$$

is called the analysis operator.

Next we will see that continuous controlled g-fusion Bessel families for H become continuous controlled g-fusion frames for H under some sufficient conditions.

Theorem 6. *Let the families* $\Lambda_{TU} = \{(F(x), \Lambda_x, v(x))\}_{x \in X}$ *and* $\Gamma_{TU} = \{(F(x), \Gamma_x, v(x))\}_{x \in X}$ *be two continuous* (T, U)*-controlled g-fusion Bessel families for* H *with bounds* B *and* D*, respectively. Suppose that* T_C *and* T_C' *are their synthesis operators such that* $T_C' T_C^* = K^*$*. Then* Λ_{TU} *and* Γ_{TU} *are a continuous* (T, U)*-controlled* K*-g-fusion frame and a continuous* (T, U)*-controlled* K^**-g-fusion frame for* H*, respectively.*

Proof. For each $f \in H$, we have

$$\|K^* f\|^4 = \langle K^* f, K^* f \rangle^2 = \left\langle T_C^* f, \left(T_C'\right)^* K^* f \right\rangle^2$$
$$\leq \|T_C^* f\|^2 \left\|\left(T_C'\right)^* K^* f\right\|^2$$
$$= \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} U f, \Lambda_x P_{F(x)} T f \right\rangle d\mu_x \times$$
$$\int_X v^2(x) \left\langle \Gamma_x P_{F(x)} U K^* f, \Gamma_x P_{F(x)} T K^* f \right\rangle d\mu_x$$
$$\leq D \|K^* f\|^2 \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} U f, \Lambda_x P_{F(x)} T f \right\rangle d\mu_x$$
$$\Rightarrow \frac{1}{D} \|K^* f\|^2 \leq \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} U f, \Lambda_x P_{F(x)} T f \right\rangle d\mu_x.$$

This shows that Λ_{TU} is a continuous (T, U)-controlled K-g-fusion frame for H with bounds $1/D$ and B. Similarly, it can be shown that Γ_{TU} is a continuous (T, U)-controlled K^*-g-fusion frame for H. □

In the following theorem, we will see that any continuous controlled K-g-fusion frame is a continuous K-g-fusion frame and conversely any continuous K-g-fusion frame is a continuous controlled K-g-fusion frame under some sufficient conditions.

Theorem 7. *Let* $T, U \in \mathcal{G}\mathcal{B}^+(H)$ *and* $S_{gF} T = T S_{gF}$*. If the operator* K *commutes with* T *and* U*, then* Λ_{TU} *is a continuous* (T, U)*-controlled* K*-g-fusion frame for* H *if and only if* Λ_{TU} *is a continuous* K*-g-fusion frame for* H*, where* S_{gF} *is the continuous g-fusion frame operator defined by*

$$\langle S_{gF} f, f \rangle = \int_X v^2(x) \left\langle P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} f, f \right\rangle d\mu_x, \quad f \in H.$$

Proof. First we suppose that Λ_{TU} is a continuous K-g-fusion frame for H with bounds A and B. Then, for each $f \in H$, we have

$$A \|K^* f\|^2 \leq \int_X v^2(x) \left\|\Lambda_x P_{F(x)} f\right\|^2 d\mu_x \leq B \|f\|^2.$$

Now, according to Lemma 3.10 of [3], we can deduce that

$$m\,m'\,A\,K\,K^{*} \,\leq\, T\,S_{g\,F}\,U \,\leq\, M\,M'\,B\,I_{H},$$

where m, m' and M, M' are positive constants. Then for each $f \in H$, we have

$$\begin{aligned} m\,m'\,A\,\|\,K^{*}\,f\,\|^{2} &\leq \int\limits_{X} v^{2}\,(\,x\,)\,\left\langle\,T\,P_{F\,(\,x\,)}\,\Lambda_{x}^{*}\,\Lambda_{x}\,P_{F\,(\,x\,)}\,U\,f,\;f\,\right\rangle\,d\mu_{x} \\ &\leq M\,M'\,B\,\|\,f\,\|^{2}. \end{aligned}$$

This shows that

$$\begin{aligned} m\,m'\,A\,\|\,K^{*}\,f\,\|^{2} &\leq \int\limits_{X} v^{2}\,(\,x\,)\,\left\langle\,\Lambda_{x}\,P_{F\,(\,x\,)}\,U\,f,\;\Lambda_{x}\,P_{F\,(\,x\,)}\,T\,f\,\right\rangle\,d\mu_{x} \\ &\leq M\,M'\,B\,\|\,f\,\|^{2}. \end{aligned}$$

Hence $\Lambda_{T\,U}$ is a continuous $(\,T,\,U\,)$-controlled K-g-fusion frame for H.

Conversely, suppose that $\Lambda_{T\,U}$ is a continuous $(\,T,\,U\,)$-controlled K-g-fusion frame for H with bounds A and B. Now, for each $f \in H$, we have

$$\begin{aligned} &A\,\|\,K^{*}\,f\,\|^{2} \,=\, A\,\left\|\,(\,T\,U\,)^{1/2}\,(\,T\,U\,)^{-1/2}\,K^{*}\,f\,\right\|^{2} \\ &=\, A\,\left\|\,(\,T\,U\,)^{1/2}\,K^{*}\,(\,T\,U\,)^{-1/2}\,f\,\right\|^{2} \\ &\leq C\int\limits_{X} v^{2}\,(\,x\,)\left\langle\,\Lambda_{x}P_{F\,(\,x\,)}U(\,T\,U\,)^{-1/2}f,\;\Lambda_{x}P_{F\,(\,x\,)}T(\,T\,U\,)^{-1/2}f\,\right\rangle\,d\mu_{x} \\ &=\, C\int\limits_{X} v^{2}\,(\,x\,)\,\left\langle\,\Lambda_{x}\,P_{F\,(\,x\,)}\,U^{1/2}\,T^{-1/2}\,f,\;\Lambda_{x}\,P_{F\,(\,x\,)}\,T^{1/2}\,U^{-1/2}\,f\,\right\rangle\,d\mu_{x} \\ &= C\int\limits_{X} v^{2}\,(\,x\,)\left\langle\,U^{-1/2}\,T^{1/2}\,P_{F\,(\,x\,)}\,\Lambda_{x}^{*}\,\Lambda_{x}\,P_{F\,(\,x\,)}\,U^{1/2}\,T^{-1/2}\,f,\;f\,\right\rangle d\mu_{x} \\ &= C\,\left\langle\,U^{-1/2}\,T^{1/2}\,S_{g\,F}\,U^{1/2}\,T^{-1/2}\,f,\;f\,\right\rangle = C\,\left\langle\,S_{g\,F}\,f,\;f\,\right\rangle \\ &=\, C\int\limits_{X} v^{2}\,(\,x\,)\,\left\langle\,P_{F\,(\,x\,)}\,\Lambda_{x}^{*}\,\Lambda_{x}\,P_{F\,(\,x\,)}\,f,\;f\,\right\rangle\,d\mu_{x}, \end{aligned}$$

where $C \,=\, \left\|\,(\,T\,U\,)^{1/2}\,\right\|^{2}$. This implies that

$$\frac{A}{\left\|\,(\,T\,U\,)^{1/2}\,\right\|^{2}}\,\|\,K^{*}\,f\,\|^{2} \,\leq\, \int\limits_{X} v^{2}\,(\,x\,)\,\left\|\,\Lambda_{x}\,P_{F\,(\,x\,)}\,f\,\right\|^{2}\,d\mu_{x}.$$

On the other hand, it is easy to verify that

$$\int_X v^2(x)\,\left\|\Lambda_x P_{F(x)} f\right\|^2 d\mu_x$$
$$= \left\langle (TU)^{-1/2}(TU)^{1/2} S_{gF} f,\, f \right\rangle$$
$$= \left\langle (TU)^{1/2} S_{gF} f,\, (TU)^{-1/2} f \right\rangle$$
$$= \left\langle S_{gF}(TU)(TU)^{-1/2} f,\, (TU)^{-1/2} f \right\rangle$$
$$= \left\langle T S_{gF} U (TU)^{-1/2} f,\, (TU)^{-1/2} f \right\rangle$$
$$= \left\langle S_C (TU)^{-1/2} f,\, (TU)^{-1/2} f \right\rangle$$
$$\leq B \left\|(TU)^{-1/2}\right\|^2 \|f\|^2.$$

Thus, Λ_{TU} is a continuous K-g-fusion frame for H. This completes the proof. □

In the next two theorems, we will construct a continuous controlled g-fusion frame of new type from a given continuous controlled K-g-fusion frame by using an invertible bounded linear operator.

Theorem 8. *Let Λ_{TU} be a continuous (T, U)-controlled K-g-fusion frame for H with bounds A, B and $V \in \mathcal{B}(H)$ be an invertible operator on H such that V^* commutes with T and U. Then the family given by $\Gamma_{TU} = \left\{\left(VF(x),\, \Lambda_x P_{F(x)} V^*,\, v(x)\right)\right\}_{x \in X}$ is a continuous (T, U)-controlled VKV^*-g-fusion frame for H.*

Proof. Since $P_{F(x)} V^* = P_{F(x)} V^* P_{VF(x)}$ for all $x \in X$, the mapping $f \to P_{VF(x)} f$, $f \in H$ is weakly measurable. Now, for each $f \in H$, using Theorem 5, we have

$$\int_X v^2(x)\,\left\langle \Lambda_x P_{F(x)} V^* P_{VF(x)} U f,\, \Lambda_x P_{F(x)} V^* P_{VF(x)} T f \right\rangle d\mu_x$$
$$= \int_X v^2(x)\,\left\langle \Lambda_x P_{F(x)} V^* U f,\, \Lambda_x P_{F(x)} V^* T f \right\rangle d\mu_x$$
$$= \int_X v^2(x)\,\left\langle \Lambda_x P_{F(x)} U V^* f,\, \Lambda_x P_{F(x)} T V^* f \right\rangle d\mu_x$$
$$\leq B\,\|V^* f\|^2 \leq B\,\|V\|^2\,\|f\|^2.$$

On the other hand, for each $f \in H$, we get

$$\frac{A}{\|V\|^2}\,\|(VKV^*)^* f\|^2 = \frac{A}{\|V\|^2}\,\|VK^*V^* f\|^2$$

$$\begin{aligned}
&\leq A \, \| K^* V^* f \|^2 \\
&= \int\limits_X v^2(x) \left\langle \Lambda_x P_{F(x)} U V^* f, \Lambda_x P_{F(x)} T V^* f \right\rangle d\mu_x \\
&= \int\limits_X v^2(x) \left\langle \Lambda_x P_{F(x)} V^* U f, \Lambda_x P_{F(x)} V^* T f \right\rangle d\mu_x \\
&= \int\limits_X v^2(x) \left\langle \Lambda_x P_{F(x)} V^* P_{VF(x)} U f, \Lambda_x P_{F(x)} V^* P_{VF(x)} T f \right\rangle d\mu_x.
\end{aligned}$$

Thus Γ_{TU} is a continuous (T, U)-controlled $V K V^*$-g-fusion frame for H with bounds $A / \| V \|^2$ and $B \| V \|^2$. □

Theorem 9. *Let $V \in \mathcal{B}(H)$ be an invertible operator such that $(V^{-1})^*$ commutes with T and U. Let $\Gamma_{TU} = \left\{ \left(V F(x), \Lambda_x P_{F(x)} V^*, v(x) \right) \right\}_{x \in X}$ be a continuous (T, U)-controlled K-g-fusion frame for H, for some $K \in \mathcal{B}(H)$. Then Λ_{TU} is a continuous (T, U)-controlled $V^{-1} K V$-g-fusion frame for H.*

Proof. Since Γ_{TU} is a continuous (T, U)-controlled K-g-fusion frame for H, for each $f \in H$, there exist constants $A, B > 0$ such that

$$\begin{aligned}
&A \, \| K^* f \|^2 \\
&\leq \int\limits_X v^2(x) \left\langle \Lambda_x P_{F(x)} V^* P_{VF(x)} U f, \Lambda_x P_{F(x)} V^* P_{VF(x)} T f \right\rangle d\mu_x \\
&\leq B \, \| f \|^2. \qquad (4)
\end{aligned}$$

Now, for each $f \in H$, using Theorem 5, we have

$$\begin{aligned}
&\frac{A}{\| V \|^2} \left\| \left(V^{-1} K V \right)^* f \right\|^2 = \frac{A}{\| V \|^2} \left\| V^* K^* (V^{-1})^* f \right\|^2 \\
&\leq A \left\| K^* \left(V^{-1} \right)^* f \right\|^2 \\
&\leq \int\limits_X v^2(x) \left\langle \Lambda_x P_{F(x)} V^* U \left(V^{-1} \right)^* f, \Lambda_x P_{F(x)} V^* T \left(V^{-1} \right)^* f \right\rangle d\mu_x \\
&= \int\limits_X v^2(x) \left\langle \Lambda_x P_{F(x)} V^* \left(V^{-1} \right)^* U f, \Lambda_x P_{F(x)} V^* \left(V^{-1} \right)^* T f \right\rangle d\mu_x \\
&= \int\limits_X v^2(x) \left\langle \Lambda_x P_{F(x)} U f, \Lambda_x P_{F(x)} T f \right\rangle d\mu_x.
\end{aligned}$$

On the other hand, for each $f \in H$, we have

$$\int_X v^2(x) \left\langle \Lambda_x P_{F(x)} U f, \Lambda_x P_{F(x)} T f \right\rangle d\mu_x$$
$$= \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} V^* U \left(V^{-1}\right)^* f, \Lambda_x P_{F(x)} V^* T \left(V^{-1}\right)^* f \right\rangle d\mu_x$$
$$\leq B \left\| \left(V^{-1}\right)^* f \right\|^2 \leq B \left\| V^{-1} \right\|^2 \| f \|^2 \text{ [by (4)]}.$$

Thus, Λ_{TU} is a continuous (T, U)-controlled $V^{-1} K V$-g-fusion frame for H. □

In the following theorem, we will see that every continuous controlled g-fusion frame is a continuous controlled K-g-fusion frame and the converse is also true under some condition.

Theorem 10. *Let $K \in \mathcal{B}(H)$. Then*

- (i) *every continuous (T, U)-controlled g-fusion frame is a continuous (T, U)-controlled K-g-fusion frame,*
- (ii) *if $\mathcal{R}(K)$ is closed, every continuous (T, U)-controlled K-g-fusion frame is a continuous (T, U)-controlled g-fusion frame for $\mathcal{R}(K)$.*

Proof. (i) Let Λ_{TU} be a continuous (T, U)-controlled g-fusion frame for H with bounds A and B. Then, for each $f \in H$, we have

$$\frac{A}{\| K \|^2} \| K^* f \|^2 \leq A \| f \|^2$$
$$\leq \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} U f, \Lambda_x P_{F(x)} T f \right\rangle d\mu_x \leq B \| f \|^2.$$

Hence Λ_{TU} is a continuous (T, U)-controlled K-g-fusion frame for H with bounds $\dfrac{A}{\| K \|^2}$ and B.

(ii) Let Λ_{TU} be a continuous (T, U)-controlled K-g-fusion frame for H with bounds A and B. Since $\mathcal{R}(K)$ is closed, by Theorem 1, there exists an operator $K^\dagger \in \mathcal{B}(H)$ such that $K K^\dagger f = f \; \forall f \in \mathcal{R}(K)$. Then for each $f \in \mathcal{R}(K)$,

$$\frac{A}{\| K^\dagger \|^2} \| f \|^2 \leq A \| K^* f \|^2$$
$$\leq \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} U f, \Lambda_x P_{F(x)} T f \right\rangle d\mu_x \leq B \| f \|^2.$$

Thus Λ_{TU} is a continuous (T, U)-controlled g-fusion frame for $\mathcal{R}(K)$ with bounds $\dfrac{A}{\|K^{\dagger}\|^2}$ and B. □

Theorem 11. *Let $K \in \mathcal{B}(H)$, $T, U \in \mathcal{GB}^+(H)$ and Λ_{TU} be a continuous (T, U)-controlled K-g-fusion frame for H with frame bounds A, B. If $V \in \mathcal{B}(H)$ with $\mathcal{R}(V) \subset \mathcal{R}(K)$, then Λ_{TU} is a continuous (T, U)-controlled V-g-fusion frame for H.*

Proof. Since Λ_{TU} is a continuous (T, U)-controlled K-g-fusion frame for H, for each $f \in H$, we have

$$A\,\|K^* f\|^2 \leq \int_X v^2(x)\left\langle \Lambda_x P_{F(x)} U f,\, \Lambda_x P_{F(x)} T f\right\rangle d\mu_x \,\leq\, B\,\|f\|^2.$$

Since $\mathcal{R}(V) \subset \mathcal{R}(K)$, by Theorem 2, there exists some $\lambda > 0$ such that $V V^* \leq \lambda K K^*$. Thus, for each $f \in H$, we have

$$\begin{aligned}&\frac{A}{\lambda}\,\|V^* f\|^2 \,\leq\, A\,\|K^* f\|^2\\ &\leq \int_X v^2(x)\left\langle \Lambda_x P_{F(x)} U f,\, \Lambda_x P_{F(x)} T f\right\rangle d\mu_x \,\leq\, B\,\|f\|^2.\end{aligned}$$

Hence Λ_{TU} is a continuous (T, U)-controlled V-g-fusion frame for H. □

In the following theorem, we will construct a continuous controlled K-g-fusion frame by using a continuous controlled g-fusion frame under some sufficient conditions.

Theorem 12. *Let $K \in \mathcal{B}(H)$ be an invertible operator on H and Λ_{TU} be a continuous (T, U)-controlled g-fusion frame for H with frame bounds A, B and S_C be the frame operator. Suppose $S_C^{-1} K^*$ commutes with T and U. Then $\Gamma_{TU} = \left\{\left(K S_C^{-1} F(x),\, \Lambda_x P_{F(x)} S_C^{-1} K^*,\, v(x)\right)\right\}_{x \in X}$ is a continuous (T, U)-controlled K-g-fusion frame for H with the corresponding frame operator $K S_C^{-1} K^*$.*

Proof. Let $V = K S_C^{-1}$. Then V is invertible on H and $V^* = S_C^{-1} K^*$. It is easy to verify that

$$\|K^* f\|^2 \,\leq\, B^2\,\left\|S_C^{-1} K^* f\right\|^2 \quad \forall\, f \in H. \tag{5}$$

Now, for each $f \in H$, using Theorem 5, we have

$$\begin{aligned}&\int_X v^2(x)\left\langle \Lambda_x P_{F(x)} V^* P_{V F(x)} U f,\, \Lambda_x P_{F(x)} V^* P_{V F(x)} T f\right\rangle d\mu_x\\ &= \int_X v^2(x)\left\langle \Lambda_x P_{F(x)} V^* U f,\, \Lambda_x P_{F(x)} V^* T f\right\rangle d\mu_x\end{aligned}$$

$$= \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} U S_C^{-1} K^* f, \, \Lambda_x P_{F(x)} T S_C^{-1} K^* f \right\rangle d\mu_x$$
$$\leq B \, \| S_C^{-1} \|^2 \, \| K^* f \|^2$$
$$\leq \frac{B}{A^2} \| K \|^2 \| f \|^2 \; [\text{ using } B^{-1} I_H \leq S_C^{-1} \leq A^{-1} I_H \,].$$

On the other hand, for each $f \in H$, we have

$$\int_X v^2(x) \left\langle \Lambda_x P_{F(x)} V^* P_{V F(x)} U f, \, \Lambda_x P_{F(x)} V^* P_{V F(x)} T f \right\rangle d\mu_x$$
$$= \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} U S_C^{-1} K^* f, \, \Lambda_x P_{F(x)} T S_C^{-1} K^* f \right\rangle d\mu_x$$
$$\geq A \, \left\| S_C^{-1} K^* f \right\|^2 \geq \frac{A}{B^2} \, \| K^* f \|^2 \; [\text{ by } (5) \,].$$

Thus Γ_{TU} is a continuous (T, U)-controlled K-g-fusion frame for H. Furthermore, for each $f \in H$, we have

$$\int_X v^2(x) \left\langle \Lambda_x P_{F(x)} V^* P_{V F(x)} U f, \, \Lambda_x P_{F(x)} V^* P_{V F(x)} T f \right\rangle d\mu_x$$
$$= \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} U S_C^{-1} K^* f, \, \Lambda_x P_{F(x)} T S_C^{-1} K^* f \right\rangle d\mu_x$$
$$= \left\langle S_C S_C^{-1} K^* f, \, S_C^{-1} K^* f \right\rangle = \left\langle K S_C^{-1} K^* f, \, f \right\rangle.$$

This implies that $K S_C^{-1} K^*$ is the corresponding frame operator of Γ_{TU}. □

In the following theorem, we give a necessary and sufficient condition for continuous controlled g-fusion Bessel family to be a continuous controlled K-g-fusion frame with the help of quotient operator.

Theorem 13. *Let $K \in \mathcal{B}(H)$ and Λ_{TU} be a continuous (T, U)-controlled g-fusion Bessel family in H with frame operator S_C. Then Λ_{TU} is a continuous (T, U)-controlled K-g-fusion frame for H if and only if the quotient operator $\left[K^* / S_C^{1/2} \right]$ is bounded.*

Proof. First, we suppose that Λ_{TU} is a continuous (T, U)-controlled K-g-fusion frame for H with bounds A and B. Then for each $f \in H$, we have

$$A \| K^* f \|^2 \leq \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} U f, \, \Lambda_x P_{F(x)} T f \right\rangle d\mu_x \leq B \, \| f \|^2.$$

Thus, for each $f \in H$, we have

$$A \| K^* f \|^2 \leq \langle S_C f, f \rangle = \left\| S_C^{1/2} f \right\|^2.$$

Now, it is easy to verify that the quotient operator $T : \mathcal{R}\left(S_C^{1/2} \right) \to \mathcal{R}(K^*)$ defined by $T \left(S_C^{1/2} f \right) = K^* f$ for every $f \in H$ is well-defined and bounded.

Conversely, suppose that the quotient operator $\left[K^* / S_C^{1/2} \right]$ is bounded. Then, for each $f \in H$, there exists some $B > 0$ such that

$$\| K^* f \|^2 \leq B \left\| S_C^{1/2} f \right\|^2 = B \langle S_C f, f \rangle$$
$$\Rightarrow \| K^* f \|^2 \leq B \int_X v^2 (x) \left\langle \Lambda_x P_{F(x)} U f, \Lambda_x P_{F(x)} T f \right\rangle d\mu_x.$$

Thus Λ_{TU} is a continuous (T, U)-controlled K-g-fusion frame for H. □

Now, we establish that a quotient operator will be bounded if and only if a continuous controlled K-g-fusion frame becomes continuous controlled $V K$-g-fusion frame, for some $V \in \mathcal{B}(H)$.

Theorem 14. *Let $K \in \mathcal{B}(H)$ and Λ_{TU} be a continuous (T, U)-controlled K-g-fusion frame for H with frame operator S_C. Let $V \in \mathcal{B}(H)$ be an invertible operator on H such that V^* commutes with T and U. Then the following statements are equivalent.*

- (i) *$\Gamma_{TU} = \left\{ \left(V F (x), \Lambda_x P_{F(x)} V^*, v (x) \right) \right\}_{x \in X}$ is a continuous (T, U)-controlled $V K$-g-fusion frame for H.*
- (ii) *The quotient operator $\left[(V K)^* / S_C^{1/2} V^* \right]$ is bounded.*
- (iii) *The quotient operator $\left[(V K)^* / (V S_C V^*)^{1/2} \right]$ is bounded.*

Proof. $(i) \Rightarrow (ii)$ Suppose Γ_{TU} is a continuous (T, U)-controlled $V K$-g-fusion frame with bounds A and B. Then, for each $f \in H$, we have

$$A \| (V K)^* f \|^2$$
$$\leq \int_X v^2 (x) \left\langle \Lambda_x P_{F(x)} V^* P_{V F(x)} U f, \Lambda_x P_{F(x)} V^* P_{V F(x)} T f \right\rangle d\mu_x$$
$$\leq B \| f \|^2.$$

By Theorem 5, for each $f \in H$, we have

$$\int_X v^2(x) \left\langle \Lambda_x P_{F(x)} V^* P_{V F(x)} U f, \Lambda_x P_{F(x)} V^* P_{V F(x)} T f \right\rangle d\mu_x$$

$$= \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} V^* U f, \Lambda_x P_{F(x)} V^* T f \right\rangle d\mu_x$$

$$= \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} U V^* f, \Lambda_x P_{F(x)} T V^* f \right\rangle d\mu_x$$

$$= \langle S_C V^* f, V^* f \rangle. \tag{6}$$

Thus, for each $f \in H$, we have

$$A \| (V K)^* f \|^2 \leq \langle S_C V^* f, V^* f \rangle = \left\| S_C^{1/2} V^* f \right\|^2.$$

We define an operator

$$T : \mathcal{R}\left(S_C^{1/2} V^* \right) \to \mathcal{R}((V K)^*)$$

by

$$T\left(S_C^{1/2} V^* f \right) = (V K)^* f \;\; \forall f \in H.$$

It is easy verify that the quotient operator T is well-defined and bounded.

$(ii) \Rightarrow (iii)$ It is obvious.

$(iii) \Rightarrow (i)$ Suppose that the quotient operator

$$\left[(V K)^* / (V S_C V^*)^{1/2} \right]$$

is bounded. Then, for each $f \in H$, there exists $B > 0$ such that

$$\| (V K)^* f \|^2 \leq B \left\| (V S_C V^*)^{1/2} f \right\|^2.$$

Now, by (6), for each $f \in H$, we have

$$\int_X v^2(x) \left\langle \Lambda_x P_{F(x)} V^* P_{V F(x)} U f, \Lambda_x P_{F(x)} V^* P_{V F(x)} T f \right\rangle d\mu_x$$

$$= \langle S_C V^* f, V^* f \rangle = \left\| (V S_C V^*)^{1/2} f \right\|^2 \geq \frac{1}{B} \| (V K)^* f \|^2.$$

On the other hand, for each $f \in H$, we have

$$\int_X v^2(x) \left\langle \Lambda_x P_{F(x)} V^* P_{V F(x)} U f, \Lambda_x P_{F(x)} V^* P_{V F(x)} T f \right\rangle d\mu_x$$

$$= \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} U V^* f, \Lambda_x P_{F(x)} T V^* f \right\rangle d\mu_x$$

$$\leq D \| U^* f \|^2 \leq D \| U \|^2 \| f \|^2.$$

Hence Γ_{TU} is a continuous (T, U)-controlled $V K$-g-fusion frame for H. This completes the proof. □

4. Stability of a dual continuous controlled g-fusion frame

In frame theory, one of the most important problems is the stability of a frame under some perturbation. Casazza and Chirstensen [6] have generalized the Paley–Wiener perturbation theorem to perturbation of frame in a Hilbert space. Ghosh and Samanta [14] discussed stability of a dual g-fusion frame in a Hilbert space. In this section, we give an important result on stability of perturbation of a continuous controlled K-g-fusion frame and a dual continuous controlled g-fusion frame.

The following theorem provides a sufficient condition on a family Λ_{TU} to be a continuous controlled K-g-fusion frame in the presence of another continuous controlled K-g-fusion frame.

Theorem 15. *Let Λ_{TU} be a continuous (T, U)-controlled g-fusion frame for H and S_C be the frame operator. Assume that S_C^{-1} commutes with T and U. Then $\Gamma_{TU} = \left\{ \left(S_C^{-1} F(x), \Lambda_x P_{F(x)} S_C^{-1}, v(x) \right) \right\}_{x \in X}$ is a continuous (T, U)-controlled g-fusion frame for H with the corresponding frame operator S_C^{-1}.*

Proof. Proof of this theorem directly follows from Theorem 12, by putting $K = I_H$. □

The family Γ_{TU} defined in Theorem 15 is called the canonical dual continuous (T, U)-controlled g-fusion frame of Λ_{TU}. We now give the stability result of the dual continuous controlled g-fusion frame.

Theorem 16. *Let Λ_{TU} and Γ_{TU} be two continuous (T, U)-controlled g-fusion frames for H with bounds A_1, B_1 and A_2, B_2 having their corresponding frame operators S_C and $S_{C'}$, respectively. Consider $\Delta_{TU} = \{ (X(x), \Delta_x, v(x)) \}_{x \in X}$ and $\Theta_{TU} = \{ (Y(x), \Theta_x, v(x)) \}_{x \in X}$ as the canonical dual continuous (T, U)-controlled g-fusion frames of Λ_{TU} and Γ_{TU}, respectively. Assume that S_C^{-1} and $S_{C'}^{-1}$ commute with both T and U. Then the following statements hold.*

(i) If the condition

$$\left| \int_X v^2(x) \left(\langle L_x U f, L_x T f \rangle - \langle M_x U f, M_x T f \rangle \right) d\mu_x \right| \leq D \| f \|^2$$

holds for each $f \in H$ and for some $D > 0$ then for all $f \in H$, we have

$$\left| \int_X v^2(x) \left(\langle D_x U f, D_x T f \rangle - \langle E_x U f, E_x T f \rangle \right) d\mu_x \right|$$

$$\leq \frac{D}{A_1 A_2} \| f \|^2,$$

where $\Lambda_x P_{F(x)} = L_x$, $\Gamma_x P_{G(x)} = M_x$ *and* $\Delta_x P_{X(x)} = D_x$, $\Theta_x P_{Y(x)} = E_x$.

(ii) *If for each* $f \in H$, *there exists* $D > 0$ *such that*

$$\left| \int_X v^2(x) \left\langle T^* \left(P_{F(x)} \Lambda_x^* L_x - P_{G(x)} \Gamma_x^* M_x \right) U f, g \right\rangle d\mu_x \right|$$

$$\leq D \| f \|^2,$$

then

$$\left| \int_X v^2(x) \left\langle T^* \left(P_{X(x)} \Delta_x^* D_x - P_{Y(x)} \Theta_x^* E_x \right) U f, g \right\rangle d\mu_x \right|$$

$$\leq \frac{D}{A_1 A_2} \| f \|^2.$$

Proof. (i) Since $S_C - S_{C'}$ is self-adjoint, we have

$$\| S_C - S_{C'} \| = \sup_{\| f \| = 1} | \langle (S_C - S_{C'}) f, f \rangle |$$

$$= \sup_{\| f \| = 1} | \langle S_C f, f \rangle - \langle S_{C'} f, f \rangle |$$

$$= \sup_{\|f\|=1} \left| \int_X v^2(x) (\langle L_x U f, L_x T f \rangle - \langle M_x U f, M_x T f \rangle) d\mu_x \right|$$

$$\leq \sup_{\| f \| = 1} D \| f \|^2 = D.$$

Then

$$\begin{aligned} \| S_C^{-1} - S_{C'}^{-1} \| &\leq \| S_C^{-1} \| \| S_C - S_{C'} \| \| S_{C'}^{-1} \| \\ &\leq \frac{1}{A_1} D \frac{1}{A_2} = \frac{D}{A_1 A_2}. \end{aligned} \tag{7}$$

Now, for each $f \in H$, we have

$$\int_X v^2(x) \left\langle \Delta_x P_{X(x)} U f, \Delta_x P_{X(x)} T f \right\rangle d\mu_x$$

$$= \int_X v^2(x) \left\langle L_x S_C^{-1} P_{S_C^{-1} F(x)} U f, L_x S_C^{-1} P_{S_C^{-1} F(x)} T f \right\rangle d\mu_x$$

$$= \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} S_C^{-1} U f, \Lambda_x P_{F(x)} S_C^{-1} T f \right\rangle d\mu_x$$

$$= \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} U S_C^{-1} f,\, \Lambda_x P_{F(x)} T S_C^{-1} f \right\rangle d\mu_x$$

$$= \int_X v^2(x) \left\langle T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U S_C^{-1} f,\, S_C^{-1} f \right\rangle d\mu_x$$

$$= \left\langle S_C S_C^{-1} f,\, S_C^{-1} f \right\rangle = \left\langle f,\, S_C^{-1} f \right\rangle.$$

Similarly, it can be shown that

$$\int_X v^2(x) \left\langle \Theta_x P_{Y(x)} U f,\, \Theta_x P_{Y(x)} T f \right\rangle d\mu_x$$
$$= \left\langle f,\, S_{C'}^{-1} f \right\rangle.$$

Therefore, for each $f \in H$, we have

$$\left| \int_X v^2(x) \left(\langle D_x U f,\, D_x T f \rangle - \langle E_x U f,\, E_x T f \rangle \right) d\mu_x \right|$$
$$= \left| \left\langle f,\, S_C^{-1} f \right\rangle - \left\langle f,\, S_{C'}^{-1} f \right\rangle \right| = \left| \left\langle f,\, \left(S_C^{-1} - S_{C'}^{-1} \right) f \right\rangle \right|$$
$$\leq \left\| S_C^{-1} - S_{C'}^{-1} \right\| \, \| f \|^2 \leq \frac{D}{A_1 A_2} \| f \|^2.$$

Proof of (ii). In this case, we also find that

$$\| S_C - S_{C'} \| = \sup_{\| f \| = 1} | \langle (S_C - S_{C'}) f,\, f \rangle |$$
$$= \sup_{\| f \| = 1} | \langle S_C f,\, f \rangle - \langle S_{C'} f,\, f \rangle |$$
$$= \sup_{\|f\|=1} \left| \int_X c_x \left\langle T^* \left(P_{F(x)} \Lambda_x^* L_x - P_{G(x)} \Gamma_x^* M_x \right) U f,\, g \right\rangle d\mu_x \right|$$
$$\leq \sup_{\| f \| = 1} D \| f \|^2 = D,\; c_x = v^2(x).$$

Then, for each $f \in H$, we have

$$\left| \int_X v^2(x) \left\langle T^* \left(P_{X(x)} \Delta_x^* D_x - P_{Y(x)} \Theta_x^* E_x \right) U f,\, g \right\rangle d\mu_x \right|$$
$$= \left| \left\langle \left(S_C^{-1} - S_{C'}^{-1} \right) f,\, f \right\rangle \right| \leq \left\| S_C^{-1} - S_{C'}^{-1} \right\| \, \| f \|^2$$
$$\leq \frac{D}{A_1 A_2} \| f \|.$$

This completes the proof. □

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