ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 28, Number 1, June 2024
Available online at https://ojs.utlib.ee/index.php/ACUTM

On h-almost conformal η-Ricci-Bourguignon solitons in a perfect fluid spacetime

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Abstract. The primary object of the paper is to study h-almost conformal η-Ricci-Bourguignon soliton in an almost pseudo-symmetric Lorentzian Kähler spacetime manifold when some different curvature tensors vanish identically. We have also explored the conditions under which an h-almost conformal Ricci-Bourguignon soliton is steady, shrinking or expanding in different perfect fluids such as stiff matter, dust fluid, dark fluid and radiation fluid. We have observed in a perfect fluid spacetime with h-almost conformal η-Ricci-Bourguignon soliton to be a manifold of constant Riemannian curvature under some certain conditions. We have gone on to refine the classification of the potential function with respect to gradient h-almost conformal η-Ricci-Bourguignon soliton in a perfect fluid spacetime with torse-forming vector field ξ. Finally, we have developed an example of h-almost conformal η-Ricci-Bourguignon soliton.

1. Introduction

In 1915, Albert Einstein enunciated in a seminal paper the concept of general relativity which establishes the fundamental relationship between the physics and the geometry of spacetimes. Spacetime symmetries are used in the study of exact solutions of Einstein's field equations of general relativity. A special subcategory of a pseudo Riemannian manifold, called the Lorentzian manifold, plays a pivotal role in the study of general relativity. The spacetime of general relativity and cosmology can be modeled as a connected 4-dimensional time-oriented Lorentzian manifold with the Lorentzian metric g having a signature $(-,+,+,+)$. Lorentzian manifolds are called

Received October 28, 2023.
2020 *Mathematics Subject Classification.* 53C20, 53C21, 53C50, 53E20.
Key words and phrases. Ricci-Bourguignon soliton, torse-forming vector field, perfect fluid spacetime, almost pseudo-symmetric manifold.
https://doi.org/10.12697/ACUTM.2024.28.06

perfect fluid spacetimes if Ricci tensor is of the form

$$S = ag + b\eta \otimes \eta, \tag{1}$$

where a, b are scalar functions, η is a 1-form and $g(X, \xi) = \eta(X)$ for all X, and $g(\xi, \xi) = -1$.

A perfect fluid having no shear, stress, viscosity or heat conduction, can be completely identified by its rest-frame mass density and isotropic pressure and it is also characterized by an energy-momentum tensor T of type $(0, 2)$ which is of the form [16]:

$$T(X, Y) = \rho g(X, Y) + (\sigma + \rho)\eta(X)\eta(Y), \tag{2}$$

where σ is the energy density, ρ is the isotropic pressure, $\eta(X) = g(X, \xi)$ is a 1-form such that $g(\xi, \xi) = -1$. If $\rho = 0$, the perfect fluid spacetime will be a dust matter fluid [8]. The perfect fluid represents radiation fluid if $\sigma = 3p$ [8]; if $\rho = -\sigma$, then perfect fluid is known as dark energy era [8]. If $\rho = \sigma$, then a perfect fluid is referred to as stiff matter. In [23], Zeldovich introduced the equation of state of stiff matter fluid to explain a cold gas of baryons and applied it to his cosmological model.

The Einstein's field equations upon adding a cosmological constant governing the perfect fluid motion [16] can be defined as

$$\kappa T(X, Y) = S(X, Y) + (\alpha - \frac{r}{2})g(X, Y), \tag{3}$$

for any $X, Y \in \chi(M)$, where r is the scalar curvature of g, α is the cosmological constant, κ is the gravitational constant with $\kappa \approx 8\pi G$, G is the universal gravitational constant.

Using the equations (2) and (3), we have

$$S(X, Y) = (\kappa p - \alpha + \frac{r}{2})g(X, Y) + \kappa(\sigma + p)\eta(X)\eta(Y). \tag{4}$$

Let (M^4, g) be a perfect fluid spacetime satisfying (4). After contracting (4) with $g(\xi, \xi) = -1$, we derive

$$r = 4\alpha + \kappa(\sigma - 3\rho). \tag{5}$$

A non-flat $n(> 2)$-dimensional Riemannian manifold is said to be an almost pseudo-symmetric manifold [9] if its curvature tensor R satisfies the condition

$$\begin{aligned}(\nabla_V R)(X, Y, Z, W) \;=\; & [A(V) + B(V)]R(X, Y, Z, W) + A(X)R(V, Y, Z, W) \\ & + A(Y)R(X, V, Z, W) + A(Z)R(X, Y, V, W) \\ & + A(W)R(X, Y, Z, V),\end{aligned} \tag{6}$$

where A, B are two nonzero 1-forms defined by $g(V, \varsigma) = A(V)$, $B(V) = g(V, \theta)$, where $X, Y, Z, V, W \in \chi(M)$. Here ς and θ are called the associated vector fields corresponding to the 1-forms A and B respectively.

The concept of a soliton flows from the idea of a 'solitary wave', which manifests itself when the dispersive and nonlinear effects of the medium of translation cancel each other out. In mathematical terms, therefore, a soliton is a particular solution of a set of nonlinear partial differential equations that represent a system of superposed waves in a particular medium. In that sense, a Ricci flow is a soliton for a Riemannian metric. The concept of a Ricci flow was formulated in the early 1980s by R. Hamilton, who was motivated by Eells and Sampson's work on harmonic map heat flow [12, 14]. Ricci flow is an evolution equation on a smooth manifold M with a Riemannian metric $g(t)$ defined as

$$\frac{\partial}{\partial t}g(t) = -2S.$$

Ricci soliton, which is a natural generalization of an Einstein manifold, is defined on a semi-Riemannian manifold (M, g) by

$$S + \frac{1}{2}\pounds_Y g = \mu g,$$

where $\pounds_Y$ is the Lie derivative along the vector field Y, S is the Ricci tensor of (M, g) and μ is a real constant. If $Y = \nabla f$ for some function f on M, the Ricci soliton transforms into a gradient Ricci soliton. A soliton becomes shrinking, steady or expanding when $\mu > 0$, $\mu = 0$ or $\mu < 0$, respectively.

Basu and Bhattacharyya [3] constructed the notion of *conformal Ricci soliton*, defined as

$$\pounds_V g + 2S + [2\mu - (p + \frac{2}{n})]g = 0, \tag{7}$$

where $\pounds_V$ is the Lie derivative along the vector field V, p is a scalar non-dynamical field (time dependent scalar field), μ is constant and n is the dimension of the manifold.

In 1979, the idea of the Ricci-Bourguignon flow (or RB flow) as a generalization of Ricci flow was developed by Jean-Pierre Bourguignon [4] using some unpublished work of Lichnerowicz and a paper of Aubin [1]. The Ricci-Bourguignon flow is an evolution equation for metrics on a Riemannian manifold given by

$$\frac{\partial}{\partial t}g(t) = -2(S - r\Lambda g), \tag{8}$$

where $\Lambda \in \mathbb{R}$ is a constant and r is the scalar curvature of the Riemannian metric g. It should be observed that the right hand side of the evolution equation (8) is of special interest for special values of Λ [11], in particular:
1) $\Lambda = \frac{1}{2}$, the Einstein tensor $S - \frac{r}{2}g$ (Einstein soliton),
2) $\Lambda = \frac{1}{n}$, the traceless Ricci tensor $S - \frac{r}{n}g$,
3) $\Lambda = \frac{1}{2(n-1)}$, the Schouten tensor $S - \frac{r}{2(n-1)}g$ (Schouten soliton),
4) $\Lambda = 0$, the Einstein tensor S (Ricci soliton).

Dwivedi [11] introduced the concept of Ricci-Bourguignon solitons which generalize Ricci solitons. In the paper, the author explained integral formulas for compact gradient Ricci-Bourguignon solitons and compact gradient Ricci-Bourguignon almost solitons.

A Riemannian manifold (M, g) is called a *Ricci-Bourguignon soliton* (or RB soliton) if there exists a smooth vector field V satisfying the equation

$$S + \frac{1}{2}\pounds_V g = (\mu + r\Lambda)g \tag{9}$$

for some real constant μ and the Lie derivative $\pounds_V g$. Ricci-Bourguignon soliton appears as a self-similar solution to Ricci-Bourguignon flow and often arises as a limit of dilation of singularities in the Ricci-Bourguignon flow [5]. The Ricci-Bourguignon soliton is said to be shrinking, steady or expanding if μ is positive, zero or negative, respectively.

If the vector field V is the gradient of a smooth function f, then g is called a *gradient Ricci-Bourguignon soliton* and the equation (9) becomes

$$\nabla\nabla f + S = (\mu + r\Lambda)g. \tag{10}$$

Proceeding from the identities (7) and (9) above, we will now introduce new entities: (a) h-almost conformal Ricci-Bourguignon soliton which generalizes both conformal soliton and Ricci-Bourguignon soliton, and (b) h-almost conformal η-Ricci-Bourguignon soliton which generalizes both conformal soliton and η-Ricci-Bourguignon soliton.

An n-dimensional complete Riemannian or pseudo-Riemannian manifold (M, g) is said to be an *h-almost conformal Ricci-Bourguignon soliton*, denoted by (M^n, g, h, V, μ), if there exists a smooth vector field V satisfying the equation

$$S + \frac{h}{2}\pounds_V g = (\mu - \frac{1}{2}(p + \frac{2}{n}) + r\Lambda)g \tag{11}$$

for some smooth functions h and μ and the Lie derivative $\pounds_V g$. The h-almost conformal Ricci-Bourguignon soliton is said to be shrinking, steady or expanding if μ is positive, zero or negative, respectively.

If the vector field V is the gradient of a smooth function f, then the soliton equation becomes

$$h\nabla\nabla f + S = (\mu - \frac{1}{2}(p + \frac{2}{n}) + r\Lambda)g, \tag{12}$$

and the soliton is called an *h-almost gradient conformal Ricci-Bourguignon soliton*.

An n-dimensional complete Riemannian or pseudo-Riemannian manifold (M, g) is said to be an *h-almost conformal η-Ricci-Bourguignon soliton*, denoted by $(M^n, g, h, \xi, \mu, \beta)$, if there exists a smooth vector field V satisfying the equation

$$S + \frac{h}{2}\pounds_V g = (\mu - \frac{1}{2}(p + \frac{2}{n}) + r\Lambda)g + \beta\eta \otimes \eta, \tag{13}$$

where h and μ are smooth functions, β is a real constant and η is a 1-form. The h-almost conformal Ricci-Bourguignon soliton is said to be shrinking, steady or expanding if μ is positive, zero or negative, respectively.

If we consider the soliton vector field as a gradient of a smooth function f, then the soliton equation becomes

$$h\nabla\nabla f + S = (\mu - \frac{1}{2}(p + \frac{2}{n}) + r\Lambda)g + \beta\eta \otimes \eta, \tag{14}$$

and the soliton is called a *gradient h-almost conformal η-Ricci-Bourguignon soliton.*

In 2020, Siddiqi and Siddiqui [20] investigated the geometrical features of a perfect fluid spacetime in terms of conformal Ricci soliton and conformal η-Ricci soliton with torse-forming vector field ξ. Praveena et al. [19] published their study on solitons of almost pseudo symmetric Kählerian space-time manifold. In the paper they showed that solitons are steady, expanding or shrinking under different relations of isotropic pressure, the cosmological constant, energy density and gravitational constant. In 2022, Azami [2] constructed a noncompact, complete nontrivial gradient h-Ricci-Bourguignon soliton isometric to Euclidean space and also showed that a compact nontrivial h-almost Ricci-Bourguignon soliton is isometric to a Euclidean sphere under some certain conditions. Chaturvedi et al. [6] explored η-Ricci-Yamabe solitons in a Bochner flat Lorentzian Kähler space-time manifolds. Dey and Roy [10] have provided the ideas of some characterization of general relativistic spacetime with an η-Ricci-Bourguignon soliton. Chaubey and Suh [7] have explored the properties of Fischer-Marsden conjecture and Ricci-Bourguignon solitons within the framework of generalized Sasakian-space-forms with β-Kenmotsu structure.

Motivated by the above outcomes and explorations, we discover in this paper the properties of perfect fluid spacetime if the Lorentzian metrics are h-almost conformal η-Ricci-Bourguignon soliton and gradient h-almost conformal η-Ricci-Bourguignon soliton.

The sections of this paper are ordered as follows: in Section 2, basic properties of a Lorentzian Kähler spacetime manifold are given. The next section investigates cases where an h-almost conformal Ricci-Bourguignon soliton on an almost pseudo symmetric Lorentzian Kähler spacetime manifold is steady, shrinking or expanding; in different perfect fluids like stiff matter, dust fluid, dark fluid and radiation fluid when the spacetime is quasi-conformally flat, conharmonically flat, pseudo-projectively flat and W_2-flat. In Section 4, we find that a perfect fluid spacetime with an h-almost conformal η-Ricci-Bourguignon soliton is a manifold of constant Riemannian curvature under the property $Q \cdot P = 0$ when ξ is a torse-forming vector field. Also, we evolve the classification of the potential function of a gradient h-almost conformal η-Ricci-Bourguignon soliton in a perfect fluid spacetime with torse-forming vector field ξ. In the last section, we construct an example of an h-almost conformal η-Ricci-Bourguignon soliton.

2. Basic properties of Lorentzian Kähler spacetime manifolds

An n-dimensional pseudo-Riemannian manifold (M, g) endowed with a Lorentzian metric g is said to be a *Lorentzian Kähler manifold* if the following conditions hold:

$$J^2 Z = -Z, g(JZ, JY) = g(Z, Y), \tag{15}$$

$$(\nabla_Z J)(Y) = 0, g(JX, Y) = -g(X, JY), \tag{16}$$

where J is a $(1, 1)$ tensor. We know that in a Kähler manifold the Riemannian curvature tensor R and the Ricci tensor S fulfils the conditions:

$$R(JX, JY, Z, W) = R(X, Y, Z, W), \tag{17}$$

$$S(JX, JY) = S(X, Y), S(JX, Y) = -S(X, JY). \tag{18}$$

Let us assume $\{e_i\}_{1 \leq i \leq 4}$ to be an orthonormal frame field, that is $g(e_i, e_j) = \epsilon_{ij}\delta_{ij}, i, j \in 1, 2, 3, 4,$ with $\epsilon_{11} = -1, \epsilon_{ii} = -1, i \in (2, 3, 4), \epsilon_{ij} = 0, i, j \in 1, 2, 3, 4, i \neq j$. Let $\xi = \sum_{i=1}^{n} \xi^i e_i$, then we can write

$$-1 = g(\xi, \xi) = \sum_{1 \leq i,j \leq 4} \xi^i \xi^j g(e_i, e_j) = \sum_{i=1}^{4} \epsilon_{ii} {\xi^i}^2 \tag{19}$$

and

$$\eta(e_i) = g(e_i, \xi) = \sum_{j=1}^{4} \xi^i g(e_i, e_j) = \epsilon_{ii}\xi^i. \tag{20}$$

3. Almost pseudo symmetric Lorentzian Kähler spacetime manifolds

In this section, we will investigate h-almost conformal η-Ricci-Bourguignon soliton of almost pseudo symmetric and some different curvature tensor in a 4-dimensional Lorentzian Kähler spacetime manifold (M^4, g) whose timelike velocity vector field is ξ. Now taking the place of the potential vector field $V = \xi$, the equation (13) becomes

$$S(X, Y) + \frac{h}{2}\pounds_\xi g(X, Y) = (\mu - \frac{1}{2}(p + \frac{2}{n}) + r\Lambda)g(X, Y) + \beta\eta(X)\eta(Y). \tag{21}$$

The following definitions will be useful to prove the main results in this section.

Definition 3.1 ([22])**.** A 4-dimensional Riemannian manifold (M, g) is said to be *quasi-conformally flat* if its quasi-conformal curvature tensor

$$\begin{aligned} C_\star(X,Y)Z &= a_0 R(X,Y)Z + a_1[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad -g(X,Z)QY] - \frac{1}{4}(\frac{a_0}{3} + 2a_1)r[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{22}$$

for all vector fields X, Y, Z and with a_0, a_1 being constants, vanishes identically.

Definition 3.2 ([15])**.** A 4-dimensional Riemannian manifold (M, g) is said to be *conharmonically flat* if its conharmonic curvature tensor

$$\begin{aligned} L(X,Y)Z &= R(X,Y)Z + \frac{1}{2}[S(X,Z)Y - S(Y,Z)X \\ &\quad +g(X,Z)QY - g(Y,Z)QX], \end{aligned} \tag{23}$$

for all vector fields X, Y, Z, vanishes identically.

Definition 3.3 ([18])**.** A 4-dimensional Riemannian manifold (M, g) is said to be *pseudo-projectively flat* if its pseudo-projective curvature tensor

$$\begin{aligned} P_\star(X,Y)Z &= a_0 R(X,Y)Z + a_1[S(Y,Z)X - S(X,Z)Y] \\ &\quad -\frac{r}{4}(\frac{a_0}{3} + a_1)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{24}$$

for all vector fields X, Y, Z and with a_0, a_1 being constants, vanishes identically.

Definition 3.4 ([17])**.** A 4-dimensional Riemannian manifold (M, g) is said to be W_2*-flat* if its W_2 curvature tensor

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{3}[g(X,Z)QY - g(Y,Z)QX], \tag{25}$$

for all vector fields X, Y, Z, vanishes identically.

Now we prove our main theorems.

Theorem 3.5. *In an almost pseudo quasi-conformally flat symmetric Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton* $(g, \xi, h, \mu, \Lambda)$ *is*
shrinking, if $p > \frac{1}{2}[(\kappa(3\rho - \sigma) - 4\alpha)(\frac{a_0}{2a_1} - 4\Lambda + 1) - 1]$,
steady, if $p = \frac{1}{2}[(\kappa(3\rho - \sigma) - 4\alpha)(\frac{a_0}{2a_1} - 4\Lambda + 1) - 1]$,
expanding, if $p < \frac{1}{2}[(\kappa(3\rho - \sigma) - 4\alpha)(\frac{a_0}{2a_1} - 4\Lambda + 1) - 1]$.

Proof. Taking covariant derivative of the equation (17) we have

$$(\nabla_V R)(X,Y,Z,W) = (\nabla_V R)(JX,JY,Z,W). \tag{26}$$

Using (6) in the equation (26) we obtain

$$\begin{aligned} &A(X)R(V,Y,Z,W)+A(Y)R(X,V,Z,W)\\ &\qquad = \quad A(JX)R(V,JY,Z,W)+A(JY)R(JX,V,Z,W). \end{aligned} \tag{27}$$

Using Definition 3.1 of quasi-conformally flat manifold, the equation (27) gives

$$\begin{aligned} &A(X)[-a_1(S(Y,Z)g(V,W)-S(V,Z)g(Y,W)\\ &\quad +g(Y,Z)S(V,W)-g(V,Z)S(Y,W))\\ &\quad +\frac{r}{4}(\frac{a_0}{3}+2a_1)(g(Y,Z)g(V,W)-g(V,Z)g(Y,W))]\\ &\quad +A(Y)[-a_1(S(V,Z)g(X,W)-S(X,Z)g(V,W)\\ &\quad +g(V,Z)S(X,W)-g(X,Z)S(V,W))\\ &\quad +\frac{r}{4}(\frac{a_0}{3}+2a_1)(g(V,Z)g(X,W)-g(X,Z)g(V,W))]\\ &= A(JX)[-a_1(S(JY,Z)g(V,W)-S(V,Z)g(JY,W)\\ &\quad +g(JY,Z)S(V,W)-g(V,Z)S(JY,W))\\ &\quad +\frac{r}{4}(\frac{a_0}{3}+2a_1)(g(JY,Z)g(V,W)-g(V,Z)g(JY,W))]\\ &\quad +A(JY)[-a_1(S(V,Z)g(JX,W)-S(JX,Z)g(V,W)\\ &\quad +g(V,Z)S(JX,W)-g(JX,Z)S(V,W))\\ &\quad +\frac{r}{4}(\frac{a_0}{3}+2a_1)(g(V,Z)g(JX,W)-g(JX,Z)g(V,W))]. \end{aligned} \tag{28}$$

On contracting $X=\varsigma=e_i$, $1\leq i\leq 4$, in the equation (28) we derive

$$\begin{aligned} &-4a_1[S(Y,Z)g(V,W)-S(V,Z)g(Y,W)+g(Y,Z)S(V,W)\\ &-g(V,Z)S(Y,W)]+r(\frac{a_0}{3}+2a_1)[g(Y,Z)g(V,W)-g(V,Z)g(Y,W)]\\ &-2a_1[S(V,Z)g(Y,W)-S(Y,Z)g(V,W)+g(V,Z)S(Y,W)\\ &-g(Y,Z)S(V,W)]+\frac{r}{2}(\frac{a_0}{3}+2a_1)[g(V,Z)g(Y,W)-g(Y,Z)g(V,W)]=0. \end{aligned} \tag{29}$$

Taking $V=W=e_i$, $1\leq i\leq 4$, we acquire

$$S(Y,Z)=\frac{r}{4}(\frac{a_0}{2a_1}+1)g(Y,Z). \tag{30}$$

Using equation (21) in the equation (30), it follows that

$$[\mu-\frac{1}{2}(p+\frac{1}{2})+r\Lambda]g(Y,Z)+\beta\eta(Y)\eta(Z)-\frac{h}{2}(\pounds_\xi g)(Y,Z)=\frac{r}{4}g(Y,Z). \tag{31}$$

Multiplying equation (31) by ϵ_{ii}, putting $Y=Z=e_i$, $1\leq i\leq 4$, and then using equations (19) and (20), we attain

$$4\mu-\beta=r(\frac{a_0}{2a_1}-4\Lambda+1)+2(p+\frac{1}{2})+hdiv\xi. \tag{32}$$

Putting $Y = Z = \xi$ in (32), we get

$$-\mu + \beta = -\frac{r}{4}(\frac{a_0}{2a_1} + 1) + r\Lambda - \frac{1}{2}(p + \frac{1}{2}). \tag{33}$$

Using equations (32) and (33), we have

$$\beta = \frac{h}{3} div\xi \tag{34}$$

and

$$\mu = \frac{h}{3} div\xi + \frac{r}{4}(\frac{a_0}{2a_1} - 4\Lambda + 1) + \frac{1}{2}(p + \frac{1}{2}). \tag{35}$$

Again using equation (5), we obtain

$$\mu = \frac{h}{3} div\xi + (\frac{4\alpha + \kappa(\sigma - 3\rho)}{4})(\frac{a_0}{2a_1} - 4\Lambda + 1) + \frac{1}{2}(p + \frac{1}{2}). \tag{36}$$

Since for the h-almost conformal Ricci-Bourguignon soliton $\beta = 0$, the equation (36) becomes

$$\mu = (\frac{4\alpha + \kappa(\sigma - 3\rho)}{4})(\frac{a_0}{2a_1} - 4\Lambda + 1) + \frac{1}{2}(p + \frac{1}{2}).$$

Hence the theorem is proved. □

Corollary 3.6. *In an almost pseudo quasi-conformally flat symmetric Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton* $(g, \xi, h, \mu, \Lambda)$ *for stiff matter is*
shrinking, if $p > (\kappa\rho - 2\alpha)(\frac{a_0}{2a_1} - 4\Lambda + 1) - \frac{1}{2}$,
steady, if $p = (\kappa\rho - 2\alpha)(\frac{a_0}{2a_1} - 4\Lambda + 1) - \frac{1}{2}$,
expanding, if $p < (\kappa\rho - 2\alpha)(\frac{a_0}{2a_1} - 4\Lambda + 1) - \frac{1}{2}$.

Corollary 3.7. *In an almost pseudo quasi-conformally flat symmetric Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton* $(g, \xi, h, \mu, \Lambda)$ *for the dark fluid is*
shrinking, if $p > 2(\kappa\rho - \alpha)(\frac{a_0}{2a_1} - 4\Lambda + 1) - \frac{1}{2}$,
steady, if $p = 2(\kappa\rho - \alpha)(\frac{a_0}{2a_1} - 4\Lambda + 1) - \frac{1}{2}$,
expanding if $p < 2(\kappa\rho - \alpha)(\frac{a_0}{2a_1} - 4\Lambda + 1) - \frac{1}{2}$.

Corollary 3.8. *In an almost pseudo quasi-conformally flat symmetric Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton* $(g, \xi, h, \mu, \Lambda)$ *for the dust fluid is*
shrinking, if $(p + \frac{1}{2}) + \frac{4\alpha+\sigma\kappa}{2}(\frac{a_0}{2a_1} - 4\Lambda + 1) > 0$,
steady, if $(p + \frac{1}{2}) + \frac{4\alpha+\sigma\kappa}{2}(\frac{a_0}{2a_1} - 4\Lambda + 1) = 0$,
expanding, if $(p + \frac{1}{2}) + \frac{4\alpha+\sigma\kappa}{2}(\frac{a_0}{2a_1} - 4\Lambda + 1) < 0$.

Corollary 3.9. *In an almost pseudo quasi-conformally flat symmetric Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton* $(g, \xi, h, \mu, \Lambda)$ *for the radiation fluid is*
shrinking, if $(p + \frac{1}{2}) + 2\alpha(\frac{a_0}{2a_1} - 4\Lambda + 1) > 0$,
steady, if $(p + \frac{1}{2}) + 2\alpha(\frac{a_0}{2a_1} - 4\Lambda + 1) = 0$,
expanding, if $(p + \frac{1}{2}) + 2\alpha(\frac{a_0}{2a_1} - 4\Lambda + 1) < 0$.

Theorem 3.10. *In an almost pseudo conharmonically flat symmetric Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton* $(g, \xi, h, \mu, \Lambda)$ *is*
shrinking, if $p > (1 + 2\Lambda)(4\alpha + \kappa(\sigma - 3\rho)) - \frac{1}{2}$,
steady, if $p = (1 + 2\Lambda)(4\alpha + \kappa(\sigma - 3\rho)) - \frac{1}{2}$,
expanding, if $p < (1 + 2\Lambda)(4\alpha + \kappa(\sigma - 3\rho)) - \frac{1}{2}$.

Proof. From Definition 3.2 of conharmonically flat manifold, the equation (26) becomes

$$\begin{aligned} A(X)[&-\frac{1}{2}(S(V,Z)g(Y,W) - S(Y,Z)g(V,W) + g(V,Z)S(Y,W) \\ &- g(Y,Z)S(V,W))] + A(Y)[-\frac{1}{2}(S(X,Z)g(V,W) \\ &- S(V,Z)g(X,W) + g(X,Z)S(V,W) - g(V,Z)S(X,W))] \\ = {}& A(JX)[-\frac{1}{2}(S(V,Z)g(JY,W) - S(JY,Z)g(V,W) \\ &+ g(V,Z)S(JY,W) - g(JY,Z)S(V,W))] \\ &+ A(JY)[-\frac{1}{2}(S(JX,Z)g(V,W) - S(V,Z)g(JX,W) \\ &+ g(JX,Z)S(V,W) - g(V,Z)S(JX,W))]. \end{aligned} \tag{37}$$

Putting $X = \varsigma = e_i$, $1 \leq i \leq 4$, we get

$$\begin{aligned} &- 2[(S(V,Z)g(Y,W) - S(Y,Z)g(V,W) + g(V,Z)S(Y,W) \\ &- g(Y,Z)S(V,W))] - [(S(Y,Z)g(V,W) - S(V,Z)g(Y,W) \\ &+ g(Y,Z)S(V,W) - g(V,Z)S(Y,W))] = 0. \end{aligned} \tag{38}$$

On contracting $V = W = e_i$, $1 \leq i \leq 4$, in the equation (38) we obtain

$$S(Y,Z) = -\frac{r}{2}g(Y,Z). \tag{39}$$

Using equation (21) in the equation (39), it can be written as

$$[\mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda]g(Y,Z) + \beta\eta(Y)\eta(Z)$$

$$- \quad \frac{h}{2}(\pounds_{\xi} g)(Y,Z) = -\frac{r}{2} g(Y,Z). \tag{40}$$

Multiplying equation (40) by ϵ_{ii} and putting $Y = Z = e_i$ $1 \leq i \leq 4$, and then using the equations (19) and (20) we get

$$4\mu - \beta = -2r - 4r\Lambda + 2(p + \frac{1}{2}) + hdiv\xi. \tag{41}$$

Putting $Y = Z = \xi$, we obtain

$$-\mu + \beta = r\Lambda + \frac{r}{2} - \frac{1}{2}(p + \frac{1}{2}). \tag{42}$$

Using equations (41) and (42) we have

$$\beta = \frac{h}{3} div\xi$$

and

$$\mu = \frac{h}{3} div\xi - r(\Lambda + \frac{1}{2}) + \frac{1}{2}(p + \frac{1}{2}).$$

Again using equation (5), the above equation can be written as

$$\mu = \frac{h}{3} div\xi - (4\alpha + \kappa(\sigma - 3\rho))(\Lambda + \frac{1}{2}) + \frac{1}{2}(p + \frac{1}{2}). \tag{43}$$

Since for the h-almost conformal Ricci-Bourguignon soliton $\beta = 0$, the equation (43) becomes

$$\mu = -(4\alpha + \kappa(\sigma - 3\rho))(\Lambda + \frac{1}{2}) + \frac{1}{2}(p + \frac{1}{2}). \tag{44}$$

Hence the theorem issues from three different signs of μ.

□

Corollary 3.11. *In an almost pseudo conharmonically flat symmetric Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton* $(g, \xi, h, \mu, \Lambda)$ *for stiff matter is*
shrinking, if $p > 4(\frac{1}{2} + \Lambda)(2\alpha - \kappa\rho) - \frac{1}{2}$,
steady, if $p = 4(\frac{1}{2} + \Lambda)(2\alpha - \kappa\rho) - \frac{1}{2}$,
expanding, if $p < 4(\frac{1}{2} + \Lambda)(2\alpha - \kappa\rho) - \frac{1}{2}$.

Corollary 3.12. *In an almost pseudo conharmonically flat symmetric Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton* $(g, \xi, h, \mu, \Lambda)$ *for the dark fluid is*
shrinking, if $p > 8(\frac{1}{2} + \Lambda)(\alpha - \kappa\rho) - \frac{1}{2}$,
steady, if $p = 8(\frac{1}{2} + \Lambda)(\alpha - \kappa\rho) - \frac{1}{2}$,
expanding, if $p < 8(\frac{1}{2} + \Lambda)(\alpha - \kappa\rho) - \frac{1}{2}$.

Corollary 3.13. *In an almost pseudo conharmonically flat symmetric Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton* $(g, \xi, h, \mu, \Lambda)$ *for the dust fluid is*
shrinking, if $p > 2(\frac{1}{2} + \Lambda)(4\alpha + \kappa\sigma) - \frac{1}{2}$,
steady, if $p = 2(\frac{1}{2} + \Lambda)(4\alpha + \kappa\sigma) - \frac{1}{2}$,
expanding, if $p < 2(\frac{1}{2} + \Lambda)(4\alpha + \kappa\sigma) - \frac{1}{2}$.

Corollary 3.14. *In an almost pseudo conharmonically flat symmetric Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton* $(g, \xi, h, \mu, \Lambda)$ *for the radiation fluid is*
shrinking, if $p > 8\alpha(\frac{1}{2} + \Lambda) - \frac{1}{2}$,
steady, if $p = 8\alpha(\frac{1}{2} + \Lambda) - \frac{1}{2}$,
expanding, if $p < 8\alpha(\frac{1}{2} + \Lambda) - \frac{1}{2}$.

Theorem 3.15. *In an almost pseudo symmetric pseudo-projectively flat Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton* $(g, \xi, h, \mu, \Lambda)$ *is*
shrinking, if $p > (\kappa(3\rho - \sigma) - 4\alpha)(\frac{1}{2}(\frac{a_0}{3a_1} + 1) - 2\Lambda) - \frac{1}{2}$,
steady, if $p = (\kappa(3\rho - \sigma) - 4\alpha)(\frac{1}{2}(\frac{a_0}{3a_1} + 1) - 2\Lambda) - \frac{1}{2}$,
expanding, if $p < (\kappa(3\rho - \sigma) - 4\alpha)(\frac{1}{2}(\frac{a_0}{3a_1} + 1) - 2\Lambda) - \frac{1}{2}$.

Proof. From Definition 3.3 of pseudo-projectively flat manifold, the equation (26) becomes

$$\begin{aligned} A(X)[&-a_1(S(Y,Z)g(V,W)-S(V,Z)g(Y,W)) \\ &+\frac{r}{4}(\frac{a_0}{3}+a_1)(g(Y,Z)g(V,W)-g(V,Z)g(Y,W))] \\ &+A(Y)[-a_1(S(V,Z)g(X,W)-S(X,Z)g(V,W)) \\ &+\frac{r}{4}(\frac{a_0}{3}+a_1)(g(V,Z)g(X,W)-g(X,Z)g(V,W))] \\ =\ & A(JX)[-a_1(S(JY,Z)g(V,W)-S(V,Z)g(JY,W)) \\ &+\frac{r}{4}(\frac{a_0}{3}+a_1)(g(JY,Z)g(V,W)-g(V,Z)g(JY,W))] \\ &+A(JY)[-a_1(S(V,Z)g(JX,W)-S(JX,Z)g(V,W)) \\ &+\frac{r}{4}(\frac{a_0}{3}+a_1)(g(V,Z)g(JX,W)-g(JX,Z)g(V,W))]. \end{aligned} \tag{45}$$

Putting $X=\varsigma=e_i$, $1\leq i\leq 4$, we get

$$\begin{aligned} &- 4a_1[S(Y,Z)g(V,W)-S(V,Z)g(Y,W)] \\ &+ r(\frac{a_0}{3}+a_1)[g(Y,Z)g(V,W)-g(V,Z)g(Y,W)] \\ &- 2a_1[S(V,Z)g(Y,W)-S(Y,Z)g(V,W)] \\ &+ \frac{r}{2}(\frac{a_0}{3}+a_1)[g(V,Z)g(Y,W)-g(Y,Z)g(V,W)]=0. \end{aligned} \tag{46}$$

Taking $V=W=e_i$, $1\leq i\leq 4$, we achieve

$$S(Y,Z)=\frac{r}{4}(\frac{a_0}{3a_1}+1)g(Y,Z). \tag{47}$$

Using equation (21) in the equation (47), we can write

$$\begin{aligned} [\mu &- \frac{1}{2}(p+\frac{1}{2})+r\Lambda]g(Y,Z)+\beta\eta(Y)\eta(Z)-\frac{h}{2}(\pounds_\xi g)(Y,Z) \\ &= \frac{r}{4}(\frac{a_0}{3a_1}+1)g(Y,Z). \end{aligned} \tag{48}$$

Multiplying equation (48) by ϵ_{ii} and putting $Y=Z=e_i$, $1\leq i\leq 4$, and then using equations (19) and (20) we get

$$4\mu-\beta=r(\frac{a_0}{3a_1}-4\Lambda+1)+2(p+\frac{1}{2})+hdiv\xi. \tag{49}$$

Putting $Y=Z=\xi$ we find

$$-\mu+\beta=-\frac{r}{4}(\frac{a_0}{3a_1}+1)+r\Lambda-\frac{1}{2}(p+\frac{1}{2}). \tag{50}$$

Using equations (49) and (50) we get

$$\beta=\frac{h}{3}div\xi,$$

$$\mu = \frac{h}{3}div\xi + \frac{r}{4}(\frac{a_0}{3a_1} - 4\Lambda + 1) + \frac{1}{2}(p + \frac{1}{2}).$$

Again using the equation (5), we can write from the above equation

$$\mu = \frac{h}{3}div\xi + \frac{(4\alpha + \kappa(\sigma - 3\rho))}{4}(\frac{a_0}{3a_1} - 4\Lambda + 1) + \frac{1}{2}(p + \frac{1}{2}). \tag{51}$$

For the h-almost conformal Ricci-Bourguignon soliton $\beta = 0$. Therefore the equation (51) becomes

$$\mu = \frac{(4\alpha + \kappa(\sigma - 3\rho))}{4}(\frac{a_0}{3a_1} - 4\Lambda + 1) + \frac{1}{2}(p + \frac{1}{2}). \tag{52}$$

Hence the theorem follows from three separate signs of μ. □

Corollary 3.16. *In an almost pseudo symmetric pseudo-projectively flat Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton $(g, \xi, h, \mu, \Lambda)$ for stiff matter is*
shrinking, if $p > (\kappa\rho - 2\alpha)((\frac{a_0}{3a_1} + 1) - 4\Lambda) - \frac{1}{2}$,
steady, if $p = (\kappa\rho - 2\alpha)((\frac{a_0}{3a_1} + 1) - 4\Lambda) - \frac{1}{2}$,
expanding, if $p < (\kappa\rho - 2\alpha)((\frac{a_0}{3a_1} + 1) - 4\Lambda) - \frac{1}{2}$.

Corollary 3.17. *In an almost pseudo symmetric pseudo-projectively flat Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton $(g, \xi, h, \mu, \Lambda)$ for the dark fluid is*
shrinking, if $p > (\kappa\rho - 4\alpha)(\frac{1}{2}(\frac{a_0}{3a_1} + 1) - 2\Lambda) - \frac{1}{2}$,
steady, if $p = (\kappa\rho - 4\alpha)(\frac{1}{2}(\frac{a_0}{3a_1} + 1) - 2\Lambda) - \frac{1}{2}$,
expanding, if $p < (\kappa\rho - 4\alpha)(\frac{1}{2}(\frac{a_0}{3a_1} + 1) - 2\Lambda) - \frac{1}{2}$.

Corollary 3.18. *In an almost pseudo symmetric pseudo-projectively flat Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton $(g, \xi, h, \mu, \Lambda)$ for the dust fluid is*
shrinking, if $p + \frac{1}{2} + (2\kappa\sigma + 8\alpha)(\frac{1}{4}(\frac{a_0}{3a_1} + 1) - \Lambda) > 0$,
steady, if $p + \frac{1}{2} + (2\kappa\sigma + 8\alpha)(\frac{1}{4}(\frac{a_0}{3a_1} + 1) - \Lambda) = 0$,
expanding, if $p + \frac{1}{2} + (2\kappa\sigma + 8\alpha)(\frac{1}{4}(\frac{a_0}{3a_1} + 1) - \Lambda) < 0$.

Corollary 3.19. *In an almost pseudo symmetric pseudo-projectively flat Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton $(g, \xi, h, \mu, \Lambda)$ for the radiation fluid is*
shrinking, if $p + \frac{1}{2} + 8\alpha(\frac{1}{4}(\frac{a_0}{3a_1} + 1) - \Lambda) > 0$,
steady, if $p + \frac{1}{2} + 8\alpha(\frac{1}{4}(\frac{a_0}{3a_1} + 1) - \Lambda) = 0$,
expanding, if $p + \frac{1}{2} + 8\alpha(\frac{1}{4}(\frac{a_0}{3a_1} + 1) - \Lambda) < 0$.

Theorem 3.20. *In an almost pseudo W_2-flat symmetric Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton $(g, \xi, h, \mu, \Lambda)$ is*
shrinking, if $p > 2(\kappa(3\rho - \sigma) - 4\alpha)(1 - \Lambda) - \frac{1}{2}$,
steady, if $p = 2(\kappa(3\rho - \sigma) - 4\alpha)(1 - \Lambda) - \frac{1}{2}$,
expanding, if $p < 2(\kappa(3\rho - \sigma) - 4\alpha)(1 - \Lambda) - \frac{1}{2}$.

Proof. From Definition 3.4 of W_2 flat manifold, the equation (26) becomes

$$\begin{aligned} A(X)[-\frac{1}{3}(S(Y,W)g(V,Z) - S(V,W)g(Y,Z)] \\ + A(Y)[-\frac{1}{3}(S(V,W)g(X,Z) - S(X,W)g(V,Z)] \\ = A(JX)[-\frac{1}{3}(S(JY,W)g(V,Z) - S(V,W)g(JY,Z)] \\ + A(JY)[-\frac{1}{3}(S(V,W)g(JX,Z) - S(JX,W)g(V,Z)]. \end{aligned} \tag{53}$$

Putting $X = \varsigma = e_i$, $1 \le i \le 4$, we get

$$\begin{aligned} (-\frac{4}{3})[(S(Y,W)g(V,Z) - S(V,W)g(Y,Z)] \\ -\frac{2}{3}[(S(V,W)g(Y,Z) - S(Y,W)g(V,Z)] = 0. \end{aligned} \tag{54}$$

On contracting $V = W = e_i$, $1 \le i \le 4$, in the equation (54) we obtain

$$S(Y,Z) = rg(Y,Z). \tag{55}$$

Using the equation (21) in the equation (55), it can be written as

$$[\mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda]g(Y,Z) + \beta\eta(Y)\eta(Z) - \frac{h}{2}(\pounds_\xi g)(Y,Z) = rg(Y,Z). \tag{56}$$

Multiplying the equation (56) by ϵ_{ii} and putting $Y = Z = e_i$, $1 \le i \le 4$ and then using the equations (19) and (20) we get

$$4\mu - \beta = 4r - 4r\Lambda + 2(p + \frac{1}{2}) + hdiv\xi. \tag{57}$$

Putting $Y = Z = \xi$ we get

$$-\mu + \beta = r\Lambda - r - \frac{1}{2}(p + \frac{1}{2}). \tag{58}$$

Using the equations (57) and (58) we have

$$\beta = \frac{h}{3}div\xi$$

and

$$\mu = \frac{h}{3}div\xi + r(1 - \Lambda) + \frac{1}{2}(p + \frac{1}{2}).$$

Again using the equation (5), the above equation can be written as

$$\mu = \frac{h}{3} div\xi + (4\alpha + \kappa(\sigma - 3\rho))(1 - \Lambda) + \frac{1}{2}(p + \frac{1}{2}). \tag{59}$$

For the h-almost conformal Ricci-Bourguignon soliton, $\beta = 0$. Therefore the equation (59) becomes

$$\mu = (4\alpha + \kappa(\sigma - 3\rho))(1 - \Lambda) + \frac{1}{2}(p + \frac{1}{2}). \tag{60}$$

Hence the theorem ensues from three various signs of μ. □

Corollary 3.21. *In an almost pseudo W_2-flat symmetric Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton $(g, \xi, h, \mu, \Lambda)$ for stiff matter is*
shrinking, if $p > 4(\kappa\rho - 2\alpha)(1 - \Lambda) - \frac{1}{2}$,
steady, if $p = 4(\kappa\rho - 2\alpha)(1 - \Lambda) - \frac{1}{2}$,
expanding, if $p < 4(\kappa\rho - 2\alpha)(1 - \Lambda) - \frac{1}{2}$.

Corollary 3.22. *In an almost pseudo W_2-flat symmetric Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton $(g, \xi, h, \mu, \Lambda)$ for the dark fluid is*
shrinking, if $p > (\kappa\rho - \alpha)(1 - \Lambda) - \frac{1}{2}$,
steady, if $p = (\kappa\rho - \alpha)(1 - \Lambda) - \frac{1}{2}$,
expanding, if $p < (\kappa\rho - \alpha)(1 - \Lambda) - \frac{1}{2}$.

Corollary 3.23. *In almost pseudo W_2-flat symmetric Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton $(g, \xi, h, \mu, \Lambda)$ for the dust fluid is*
shrinking, if $p + 2(\kappa\sigma + 4\alpha)(1 - \Lambda) + \frac{1}{2} > 0$,
steady, if $p + 2(\kappa\sigma + 4\alpha)(1 - \Lambda) + \frac{1}{2} = 0$,
expanding, if $p + 2(\kappa\sigma + 4\alpha)(1 - \Lambda) + \frac{1}{2} < 0$.

Corollary 3.24. *In almost pseudo W_2-flat symmetric Lorentzian Kähler spacetime manifold admitting Einstein field equation with cosmological constant, the h-almost conformal Ricci-Bourguignon soliton $(g, \xi, h, \mu, \Lambda)$ for the radiation fluid is*
shrinking, if $p + 8\alpha(1 - \Lambda) + \frac{1}{2} > 0$,
steady, if $p + 8\alpha(1 - \Lambda) + \frac{1}{2} = 0$,
expanding, if $p + 8\alpha(1 - \Lambda) + \frac{1}{2} < 0$.

4. The h-almost conformal η-Ricci-Bourguignon soliton and the gradient h-almost conformal η-Ricci-Bourguignon soliton on a perfect fluid spacetime with torse-forming vector field ξ

A vector field ξ on a semi-Riemannian manifold (M^4, g) is called *torse-forming* if it satisfies

$$\nabla_X \xi = X + \eta(X)\xi, \tag{61}$$

for any $X \in \chi(M)$.

From equation (61) it follows that

$$(\pounds_\xi g)(X, Y) = 2[g(X, Y) + \eta(X)\eta(Y)]. \tag{62}$$

Hence from equation (21), we can write

$$S(X, Y) = (\mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda - 2h)g(X, Y) + (\beta - 2h)\eta(X)\eta(Y) \tag{63}$$

and

$$QX = (\mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda - 2h)X + (\beta - 2h)\eta(X)\xi. \tag{64}$$

Lemma 4.1. *In perfect fluid spacetime with torse-forming vector field ξ, the following relations hold* [21] *:*

$$\nabla_\xi \xi = 0, \eta(\nabla_\xi \xi) = 0, \tag{65}$$

$$(\nabla_X \eta)(Y) = g(X, Y) + \eta(X)\eta(Y), \tag{66}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{67}$$

$$R(X, \xi)\xi = -X - \eta(X)\xi, \tag{68}$$

$$\eta(R(X, Y)Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z), \tag{69}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{70}$$

$$S(X, \xi) = 3\eta(X). \tag{71}$$

Now, we study h-almost conformal η-Ricci-Bourguignon soliton on a perfect fluid spacetime satisfying the curvature condition $Q \cdot P = 0$ where P is the projective curvature tensor defined for a 4-dimensional semi-Riemannian manifold as

$$P(X, Y)Z = R(X, Y)Z + \frac{1}{3}[S(X, Z)Y - S(Y, Z)X]. \tag{72}$$

Theorem 4.2. *Let (M^4, g) be a perfect fluid spacetime with h-almost conformal η-Ricci-Bourguignon soliton $(g, \xi, \mu, \Lambda, \beta)$, satisfying the property $Q \cdot P = 0$, where ξ is the torse-forming vector field. Then M is a manifold of constant Riemannian curvature.*

Proof. Let us assume that the curvature property $Q \cdot P = 0$ holds on M. Then for $X, Y, Z \in \chi(M)$,

$$Q(P(X,Y),Z) - P(QX,Y)Z - P(X,QY)Z - P(X,Y)QZ = 0. \tag{73}$$

Using equation (72), we have

$$\begin{aligned}(\mu \quad &- \quad \frac{1}{2}(p+\frac{1}{2}) + r\Lambda - 2h)P(X,Y)Z \\ &+ \quad (\beta - 2h)\eta(P(X,Y)Z)\xi - (\mu - \frac{1}{2}(p+\frac{1}{2}) + r\Lambda - 2h)P(X,Y)Z \\ &- \quad (\beta - 2h)\eta(X)P(\xi,Y)Z - (\mu - \frac{1}{2}(p+\frac{1}{2}) + r\Lambda - 2h)P(X,Y)Z \\ &- \quad (\beta - 2h)\eta(Y)P(X,\xi)Z - (\mu - \frac{1}{2}(p+\frac{1}{2}) + r\Lambda - 2h)P(X,Y)Z \\ &- \quad (\beta - 2h)\eta(Y)P(X,Y)\xi = 0. \end{aligned} \tag{74}$$

From identities (67)–(70), we see that

$$P(\xi,Y)Z = g(Y,Z)\xi - \frac{1}{3}S(Y,Z)\xi, \tag{75}$$

$$P(X,\xi)Z = -g(X,Z)\xi + \frac{1}{3}S(X,Z)\xi, \tag{76}$$

$$P(X,Y)\xi = 0, \tag{77}$$

and

$$\begin{aligned}\eta(R(X,Y)Z) \quad &= \quad \eta(X)g(Y,Z) - \eta(Y)g(X,Z) \\ &\quad + \frac{1}{3}[S(X,Z)\eta(Y) - S(Y,Z)\eta(X)]. \end{aligned} \tag{78}$$

Substituting equations 75–78 for (74), we get

$$-2(\mu - \frac{1}{2}(p+\frac{1}{2}) + r\Lambda - 2h)P(X,Y)Z = 0. \tag{79}$$

This implies that either $P(X,Y)Z = 0$ or $\mu - \frac{1}{2}(p+\frac{1}{2}) + r\Lambda - 2h = 0$.

Case 1: If $\mu - \frac{1}{2}(p+\frac{1}{2}) + r\Lambda - 2h = 0$, $S(X,Y) = (\beta - 2h)\eta(X)\eta(Y)$, then putting $X = Y = e_i$, $1 \leq i \leq 4$, the scalar curvature becomes $r = 2h - \beta$.

Case 2: If $P(X,Y)Z = 0$, then

$$R(X,Y)Z = \frac{1}{3}(S(Y,Z)X - S(X,Z)Y). \tag{80}$$

Replacing $X = \xi$ in the above equation, we get $g(Y,Z)\xi = \frac{1}{3}S(Y,Z)\xi$. Inner product with ξ produces

$$S(Y,Z) = 3g(Y,Z). \tag{81}$$

From the equations (80) and (81), we find $R(X,Y)Z = g(Y,Z)X - g(X,Z)Y$. Therefore the manifold of constant Riemannian curvature -1. □

Theorem 4.3. *Let (M^4, g) be a perfect fluid spacetime with gradient h-almost conformal η-Ricci-Bourguignon soliton $(g, \nabla f, \mu, \Lambda, \beta)$ with h as a nonzero constant and $\beta = \xi(\mu - a)$ where the timelike vector field ξ is the torse-forming vector field. Then either the scalar fields a, b are related by $a = b$ or f is invariant under ξ.*

Proof. Let the spacetime satisfy the gradient h-almost conformal η-Ricci-Bourguignon soliton. Then from the equation (14), we can write

$$\nabla_X Df = \frac{1}{h}[(\mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda)X + \beta\eta(X)\xi - QX]. \tag{82}$$

Taking covariant derivative of the identity (82) with respect to Y and multiplying by h the above equation becomes

$$\begin{aligned} h\nabla_Y\nabla_X Df &= -\frac{1}{h}(Yh)[(\mu - \frac{1}{2}(p + \frac{1}{2}) \\ &\quad + r\Lambda)X + \beta\eta(X)\xi - QX] + (\mu - \frac{1}{2}(p + \frac{1}{2}) \\ &\quad + r\Lambda)\nabla_Y X + \beta\nabla_Y(\eta(X)\xi) - \nabla_Y(QX) + (Y\mu)X \\ &\quad + \Lambda(Xr)Y]. \end{aligned} \tag{83}$$

Interchanging X by Y in the equation (83), we obtain

$$\begin{aligned} h\nabla_X\nabla_Y Df &= -\frac{1}{h}(Xh)[(\mu - \frac{1}{2}(p + \frac{1}{2}) \\ &\quad + r\Lambda)Y + \beta\eta(Y)\xi - QY] + (\mu - \frac{1}{2}(p + \frac{1}{2}) \\ &\quad + r\Lambda)\nabla_X Y + \beta\nabla_X(\eta(Y)\xi) - \nabla_X(QY) + (X\mu)Y \\ &\quad + \Lambda(Yr)X]. \end{aligned} \tag{84}$$

Also from the equation (82), we have

$$\nabla_{[X,Y]}Df = \frac{1}{h}((\mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda)[X, Y] + \beta\eta([X, Y])\xi - Q[X, Y]). \tag{85}$$

We know the Riemannian curvature tensor

$$hR(X, Y)Df = h\nabla_X\nabla_Y Df - h\nabla_Y\nabla_X Df - \nabla_{[X,Y]}Df. \tag{86}$$

From the identities (84), (85) and (86), the above expression will be

$$\begin{aligned} hR(X, Y)Df &= -\frac{1}{h}(Xh)[(\mu - \frac{1}{2}(p + \frac{1}{2}) \\ &\quad + r\Lambda)Y + \beta\eta(Y)\xi - QY] + \frac{1}{h}(Yh)[(\mu - \frac{1}{2}(p + \frac{1}{2}) \\ &\quad + r\Lambda)X + \beta((\nabla_X\eta)(Y))\xi + \beta\eta(Y)\nabla_X\xi - (\nabla_X Q)(Y) \\ &\quad + (X\mu)Y + \Lambda(Xr)Y - \beta((\nabla_Y\eta)(X))\xi - \beta\eta(X)\nabla_Y\xi \\ &\quad + (\nabla_Y Q)(X) - (Y\mu)X - \Lambda(Yr)X. \end{aligned} \tag{87}$$

For perfect fluid spacetime we have

$$QX = aX + b\eta(X)\xi.$$

Taking covariant derivative of the above equation with respect to Y and also using the equation (61), we get

$$\begin{aligned}(\nabla_Y Q)(X) &= (Ya)X + (Yb)\eta(X)\xi + bg(X,Y)\xi + b\eta(X)Y \\ &\quad + 2b\eta(X)\eta(Y)\xi. \end{aligned} \tag{88}$$

Taking h as a constant, from the equation (88) it follows that

$$\begin{aligned} hR(X,Y)Df &= \beta\eta(Y)X + (X\mu)Y + \Lambda(Xr)Y - \beta\eta(X)Y - (Y\mu)X \\ &\quad - \Lambda(Yr)X + (Ya)X + (Yb)\eta(X)\xi - (Xa)Y \\ &\quad - (Xb)\eta(Y)\xi. \end{aligned} \tag{89}$$

Taking inner product with Z, the equation (89) can be written as

$$\begin{aligned} hg(R(X,Y)Df,Z) &= \beta\eta(Y)g(X,Z) + (X\mu)g(Y,Z) + \Lambda(Xr)g(Y,Z) \\ &\quad - \beta\eta(X)g(Y,Z) - (Y\mu)g(X,Z) - \Lambda(Yr)g(X,Z) \\ &\quad + (Ya)g(X,Z) + (Yb)\eta(X)\eta(Z) - (Xa)g(Y,Z) \\ &\quad - (Xb)\eta(Y)\eta(Z). \end{aligned} \tag{90}$$

Putting $X = Z = e_i$, $1 \leq i \leq 4$, we obtain

$$\begin{aligned} hS(Y,Df) &= 3\beta\eta(Y) - 3(Y\mu) + \Lambda(\xi r) + 3(Ya) - (Yb) - (\xi b)\eta(Y) \\ &\quad - 4\Lambda(Yr). \end{aligned} \tag{91}$$

Taking $Y = \xi$ in the above expression we get

$$h(a-b)g(\xi,Df) = -3\beta - 3(\xi\mu) - 3\Lambda(\xi r) + 3(\xi a). \tag{92}$$

In a perfect fluid spacetime, r is a constant. So we can write

$$h(a-b)(\xi f) = -3\beta - 3(\xi\mu) + 3(\xi a). \tag{93}$$

Using the condition $\beta = \xi(\mu - a)$ we have either $a = b$ or $\xi f = 0$ implies f is invariant under velocity vector field ξ. Hence the theorem is proved. □

5. Example of the h-almost conformal η-Ricci-Bourguignon soliton

We consider the four dimensional manifold $M = \{(x,y,z,u) \in \mathbb{R}^4 : u \neq 0\}$ where (x,y,z,u) are the standard coordinates in $\mathbb{R}^4$. The vector fields

$$e_1 = e^u \frac{\partial}{\partial x}, e_2 = e^u \frac{\partial}{\partial y}, e_3 = e^u \frac{\partial}{\partial z}, e_4 = \frac{\partial}{\partial u}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g_{ij} = \begin{cases} 1 & \text{for} \quad i = j = 1, 2, 3, \\ -1 & \text{for} \quad i = j = 4, \\ 0 & \text{for} \quad i \neq j. \end{cases}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_4)$ for any $Z \in \chi(M^4)$. Now, after some calculation, we have

$$[e_1, e_3] = [e_2, e_3] = [e_1, e_2] = 0, [e_1, e_4] = -e_1, [e_2, e_4] = -e_2, [e_3, e_4] = -e_3.$$

The Riemannian connection ∇ of the metric is given by the Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ &\quad - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

By Koszul's formula we get,

$$\nabla_{e_1} e_1 = e_4, \nabla_{e_2} e_2 = e_4, \nabla_{e_3} e_3 = e_4,$$

$$\nabla_{e_1} e_4 = -e_1, \nabla_{e_2} e_4 = -e_2, \nabla_{e_3} e_4 = -e_3,$$

$$\nabla_{e_1} e_2 = \nabla_{e_1} e_3 = \nabla_{e_2} e_1 = \nabla_{e_3} e_1 = \nabla_{e_3} e_2 = \nabla_{e_2} e_3 = 0.$$

$$\nabla_{e_4} e_1 = \nabla_{e_4} e_2 = \nabla_{e_4} e_3 = \nabla_{e_4} e_4 = 0.$$

The nonzero Riemannian curvature tensors are given by
$R(e_1, e_2)e_2 = -e_1, R(e_1, e_3)e_3 = -e_1, R(e_1, e_4)e_4 = -e_1, R(e_1, e_2)e_1 = e_2,$ $R(e_1, e_3)e_1 = e_3, R(e_1, e_4)e_1 = e_4, R(e_2, e_3)e_3 = -e_2, R(e_2, e_3)e_2 = e_3,$ $R(e_2, e_4)e_2 = e_4, R(e_2, e_4)e_4 = -e_2, R(e_3, e_4)e_3 = e_4, R(e_4, e_3)e_4 = e_3.$

Now from the above results we have $S(e_i, e_i) = -3$ for $i = 1, 2, 3, 4$. Contracting this we have $r = -12$.

Also we have

$$S(X, Y) + \frac{h}{2}[g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)] = (\mu - \frac{1}{2}(p + \frac{1}{2}) + r\Lambda)g(X, Y) + \beta\eta(X)\eta(Y).$$

Here $\xi = e_4$. So from the above, we get

$$\mu = -3 - h + 12\Lambda + \frac{1}{2}(p + \frac{1}{2})$$

and

$$\mu = 3 + 12\Lambda + \beta + \frac{1}{2}(p + \frac{1}{2}).$$

Hence we have $\beta = -h - 6$ and $\mu = -h + 12\Lambda + \frac{p}{2} - \frac{11}{4}$.

The h-almost conformal Ricci-Bourguignon soliton $(g, \xi, h, \mu, \Lambda)$ is shrinking if $p > \frac{11}{2} - 24\Lambda + 2h$; steady if $p = \frac{11}{2} - 24\Lambda + 2h$; and expanding if $p < \frac{11}{2} - 24\Lambda + 2h$.

Note. A soliton is a self-reinforcing single wave which arises from a balance between nonlinear and dispersive effects that are associated with physical system. Solitons preserve their shapes and speeds while propagating

freely at constant velocity and revive it after collision with one more such wave. The results of the paper fetches the new concept of h-almost conformal η-Ricci-Bourguignon soliton in a perfect fluid spacetime. Any contribution in this direction will bring new ideas of view on the geometry of the manifold. There are some questions that arise from our article to study in further research.
(i) Which of the results of our paper are also true for nearly Kähler spacetime manifold or cokähler manifold?
(ii) What will happen in Section 4 in this paper without assuming ξ as a torse-forming vector field?

Acknowledgements

The author is highly grateful to the editor and referees for their valuable comments and suggestions for improving the paper.

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