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Statistical Killing vector fields on the Hopf hypersurfaces in the complex space forms

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Abstract. Statistical Killing vector field is introduced and Hopf hypersurfaces of complex space forms with the condition of being structural statistical Killing vector field are studied. It is shown that these hypersurfaces have at most three distinct constant principal curvatures.

1. Introduction

The geometry of statistical manifolds is in the intersection of some research areas such as information geometry, affine differential geometry and Hessian geometry. It has been shown that this branch of geometry has emerged from the study of the geometric structure of the natural differential on the manifolds of the probability distribution and includes a Riemannian metric, which is defined by Fisher information and a one-parameter family of symmetric connections, called statistical connections, dependent on the Levi-Civita connection. That is an interesting relationship between two types of connections.

In 1980, statistical structure was introduced and it has played an important role in information geometry research. Then, Furuhata [3] surveyed the hypersurfaces in the statistical manifolds, he presented the curvature condition of a statistical manifold to accept a standard hypersurface as the first step in the theory of statistical manifolds, after which he wrote about submanifolds in statistical manifolds. Sasakian statistical manifolds were studied by Furuhata et al. in [4], where a notion of Sasakian statistical structure was introduced and a real hypersurface in holomorphic statistical

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manifold was given to accept such a structure. In addition, Milijevic [7] discussed about statistical hypersurfaces with shape operators by one or two constant eigenvalues. Moreover, in [8], she studied CR statistical submanifold of maximal CR-dimension with umbilical shape operator in holomorphic statistical manifolds and stated an extension of results in the CR statistical submanifolds in complex space forms.

The CR-submanifolds of complex manifolds, with particular emphasis on CR-submanifolds of complex projective space were classified by Mirjana and Masafumi [2]. One of the significant cases here is the creation of a link between Riemannian and statistical manifolds, in order to generalize some features and results found in Riemannian manifolds to statistical manifolds under a few conditions.

In 2021, Deshmukh and Belova [10] studied Killing vector fields, the infiltration of unit Killing vector field on Riemannian manifolds, to show the necessary and sufficient conditions for a hypersurface M to be isometric to sphere.

In 1986, Kimura [6] showed that M is a homogeneous hypersurface if and only if M has constant principal curvatures and he also obtained a characterization of certain complex submanifolds in a complex projective space CP^n. Then Niebergall and Ryan [9] studied real hypersurfaces in complex space forms and discussed about the number of principal curvatures in these hypersurfaces. In 2006, Berndt and Diaz-Ramos [5] classified all of the real hypersurfaces in the complex hyperbolic space with three distinct constant principal curvatures. Then Chen and Maeda [1] collected all these spaces in Theorems A and B, when they are Hopf hypersurfaces by constant principal curvatures, and stated that these hypersurfaces are tubes of radius r over a hyperplane CP^{n-1} where $0 < r < \frac{\pi}{2}$, totally geodesic CP^k ($1 \leq k \leq n-2$) where $0 < r < \frac{\pi}{2}$, complex hyperquadric CQ^{n-1} where $0 < r < \frac{\pi}{4}$, $CP^1 \times CP^{\frac{n-1}{2}}$ where $0 < r < \frac{\pi}{4}$ and $n \geq 5$ is odd, complex Grassmann $CG_{2,5}$ where $0 < r < \frac{\pi}{4}$ and $n = 9$, Hermitian symmetric space $SO(10)/U5$ where $0 < r < \frac{\pi}{4}$ and $n = 15$, in the complex projective space. Also they are tubes of radius r over CH^k ($k = 0, n-1$) where $0 < r < \infty$, CH^k ($1 \leq k \leq n-2$) where $0 < r < \infty$, RH^n where $0 < r < \infty$ and a horosphere in CH^n in the complex hyperbolic space.

In recent years much more attention is paid to statistical structure not only by applied mathematicians but also in other sciences such as physics and computer science, because of interdisciplinary research.

In this paper, we introduce statistical Killing vector fields and we study the statistical contact Hopf hypersurfaces in the complex space forms, such that the structural vector field ξ is a statistical Killing vector field. We show that these types of hypersurfaces have three distinct constant principal curvatures in the non-flat complex space forms which are classified in references [5, 6] and [9]. Then, in a particular case, we discuss about these hypersurfaces in the Euclidean space.

2. Preliminaries

Prior to presenting the main result, we recall and introduce some definitions and lemmas which are used intensively in our study.

Let (M, g) be an n-dimensional Riemannian manifold, ∇ be a symmetric connection on M and let $\Gamma(TM)$ denote the sections of tangent bundle. The torsion tensor of ∇ is defined by

$$T^{\nabla}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]\,,$$

for any $X, Y \in \Gamma(TM)$.

A pair (∇, g) is called a *statistical structure* on M if

$$(\nabla_X g)(Y,Z) = (\nabla_Y g)(X,Z)$$

holds for $X, Y, Z \in \Gamma(TM)$. Then (M, ∇, g) is called *a statistical manifold.*

The dual connection ∇^* of ∇ is defined by

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z),$$

for any $X, Y, Z \in \Gamma(TM)$. It is not difficult to see that if (∇, g) is a statistical structure on M, so (∇^*, g) is as well. Let K be a tensor field of type $(1,2)$. We define

$$K_X Y = \nabla_X Y - \nabla^g_X Y, \tag{1}$$

where K_X is a self-adjoint operator, that is $g(K_X Y, Z) = g(Y, K_X Z)$ and ∇^g is the Levi-Civita connection in which

$$\nabla^g_X Y = \frac{1}{2}(\nabla_X Y + \nabla^*_X Y).$$

Let $(\bar{M}, \bar{g}, J)$ be a complex manifold with the complex structure J, where $J^2 = -1$ and the Hermitian metric $\bar{g}$. Let (M, g) be a Riemannian submanifold of $\bar{M}$, that is g induces a metric on M. In addition, $H_x M = JT_x M \cap T_x M$ is the holomorphic tangent space of M at $x \in M$.

The submanifold M of $\bar{M}$ is called a *CR-submanifold* if $H_x M$ has constant dimension for any $x \in M$, and the constant complex dimension is called the *CR-dimension* of M.

The complex manifold (M, g) with Hermitian metric is a *Kähler manifold* when $\nabla^g_X J = 0$ for any $X \in T(M)$.

From [2] we recall that the *Riemannian curvature tensor* for any $X, Y, Z \in \Gamma(T\bar{M})$ is defined by

$$\bar{R}^g(X,Y)Z = \bar{\nabla}^g_X \bar{\nabla}^g_Y Z - \bar{\nabla}^g_Y \bar{\nabla}^g_X Z - \bar{\nabla}^g_{[X,Y]} Z. \tag{2}$$

Let $\bar{R}$ be a curvature tensor. The *sectional curvature* of $(\bar{M}, \bar{g})$ for $\Pi = span\{X, Y\}$ is defined by

$$k(X,Y) = \frac{\bar{g}(\bar{R}(X,Y)Y,X)}{\bar{g}(X,X)\bar{g}(Y,Y) - \bar{g}(X,Y)^2},$$

for any $X, Y \in \Gamma(T\bar{M})$.

A Kähler manifold $\bar{M}$ is called a *complex space form*, when, for any $X \in \Gamma(T\bar{M})$, the holomorphic sectional curvature plans spanned by $\{X, JX\}$ are constant.

Let (M^{2n+1}, g) be a Riemannian hypersurface of the complex space form $(\bar{M}^{2n+2}(k), \bar{g}, J)$, ∇^g be the Levi-Civita connection of M^{2n+1}, and ∇ be a statistical connection; that is (M^{2n+1}, ∇, g) is a statistical hypersurface of $\bar{M}^{2n+2}(k)$. A quadruple (g, φ, ξ, η) is called an *almost contact metric structure* on M^{2n+1} if $\xi = -JN$ is the structural vector field, N is a local unit normal vector field on the hypersurface, η is a 1-form on M^{2n+1}, where $\eta(X) = g(X, \xi)$ and φ is a skew-symmetric tensor field, where $(JX)^{\top} = \varphi X$. We clearly have

$$\varphi^2 X = -X + \eta(X)\xi,$$

so $(M^{2n+1}, g, \varphi, \xi, \eta)$ is called an *almost contact manifold.*

The hypersurface M^{2n+1} is called a *Hopf hypersurface* if the structural vector field ξ is the eigenvector of A; that means, $A\xi = \alpha\xi$, where A is the shape operator of M^{2n+1} and $\alpha \in C^{\infty}(M^{2n+1})$.

A vector field U on a Riemannian manifold (M, g) is called a *Killing vector field* if we have $L_U g = 0$, where $L_U g$ is the Lie-derivative of the metric g with respect to U. We have

$$\begin{aligned} 0 &= (L_U g)(X,Y) \\ &= Ug(X,Y) - g(L_U X, Y) - g(X, L_U Y) \\ &= g(\nabla^g_X U, Y) + g(X, \nabla^g_Y U). \end{aligned} \tag{3}$$

Let a skew-symmetric tensor field Ψ of type $(1,1)$ be defined by

$$\Psi(X,Y) = g(\nabla^g_X U, Y) - g(X, \nabla^g_Y U).$$

Clearly, equation (3) is equivalent to

$$2g(\nabla^g_X U, Y) = 2g(\psi X, Y),$$

where

$$\Psi(X,Y) = 2g(\psi X, Y).$$

Therefore, the existence of a Killing vector field on a manifold is equivalent to

$$\nabla^g_X U = \psi X.$$

Definition 1. Let (M^{2n+1}, ∇, g) be a statistical manifold. Then the vector field U on M^{2n+1} is called a *statistical Killing vector field* when $\nabla_X U = \psi X$ for any $X \in M^{2n+1}$.

Clearly we have

$$\nabla^g_X U = \psi X - K_U X, \tag{4}$$

where $X \in T_p(M^{2n+1})$ and K is defined in (1).

We have the following Gauss and Wingarten formulas:

$$\bar{\nabla}^g_X Y = \nabla^g_X Y + h(X, Y), \tag{5}$$

$$\bar{\nabla}^g_X N = -AX + D_X N, \tag{6}$$

where D is a normal connection and h is the second fundamental form of M^{2n+1}. Furthermore, we have $h(X, Y) = g(AX, Y)N$.

By taking the covariant derivative from both sides of $JX = \varphi X + \eta(X)N$ and as this structure is Kählerian, we obtain

$$\nabla^g_X \xi = \varphi AX. \tag{7}$$

Based on these relations, we have

$$\begin{aligned} \bar{\nabla}^g_X(JY) = J\bar{\nabla}^g_X Y &= J(\nabla^g_X Y + h(X, Y)) \\ &= \varphi(\nabla^g_X Y) + \eta(\nabla^g_X Y)N + J(h(X, Y)) \\ &= \varphi(\nabla^g_X Y) + \eta(\nabla^g_X Y)N - g(AX, Y)\xi. \end{aligned} \tag{8}$$

On the other hand

$$\begin{aligned} \bar{\nabla}^g_X(JY) &= \bar{\nabla}^g_X(\varphi Y + \eta(Y)N) \\ &= \bar{\nabla}^g_X(\varphi Y) + \bar{g}(\bar{\nabla}^g_X \xi, Y)N + \bar{g}(\xi, \bar{\nabla}^g_X Y)N + \eta(Y)\bar{\nabla}^g_X N \\ &= (\nabla)^g_X(\varphi Y) + h(X, \varphi Y) + \bar{g}(\bar{\nabla}^g_X \xi, Y)N + \bar{g}(\xi, \bar{\nabla}^g_X Y)N \\ &\quad + \eta(Y)(-AX + D_X N). \end{aligned} \tag{9}$$

By comparing the tangential parts of (8) and (9) we have

$$\begin{aligned} \varphi(\nabla^g_X Y) - g(AX, Y)\xi &= \nabla^g_X(\varphi Y) - \eta(Y)AX \\ &= (\nabla^g_X \varphi)Y + \varphi(\nabla^g_X Y) - \eta(Y)AX. \end{aligned}$$

Therefore,

$$(\nabla^g_X \varphi)Y = \eta(Y)AX - g(AY, X)\xi. \tag{10}$$

If M^{2n+1} is a hypersurface from the complex space form $\bar{M}^{2n+2}(k)$, the following (well-known) Gauss equation is [2]

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)Z, W) &= g(R(X, Y)Z, W) - \{g(AY, Z)g(AX, W) \\ &\quad - g(AX, Z)g(AY, W)\}, \end{aligned}$$

where

$$\begin{aligned} \bar{R}(X, Y)Z &= k\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y + \bar{g}(JY, Z)JX - \bar{g}(JX, Z)JY \\ &\quad - 2\bar{g}(JX, Y)JZ\}, \end{aligned}$$

and k is a constant sectional curvature for $X, Y, Z \in \Gamma(TM^{2n+1})$. We obtain the Codazzi equation

$$\begin{aligned} g((\nabla^g_X A)Y - \nabla^g_Y A)X, Z) &= kg(\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \bar{g}(JY,Z)JX \\ &\quad -\bar{g}(JX,Z)JY - 2\bar{g}(JX,Y)JZ\}, N), \quad (11) \end{aligned}$$

where N is a local unit normal vector field in M^{2n+1}.

At the end of this section, we bring a lemma that is used to obtain the important results.

Lemma 1 ([2]). *If M^{2n+1} is a Hopf hypersurface from the complex space form $\bar{M}^{2n+2}(k)$ and if $X \in \xi^{\perp}$ is an eigenvector of the shape operator A with eigenvalue λ, then JX is an eigenvector of A with eigenvalue μ and we have $2k + \alpha\lambda = \mu(2\lambda - \alpha)$.*

3. The statistical hypersurfaces of complex space forms

We prove the following lemmas which are significant for the proof of the main theorem.

Lemma 2. *Let (M^{2n+1}, ∇) be a statistical Hopf hypersurface in the complex space form $\bar{M}^{2n+2}(k)$ and ξ be a statistical Killing vector field. Then the curvature tensor*

$$R(X,Y)\xi = R^g(X,Y)\xi + (\nabla_X K)(Y,\xi) - (\nabla_Y K)(X,\xi) - K(\nabla_X Y,\xi) + K(\nabla_Y X,\xi)$$

holds for any $X, Y \in \Gamma(TM)$.

Proof. By directly applying

$$\nabla_X \xi = \nabla^g_X \xi + K_X \xi$$

in (2), we have

$$\begin{aligned} R(X,Y)\xi &= \nabla_X(\nabla^g_Y \xi + K_Y \xi) - \nabla_Y(\nabla^g_X \xi + K_X \xi) - \nabla^g_{[X,Y]}\xi - K_{[X,Y]}\xi \\ &= R^g(X,Y)\xi + (\nabla_X K)(Y,\xi) - (\nabla_Y K)(X,\xi) - K(\nabla_X Y,\xi) \\ &\quad + K(\nabla_Y X,\xi), \end{aligned}$$

which finishes the proof. □

Remark 1. In the above Lemma, $R^g(X,Y)\xi$ is the Riemannian curvature tensor, where

$$R^g(X,Y)\xi = (\nabla^g_X \psi)Y - (\nabla^g_Y \psi)X - (\nabla^g_X K_\xi)Y + (\nabla^g_Y K_\xi)X.$$

Lemma 3. *Let (M^{2n+1}, ∇) be a statistical Hopf hypersurface in the complex space form $\bar{M}^{2n+2}(k)$ and ξ be a statistical Killing vector field. Then $K_\xi \xi = 0$.*

Proof. By applying (4) and (5), if we insert $X = \xi$, then

$$\begin{aligned} 0 &= \nabla_\xi \xi - K_\xi \xi \\ &= \psi\xi - K_\xi \xi, \end{aligned}$$

therefore, $\psi\xi = K_\xi \xi$.

On the other hand, from the equations (4) and (7) we have

$$0 = g(\varphi AX, \xi) = g(\nabla^g_X \xi, \xi) = g(\psi X - K_\xi X, \xi),$$

hence

$$g(\psi X, \xi) = g(K_\xi X, \xi).$$

Since ψ is skew-symmetric, we have $K_\xi \xi = -\psi\xi$. Therefore, $K_\xi \xi = 0$. □

Lemma 4. *Let (M^{2n+1}, ∇) be a statistical Hopf hypersurface in the complex space form $\bar{M}^{2n+2}(k)$. Then we have*

$$trace \nabla_\xi K_\xi = 0.$$

Proof. Let $\{e_i\}_{i=1}^{2n+1}$ be a parallel frame field on M^{2n+1} such that the shape operator A is diagonal on the frame field. Then, from (7) and Lemma 3, we obtain

$$\begin{aligned} trace \nabla_\xi K_\xi &= trace(\nabla^g_\xi K_\xi + K_\xi \xi) = trace \nabla^g_\xi K_\xi \\ &= \sum_{i=1}^{2n+1} g(\nabla^g_\xi K_\xi e_i, e_i) = \sum_{i=1}^{2n+1} g(\nabla^g_\xi(-\varphi A e_i + \psi e_i), e_i) \\ &= \sum_{i=1}^{2n+1} -g((\nabla^g_\xi \varphi A)e_i, e_i) + g((\nabla^g_\xi \psi)e_i, e_i). \end{aligned}$$

Considering that A is symmetric and φ, ψ are skew-symmetric, it is easy to conclude that the above equation is zero. □

Lemma 5. *Let (M^{2n+1}, ∇) be a statistical Hopf hypersurface in the complex space form $\bar{M}^{2n+2}(k)$. Then*

$$|\psi|^2 = |A|^2 - \alpha^2 - |K_\xi|^2.$$

Proof. Let $\{e_i\}_{i=1}^{2n+1}$ be a parallel frame field on M^{2n+1}, then we have

$$\begin{aligned} \sum_{i=1}^{2n+1} g(\varphi A e_i, \varphi A e_i) &= \sum_{i=1}^{2n+1} -g(\varphi^2 A e_i, A e_i) \\ &= \sum_{i=1}^{2n+1} -g(-Ae_i + \eta(Ae_i)\xi, Ae_i) \\ &= |A|^2 - |A_\xi|^2 \\ &= |A|^2 - \alpha^2. \end{aligned}$$

On the other hand,

$$\sum_{i=1}^{2n+1} g(\varphi Ae_i, \varphi Ae_i) = \sum_{i=1}^{2n+1} g(\psi e_i - K_\xi e_i, \psi e_i - K_\xi e_i) = |\psi|^2 + |K_\xi|^2.$$

Therefore, by comparing, we have

$$|\psi|^2 = |A|^2 - \alpha^2 - |K_\xi|^2.$$

□

Lemma 6. *Let* (M^{2n+1}, ∇) *be a statistical Hopf hypersurface from* C^{n+1} *and* ξ *be a structural vector field. Then we have the equation*

$$Ric(\xi, \xi) = \sum_{i=1}^{n} \alpha(\lambda_i + \mu_i).$$

Proof. Let $\{e_i, \varphi e_i, \xi\}_{i=1}^{n}$ be an adopted frame field on M^{2n+1} in which $e_1, \ldots, e_n$ are principal directions, also the shape operator A on M^{2n+1} satisfies $Ae_i = \lambda_i e_i$, $A\varphi e_i = \mu_i \varphi e_i$ and $A\xi = \alpha\xi$. If the ambient manifold is C^{n+1}, from the Codazzi formula for $X, Y \in T(M^{2n+1})$ we have $\varphi(\nabla^g_X A)Y - \varphi(\nabla^g_Y A)X = 0$. On the other hand, from the equations (7) and (2), we have

$$R^g(X, Y)\xi = (\nabla^g_X \varphi)AY - (\nabla^g_Y \varphi)AX + \varphi(\nabla^g_X A)Y - \varphi(\nabla^g_Y A)X. \quad (12)$$

Then, by taking into account the equations (10) and (12), the Ricci tensor is given by

$$\begin{aligned} Ric(\xi, \xi) &= \sum_{i=1}^{n} g(R^g(e_i, \xi)\xi, e_i) \\ &= \sum_{i=1}^{n} g((\nabla_{e_i}\varphi)A\xi - (\nabla_{e_i}K)A\xi, e_i) - g((\nabla_\xi \varphi)Ae_i - (\nabla_\xi K)Ae_i, e_i) \\ &\quad + g(\varphi((\nabla_{e_i}A)\xi - K_{e_i}A\xi), e_i) - g(\varphi((\nabla_\xi A)e_i - K_\xi Ae_i), e_i) \\ &= \sum_{i=1}^{n} \eta(A\xi)g(Ae_i, e_i) - \eta(Ae_i)g(A\xi, e_i) \\ &= \sum_{i=1}^{n} \alpha(\lambda_i + \mu_i). \end{aligned}$$

□

Proposition 1. *Let* (M^{2n+1}, ∇) *be a statistical Hopf hypersurface in the complex space form* $\bar{M}^{2n+2}(k)$*. Let the structural vector field* ξ *be a statistical*

Killing vector field on M^{2n+1}, and $X \in \xi^{\perp}$ be an eigenvector of the shape operator A, that is, $AX = \lambda X$. Then $X\alpha = \xi\lambda = 0$.

Proof. From(11) we obtain

$$\nabla_X(AY)–A\nabla_X Y–\nabla_Y(AX)+A\nabla_Y X = k\{g(X,\xi)\varphi Y–g(Y,\xi)\varphi X–2g(\varphi X,Y)\xi\}.$$

If we insert $Y = \xi$ in the above equation, it yields

$$\nabla_X(A\xi) - A\nabla_X\xi + \nabla_\xi(AX) - A\nabla_\xi X = -k\varphi X. \tag{13}$$

If we multiply both sides of (13) in X, we have

$$g(\nabla_X(A\xi) - A\nabla_X\xi + \nabla_\xi(AX) - A\nabla_\xi X, X) = 0,$$

thus

$$g((X\alpha)\xi + \alpha\varphi AX, X) - g((\xi\lambda)X + \lambda(\nabla_\xi X - K_\xi X) - A(\nabla_\xi X - K_\xi X), X) = 0,$$

and therefore $\xi\lambda = 0$. Similarly, by multiplying both sides of (13) in ξ, we have $X\alpha = 0$. □

Theorem 1. *Let (M^{2n+1}, ∇) be a statistical Hopf hypersurface in the complex space form $\bar{M}^{2n+2}(k)$, where the structural vector field ξ is a statistical Killing vector field. Then we have*

$$2|K_\xi|^2 \geq -(2n+1)k. \tag{14}$$

Proof. Let $\{e_i, \varphi e_i, \xi\}_{i=1}^n$ be an adopted frame field on M^{2n+1}. Now applying Lemma 2, Lemma 4 and Lemma 5, the Ricci tensor is computed as follows:

$$\begin{aligned} Ric(\xi,\xi) &= \sum_{i=1}^n g(R^g(e_i,\xi)\xi, e_i) \\ &= \sum_{i=1}^n \{g((\nabla^g_{e_i}\psi)\xi, e_i) - g((\nabla^g_\xi\psi)e_i, e_i) - g((\nabla^g_{e_i}K_\xi)\xi, e_i) \\ &\quad + g((\nabla^g_\xi K_\xi)e_i, e_i)\} \\ &= |\psi|^2 - |K_\xi|^2 \\ &= |A|^2 - \alpha^2 - 2|K_\xi|^2. \end{aligned} \tag{15}$$

Furthermore, from (11) and(12) we obtain the Ricci tensor

$$\begin{aligned} Ric(\xi,\xi) &= \sum_{i=1}^n \{k\{\bar{g}(e_i,e_i)\bar{g}(\xi,\xi) - \bar{g}(e_i,\xi)\bar{g}(\xi,e_i) + \bar{g}(J\xi,\xi)\bar{g}(Je_i,e_i) \\ &\quad -\bar{g}(Je_i,\xi)\bar{g}(J\xi,e_i) - 2\bar{g}(Je_i,\xi)\bar{g}(J\xi,e_i)\} \\ &\quad +\{\alpha g(Ae_i,e_i) - g(Ae_i,\xi)g(A\xi,e_i)\}\} \\ &= (2n+3)k + \sum_{i=1}^n \alpha(\lambda_i + \mu_i) \end{aligned}$$

$$= \quad (2n+1)k + 2\sum_{i=1}^{n} \lambda_i \mu_i. \tag{16}$$

Then, by comparing the coefficients of $Ric(\xi, \xi)$ in (15) and (16), we arrive at

$$|A|^2 - \alpha^2 - 2|K_\xi|^2 = (2n+1)k + 2\sum_{i=1}^{n} \lambda_i \mu_i,$$

and therefore

$$\sum_{i=1}^{n} (\lambda_i - \mu_i)^2 = 2|K_\xi|^2 + (2n+1)k,$$

so $2|K_\xi|^2 \geq -(2n+1)k$. □

Corollary 1. *Let (M^{2n+1}, ∇) be a statistical Hopf hypersurface in the complex space form $\bar{M}^{2n+2}(k)$, where the vector field ξ is a statistical Killing vector field. If the equality holds in (14), then (M^{2n+1}, ∇) has at most three distinct constant principal curvatures.*

Proof. If we have $\lambda_i = \mu_i$, from Lemma 1 we obtain

$$\lambda_i^2 - \alpha\lambda_i - k = 0. \tag{17}$$

Hence we have at most 3 eigenvalues. If we take the derivative from (17) with respect to ξ, we get $-(\xi\alpha)\lambda_i = 0$. Because of $\lambda_i \neq 0$, $\xi\alpha = 0$. From Proposition 1 we deduce that α is a constant. Since α is a constant and the eigenvalues of A span $T_p(M^{2n+1})$, the result is valid. It is worth noting that these hypersurfaces are classified in references [5, 6, 9]. □

Corollary 2. *Let (M^{2n+1}, ∇) be a statistical Hopf hypersurface of C^{n+1}, and ξ be a statistical Killing vector field. If $K(., \xi) = 0$, then M^{2n+1} is either a totally umbilical hypersurface or M^{2n+1} is a totally geodesic hypersurface.*

Proof. If the ambient manifold is C^{n+1} from Theorem 1, we have

$$0 \leq \sum_{i=1}^{n} (\lambda_i - \mu_i)^2 = 2|K_\xi|^2.$$

Then, from [2], if the ambient manifold is a Euclidean space and M^{2n+1} has two distinct constant principal curvatures, then one of them must be zero. From (17), in the Euclidean space we have $\lambda_i(\lambda_i - \alpha) = 0$, where we deduce that $\lambda_i = 0$ or $\lambda_i = \alpha$. Then we have one of the following.

1. When $\lambda_i \neq 0$, then $\lambda_i = \alpha$ for any $i = 1, 2, ..., n$, hence the hypersurface is totally umbilical and isometric to a sphere.
2. When $\lambda_i = 0$ and $\alpha \neq 0$ for some $i = 1, 2, ..., n$, then the hypersurface is isometric to a cylinder.
3. When $\lambda_i = 0$ and $\alpha = 0$ for any $i = 1, 2, ..., n$, then the hypersurface is totally geodesic.

□

Let $P^n(H)$ be a Hyperbolic projective space. Then we have the following corollary.

Corollary 3. *There exists no totally umbilical statistical Hopf hypersurface with the statistical Killing vector field* ξ *in the* $P^n(H)$ *by the sectional curvature* $k < \frac{-2}{2n+1} \mid K_\xi \mid^2$.

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