Equivalent notions in the context of compatible 
Endo-Lie algebras

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Abstract. In this article, we introduce a notion of compatibility be-
tween two Endo-Lie algebras defined on the same linear space. Compat-
ibility means that any linear combination of the two structures always
induces a new Endo-Lie algebra structure. In this case of compati-
bility, we show that the notions of bialgebras, standard Manin triples
and matched pairs are equivalent. We find this equivalence for the case
of compatible Lie algebras since this is a particular case of compatible
Endo-Lie algebras.

1. Introduction

An Endo-Lie algebra $[1]$ is a triple $(g, [\cdot, \cdot]_g, \phi)$, or simply $(g, \phi)$, where
$(g, [\cdot, \cdot]_g)$ is a Lie algebra and $\phi$ is a Lie algebra endomorphism. Two struc-
tures of Endo-Lie algebras, $(g, [\cdot, \cdot]_{g1}, \phi)$ and $(g, [\cdot, \cdot]_{g2}, \phi)$, are compatible if, for
all $k_1, k_2 \in \mathbb{K}$, $(g, k_1[\cdot, \cdot]_{g1} + k_2[\cdot, \cdot]_{g2}, \phi)$ is still an Endo-Lie algebra, in this case,
the compatibility of the two algebras is noted by $(g, [\cdot, \cdot]_{g1}, \phi)$ or $(g, \phi)$
and we call $(g, [\cdot, \cdot]_{g1}, [\cdot, \cdot]_{g2}, \phi)$ a compatible Endo-Lie algebra (for short CE-Lie
algebra). If we adopt the notation $(g, \phi)$, the context will indicate whether
the algebra is an Endo-Lie or a CE-Lie algebra. Note that if $\phi = id_g$, then we
find the definition of compatible Lie algebra, denoted by $(g, [\cdot, \cdot]_{g1}, [\cdot, \cdot]_{g2})$ or $g$
if no confusion is to be expected. A CE-Lie algebra $(g, [\cdot, \cdot]_{g1}, [\cdot, \cdot]_{g2}, \phi)$ is there-
fore a compatible Lie algebra $(g, [\cdot, \cdot]_{g1}, [\cdot, \cdot]_{g2})$ with a compatible Lie algebra
endomorphism $\phi$: $\phi([x, y]_{g1}) = [\phi(x), \phi(y)]_{g1}$ and $\phi([x, y]_{g2}) = [\phi(x), \phi(y)]_{g2}$
for all $x, y \in g$. This notion of compatible Lie algebra was introduced by
Golubchik and Sokolov $[2]$ and is characterised by

$$C_{x, y, z} ( [x, [y, z]_{g1}]_{g2} + [x, [y, z]_{g2}]_{g1} ) = 0, \forall x, y, z \in g. \quad (1)$$

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This paper is structured as follows. In the second section, we illustrate the notion of CE-Lie algebra with an example. In the third section, we study the notion of CE-Lie algebra representation. This notion is characterized by the following result: \((V, \phi_V, \rho_1, \rho_2)\) is a representation of a CE-Lie algebra \((g, [,]_1, [,]_2, \phi_g)\) if and only if \((V, \phi_V, \rho_1)\) and \((V, \phi_V, \rho_2)\) are representations of Endo-Lie algebras \((g, [,]_1, \phi_g)\) and \((g, [,]_2, \phi_g)\) respectively, and
\[\rho_1([x, y]_{g_2}) + \rho_2([x, y]_{g_1}) = [\rho_1(x), \rho_2(y)]_{gl(V)} + [\rho_2(x), \rho_1(y)]_{gl(V)}, \forall x, y \in g.\]

Unlike the case of compatible Lie algebras, if \((V, \phi_V, \rho_1, \rho_2)\) is a representation of a CE-Lie algebra, then the quadruple \((V^*, \phi_V^*, \rho_1^*, \rho_2^*)\) is not in general a representation of \((g, [,]_1, [,]_2, \phi_g)\). We then show that if \((V, \rho_1, \rho_2)\) is a representation of compatible Lie algebras \((g, [,]_{g_1}, [,]_{g_2})\) and for \(\beta \in gl(V)\), the triplet \((V^*, \beta^*, \rho_1^*, \rho_2^*)\) is a representation of a CE-Lie algebra \((g, [,]_{g_1}, [,]_{g_2})\) if and only if \(\beta(\rho_1(\phi_g(x)v)) = \rho(\phi_g(x)v) = \phi(g)(\beta(v)). \forall x \in g, v \in V.\)

We say that \(\beta\) dually represents \((g, [,]_{g_1}, [,]_{g_2}, \phi_g)\) on \((V, \rho_1, \rho_2)\).

The notion of matched pairs of two CE-Lie algebras is essential. Proposition 4 characterises this notion. Let two CE-Lie algebras \(g\) and \(h\) where \(g\) is an \(h\)-bimodule and \(h\) is a \(g\)-bimodule via two representations \((g, \phi_g, \rho_h, \mu_h)\) and \((h, \phi_h, \rho_g, \mu_g)\). Theorem 1 shows the following result: if the sextuple \((g, \phi_g, \rho_1, \rho_2)\) is a representation of a CE-Lie algebra, then the quadruple \((g, \phi_g, \rho_1, \rho_2)\) is a representation of endo-Lie algebras \((g, \phi_g, \rho_1, \rho_2)\) and \((h, \phi_h, \rho_g, \mu_g)\). Theorem 1 shows the following result: if the sextuple \((g, \phi_g, \rho_1, \rho_2)\) is a representation of a CE-Lie algebra, then the quadruple \((g, \phi_g, \rho_1, \rho_2)\) is a representation of endo-Lie algebras \((g, \phi_g, \rho_1, \rho_2)\) and \((h, \phi_h, \rho_g, \mu_g)\). Theorem 1 shows the following result: if the sextuple \((g, \phi_g, \rho_1, \rho_2)\) is a representation of a CE-Lie algebra, then the quadruple \((g, \phi_g, \rho_1, \rho_2)\) is a representation of endo-Lie algebras \((g, \phi_g, \rho_1, \rho_2)\) and \((h, \phi_h, \rho_g, \mu_g)\). Theorem 1 shows the following result: if the sextuple \((g, \phi_g, \rho_1, \rho_2)\) is a representation of a CE-Lie algebra, then the quadruple \((g, \phi_g, \rho_1, \rho_2)\) is a representation of endo-Lie algebras \((g, \phi_g, \rho_1, \rho_2)\) and \((h, \phi_h, \rho_g, \mu_g)\).
CE-Lie algebras. A few more structures on \((g \bowtie g^*, g, g^*)\), and we get the notion of Manin triple of CE-Lie algebras: \(((g \bowtie g^*, \phi \oplus \psi^*), (g, \phi), (g^*, \psi^*))\) is a Manin triple of CE-Lie algebras if \((g \bowtie g^*, g, g^*)\) is a standard Manin triple of compatible Lie algebras \(g\) and \(g^*\) such that \(g \bowtie g^*\) is quadratic concerning the natural scalar product and \((g \bowtie g^*, \phi \oplus \psi^*)\) is a CE-Lie algebra. By Theorem 2 and Lemma 2, \(((g \bowtie g^*, \phi \oplus \psi^*), (g, \phi), (g^*, \psi^*))\) is a Manin triple of CE-Lie algebras if and only if \(((g, \phi), (g^*, \psi^*), \text{ad}_1^g, \text{ad}_2^g, \text{Ad}_1^g, \text{Ad}_2^g)\) is a matched pair of CE-Lie algebras. This result is the subject of Theorem 3.

Finally, in the fifth section, we study, in any dimension, the CE-Lie bialgebra structure. Then we show that in finite dimension, we have the following result: \(((g, \phi), (g^*, \psi^*), \text{ad}_1^g, \text{ad}_2^g, \text{Ad}_1^g, \text{Ad}_2^g)\) is a matched pair of CE-Lie algebras if and only if \(((g, \phi)\), \(\text{ad}_1^g, \text{ad}_2^g, \text{Ad}_1^g, \text{Ad}_2^g)\) is a Manin triple of CE-Lie algebras. This result is the subject of Theorem 4. Combining all these results, we have that the expressions \((i)\) \(((g, \phi), (g^*, \psi^*), \text{ad}_1^g, \text{ad}_2^g, \text{Ad}_1^g, \text{Ad}_2^g)\) is a matched pair of CE-Lie algebras, \((ii)\) there is a structure of Manin triple of CE-Lie algebras \((g, g^*, \phi \oplus \psi^*), (g, \phi), (g^*, \psi^*)\) and \((iii)\) the triple \(((g, \phi), \Delta_1, \Delta_2, \psi)\) is a CE-Lie bialgebra, are equivalent. Note that if we take \(\phi = \text{id}_g\) and \(\psi^* = \text{id}_{g^*}\), the results obtained are those obtained in the case of compatible Lie algebras, see \([4]\).

2. Compatibility through an example

Definition 1. Let \((g, [\cdot, \cdot]_g, \phi)\) and \((g, [\cdot, \cdot]_{g2}, \phi)\) be two Endo-Lie algebras over a field \(K\). They are called compatible if for any \(k_1, k_2 \in K\), the following bilinear operation

\[
[x, y] = k_1 [x, y]_{g1} + k_2 [x, y]_{g2}, \quad \forall x, y \in g,
\]

defines an Endo-Lie algebra structure on \(g\). We denote the two compatible Endo-Lie algebras by \((g, [\cdot, \cdot]_{g1}, [\cdot, \cdot]_{g2}, \phi)\) and call \((g, [\cdot, \cdot]_{g1}, [\cdot, \cdot]_{g2}, \phi)\) a compatible Endo-Lie algebra.

Let \((g, [\cdot, \cdot]_{g1}, [\cdot, \cdot]_{g2}, \phi)\) be a compatible Endo-Lie algebra. We denote the Endo-Lie algebra defined by equation (2) by \((g, k_1 [\cdot, \cdot]_{g1} + k_2 [\cdot, \cdot]_{g2}, \phi)\) for any \(k_1, k_2 \in K\). A compatible Endo-Lie sub-algebra of \((g, [\cdot, \cdot]_{g1}, [\cdot, \cdot]_{g2}, \phi)\) is a sub-space of \(g\), which is an Endo-Lie sub-algebra of \((g, k_1 [\cdot, \cdot]_{g1} + k_2 [\cdot, \cdot]_{g2}, \phi)\), for any \(k_1, k_2 \in K\).

Example 1. Consider two Lie operations \([\cdot, \cdot]_1\) and \([\cdot, \cdot]_2\) on a three dimensional vector space \(g\) over \(\mathbb{R}\). We want to determine, if they exist, the \(\phi\) maps for which \((g, [\cdot, \cdot]_{g1}, \phi), (g, [\cdot, \cdot]_{g2}, \phi)\) are two Endo-Lie algebras and \((g, [\cdot, \cdot]_1, [\cdot, \cdot]_2, \phi)\) is a CE-Lie algebra. Let \(\{x, y, z\}\) be a basis of \(g\):

\[
[x, y]_{g1} = z, \quad [y, x]_{g1} = -z, \quad [x, y]_{g2} = z, \quad [y, x]_{g2} = -z,
\]
\[
[z, x]_{g^2} = y, \quad [x, z]_{g^2} = -y, \\
[y, z]_{g^2} = x, \quad [z, y]_{g^2} = -x.
\]

Let us put \( \text{Mat}[\phi, \{x, y, z\}] = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}, \)

where \( \phi(x), \phi(y) \) and \( \phi(z) \) are the column vectors \( (a_{ij} \in \mathbb{R}, i \text{ is the } i\text{th row and } j \text{ is the } j\text{th column}) \).

Using equation (1), we show that \( (g, [\cdot, \cdot], [, ]_{g^2}) \) is a compatible Lie algebra. Then \( \phi \) is an algebra endomorphism of \( (g, [\cdot, \cdot], [, ]_{g^2}) \) if and only if

\[
\begin{align*}
\phi(z) &= [\phi(x), \phi(y)]_{g^2}, \\
\phi(z) &= [\phi(x), \phi(y)]_{g^2}, \\
-\phi(y) &= [\phi(x), \phi(z)]_{g^2}, \\
\phi(x) &= [\phi(y), \phi(z)]_{g^2}.
\end{align*}
\]

\[
\begin{align*}
(1) \quad & a_{13} = a_{23} = a_{31} = a_{32} = 0, \\
(2) \quad & a_{33} = a_{11}a_{22} - a_{12}a_{21}, \\
(3) \quad & a_{11} = a_{22}a_{33}, \\
(4) \quad & a_{12} = -a_{21}a_{33}, \\
(5) \quad & a_{21} = -a_{12}a_{33}.
\end{align*}
\]

Let us calculate \((2) \times a_{11} \text{ and } (4) \times a_{21} \), then use (1) to find \( a_{33}(a_{11}^2 + a_{21}^2 - 1) = 0 \).

**Case 1.** If \( a_{33} = 0 \), we have \( a_{11} = a_{12} = a_{22} = a_{21} = 0 \), hence \( \phi \) is zero.

**Case 2.** If \( a_{11}^2 + a_{21}^2 = 1 \), then by equalities (4) and (5), we have \( a_{12}(1 - a_{33}^2) = 0 \).

**Case 1.** If \( a_{33} = \pm 1 \), we have

\[
\begin{align*}
\text{Mat}[\phi, \{x, y, z\}] &= \begin{pmatrix}
\alpha & -\beta & 0 \\
\beta & \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}, \\
\text{Mat}[\phi, \{x, y, z\}] &= \begin{pmatrix}
\alpha & \beta & 0 \\
\beta & -\alpha & 0 \\
0 & 0 & -1
\end{pmatrix},
\end{align*}
\]

\( \alpha^2 + \beta^2 = 1 \).

**Case 2.** If \( a_{12} = 0 \), then \( a_{11}^2 = 1 \). Possible solutions are

\[
\begin{align*}
\text{Mat}[\phi, \{x, y, z\}] &= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \\
\text{Mat}[\phi, \{x, y, z\}] &= \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}, \\
\text{Mat}[\phi, \{x, y, z\}] &= \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}, \\
\text{Mat}[\phi, \{x, y, z\}] &= \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\end{align*}
\]

**Remark 1.** Case 2.2 is a sub-case of the case 2.1. The solutions are

\[
\begin{align*}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \\
\begin{pmatrix}
\alpha & -\beta & 0 \\
\beta & \alpha & 0 \\
0 & 0 & 1
\end{pmatrix} \text{ and } \\
\begin{pmatrix}
\alpha & \beta & 0 \\
\beta & -\alpha & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad \alpha^2 + \beta^2 = 1.
\end{align*}
\]

In this example, we have a trivial Lie-compatible algebra (in the context of CE-Lie algebras), a compatible Lie algebra and CE-Lie algebras.
3. Matched pair of CE-Lie algebras

Let us recall the following definitions.

**Definition 2** ([3]). We call a representation of a compatible Lie algebra \((g, [\cdot, \cdot], \phi_g)\) a triple \((V, \rho, \mu)\) where \(V\) is a vector space and \(\rho, \mu : g \to gl(V)\) are linear maps such that for any \(k_1, k_2 \in \mathbb{K}\), \((V, k_1 \rho + k_2 \mu)\) is a representation of the Lie algebra \((g, k_1 \cdot + k_2 \cdot)\).

**Definition 3** ([3]). A representation of an Endo-Lie algebra \((g, [\cdot, \cdot], \phi_g)\) is a triple \((V, \rho, \mu)\), where \((V, \rho)\) is a representation of the Lie algebra \((g, [\cdot, \cdot], \phi_g)\) and \(\phi \in gl(V)\), such that, for all \(x, v \in V\), \(\phi(V(\rho(x)v) = \rho(\phi_g(x))\phi_V(v)\).

**Definition 4.** A representation of CE-Lie algebra \((g, [\cdot, \cdot], \rho_1, \rho_2)\) on a vector space \(V\) is a quadruple \((V, \phi_V, \rho_1, \rho_2)\) such that \((V, \phi_V, k_1 \rho_1 + k_2 \rho_2)\) is a representation of the Endo-Lie algebra \((g, k_1 \cdot + k_2 \cdot, \phi_g)\), for all \(k_1, k_2 \in \mathbb{K}\).

The above definitions lead to the following results.

**Proposition 1.** Let \((g, \phi_g)\) be a CE-Lie algebra. Then \((V, \phi_V, \rho, \mu)\) is a representation of \((g, \phi_g)\) if and only if the following conditions are satisfied:

a) \((V, \rho, \mu)\) is a representation of compatible Lie algebras \(g,

b) for all \(x, v \in V\),

\[
\phi_V(\rho(x)v) = \rho(\phi_g(x))\phi_V(v),
\]

\[
\phi_V(\mu(x)v) = \mu(\phi_g(x))\phi_V(v).
\]

**Proof.** The proof is obvious. \(\square\)

**Proposition 2.** \((V, \phi_V, \rho_1, \rho_2)\) is a representation of \((g, [\cdot, \cdot], [\cdot, \cdot], \phi_g)\) if and only if, for all \(x, y \in g\), the following conditions are satisfied:

1) the triplets \((V, \phi_V, \rho_1)\) and \((V, \phi_V, \rho_2)\) are representations of Endo-Lie algebras \((g, [\cdot, \cdot], \phi_g)\) and \((g, [\cdot, \cdot], \phi_g)\), respectively;

2) \(\rho_1([x, y]_g) + \rho_2([x, y]_g) = [\rho_1(x), \rho_2(y)]_{gl(V)} + [\rho_2(x), \rho_1(y)]_{gl(V)}\).

**Proof.** “\(\Rightarrow\)” \((V, \phi_V, \rho_1)\) and \((V, \phi_V, \rho_2)\) are representations of \((g, [\cdot, \cdot], \phi_g)\) and \((g, [\cdot, \cdot], \phi_g)\), respectively, they correspond to the cases \((k_1, k_2) = (1, 0)\) and \((k_1, k_2) = (0, 1)\). Hence the result in 1) holds. For \((k_1, k_2) = (1, 1)\) we have that \((V, \phi_V, \rho_1 + \rho_2)\) is a representation of \((g, [\cdot, \cdot], [\cdot, \cdot], \phi_g)\). By Definition 2, we have

\[
(\rho_1 + \rho_2)([x, y]_g) + [x, y]_g = (\rho_1(x) + \rho_2(x))(\rho_1(y) + \rho_2(y)) - (\rho_1(y) + \rho_2(y))(\rho_1(x) + \rho_2(x)).
\]

This is equivalent to

\[
\rho_1([x, y]_g) + \rho_1([x, y]_g) + \rho_2([x, y]_g) + \rho_2([x, y]_g) = \rho_1(x)\rho_1(y) + \rho_1(x)\rho_2(y) + \rho_2(x)\rho_1(y) + \rho_2(x)\rho_2(y).
\]
Thus \( (V, \text{operator on} (\text{representation of the compatible Lie algebra} \psi)) \) adjoint representation. Indeed, check the conditions of the previous proposition. 

By hypothesis, the two members on the left are equal.

In addition, we have

\[
(k_1 \rho_1 + k_2 \rho_2)(k_1[x, y]_{g_1} + k_2[x, y]_{g_2}) = k_1^2 \rho_1([x, y]_{g_1}) + k_2^2 \rho_2([x, y]_{g_2}) + k_1 k_2(\rho_1([x, y]_{g_2}) + \rho_2([x, y]_{g_1})).
\]

On the other hand, we have

\[
(k_1 \rho_1(x) + k_2 \rho_2(x))(k_1 \rho_1(y) + k_2 \rho_2(y)) = (k_1 \rho_1(y) + k_2 \rho_2(y))(k_1 \rho_1(x) + k_2 \rho_2(x))
\]

\[
= k_1^2(\rho_1(x)\rho_1(y) - \rho_1(y)\rho_1(x)) + k_2^2(\rho_2(x)\rho_2(y) - \rho_2(y)\rho_2(x)) + k_1 k_2(\rho_1(x)\rho_2(y) - \rho_2(y)\rho_1(x) + \rho_2(y)\rho_1(x) - \rho_1(y)\rho_2(x)).
\]

By hypothesis, the two members on the left are equal.

**Example 2.** Let \((g, [,]_{g_1}, [,]_{g_2}, \phi_g)\) be a CE-Lie algebra. Denote by \(\text{ad}_1\) the adjoint representation with respect to \([,]_{g_1}\) and by \(\text{ad}_2\) that with respect to \([,]_{g_2}\). Then \((g, \phi_g, \text{ad}_1, \text{ad}_2)\) is naturally a representation of \(g\), called an adjoint representation. Indeed, check the conditions of the previous proposition.

**Proof.** Condition 1) is obvious. For condition 2), using equation (1), we have for all \(x, y, z \in g\),

\[
\text{ad}_1[x, y]_{g_2} z + \text{ad}_2[x, y]_{g_1} z = [[x, y]_{g_2}, z]_{g_1} + [[x, y]_{g_1}, z]_{g_2}
\]

\[
= [[x, z]_{g_2} y]_{g_1} - [[y, z]_{g_2} x]_{g_1} + [[x, z]_{g_1} y]_{g_2} - [[y, z]_{g_1} x]_{g_2}
\]

\[
= (\text{ad}_1 x \text{ad}_2 y - \text{ad}_2 y \text{ad}_1 x)(z) + (\text{ad}_2 x \text{ad}_1 y - \text{ad}_1 y \text{ad}_2 x)(z).
\]

Thus \((g, \phi_g, \text{ad}_1, \text{ad}_2)\) is indeed a representation of \(g\).

Two representations \((V_1, \alpha_1, \rho_1, \mu_1)\) and \((V_2, \alpha_2, \rho_2, \mu_2)\) of a CE-Lie algebra \((g, [,]_{g_1}, [,]_{g_2}, \phi)\) are called equivalent if there exists a linear isomorphism \(\psi : V_1 \rightarrow V_2\) such that, for all \(x \in g\) and \(v \in V_1\),

\[
\psi(\rho_1(x)v) = \rho_2(x)\psi(v), \psi(\mu_1(x)v) = \mu_2(x)\psi(v)\text{ and } (\psi \circ \alpha_1)(v) = (\alpha_2 \circ \psi)(v)
\]

**Proposition 3.** Let \((g, [,]_{g_1}, [,]_{g_2}, \phi_g)\) be a CE-Lie algebra, \((V, \rho, \mu)\) a representation of the compatible Lie algebra \((g, [,]_{g_1}, [,]_{g_2})\) and \(\beta\) a linear operator on \(V\). Then \((V, \beta, \rho, \mu)\) is a representation of \((g, [,]_{g_1}, [,]_{g_2}, \phi_g)\) if
and only if \((g \oplus V, [\cdot, \cdot]_1, [\cdot, \cdot]_2), \phi_g \oplus \beta)\) is a CE-Lie algebra, where \([\cdot, \cdot]_1\) and \([\cdot, \cdot]_2\) are defined by

\[
[x + u, y + v]_1 = [x, y]_{g1} + \rho(x)v - \rho(y)u, \\
[x + u, y + v]_2 = [x, y]_{g2} + \mu(x)v - \mu(y)u.
\]

In this case, it is called the semi-product of \(g\) and \(V\) and is denoted by \(g \ltimes_{\rho, \mu} V\).

**Proof.** “\(\Longrightarrow\)” By the Jacobi identity and the fact that \((V, \rho_1)\) is a representation of the Lie algebra \((g, [\cdot, \cdot]_{g1})\), we have

\[
\circlearrowleft_{x+u, y+v, z+w} [x + u, [y + v, z + w]]_1 = \\
- \rho([x, y]_{g1})(w) + \rho(x) \circ \rho(y)w - \rho(y) \circ \rho(x)w \\
- \rho([z, x]_{g1})(v) + \rho(z) \circ \rho(x)v - \rho(x) \circ \rho(z)v \\
- \rho([y, z]_{g1})(u) + \rho(y) \circ \rho(z)u - \rho(z) \circ \rho(y)u \\
+ [x, [y, z]_{g1}]_{g1} + [z, [x, y]_{g1}]_{g1} + [y, [z, x]_{g1}]_{g1} = 0.
\]

Similarly, we have \(\circlearrowleft_{x+u, y+v, z+w} [x + u, [y + v, z + w]]_2 = 0\).

On the other hand, by equation (1) and condition 2) of Proposition 2, we have

\[
\circlearrowleft_{x+u, y+v, z+w} ([x + u, [y + v, z + w]]_{g1} + [x + u, [y + v, z + w]]_{g2}) = \\
- \rho([z, x]_{g2})(v) - \mu([z, x]_{g1})v + \rho(z)(\mu(x)v - \rho(x)(\mu(z)v) \\
+ \mu(z)(\rho(x)v) - \mu(x)(\rho(z)v) \\
- \rho([y, z]_{g2})u - \mu([y, z]_{g1})u + \rho(y)(\mu(z)u - \rho(z)(\mu(y)u) \\
+ \mu(y)(\rho(z)u) - \mu(z)(\rho(y)u) \\
- \rho([x, y]_{g2})w - \mu([x, y]_{g1})w + \rho(x)(\mu(y)w - \rho(y)(\mu(x)w) \\
+ \mu(x)(\rho(y)w) - \mu(y)(\rho(x)w) \\
+ [x, [y, z]_{g1}]_{g2} + [z, [x, y]_{g1}]_{g2} + [y, [z, x]_{g1}]_{g2} \\
+ [x, [y, z]_{g2}]_{g1} + [z, [x, y]_{g2}]_{g1} + [y, [z, x]_{g2}]_{g1} = 0.
\]

We conclude that \((g \oplus V, [\cdot, \cdot]_1, [\cdot, \cdot]_2)\) is a compatible Lie algebra.

Now let us show that \(\phi_g \oplus \beta\) is an endomorphism of \(g \oplus V\). Indeed,

\[
(\phi_g \oplus \beta)([x + u, y + v])_1 = (\phi_g \oplus \beta)([x, y]_{g1} + \rho(x)v - \rho(y)u) \\
= \phi_g([x, y]_{g1}) + \beta(\rho(x)v) - \beta(\rho(y)u) \\
= [\phi_g(x), \phi_g(y)]_1 + \rho(\phi_g(x))\beta(v) - \rho(\phi_g(y))\beta(u) \\
= [\phi_g(x) + \beta(u), \phi_g(y) + \beta(v)]_1 \\
= ([\phi_g + \beta](x + u), (\phi_g + \beta)(y + v)]_1.
\]

In the same way, we proceed for \([\cdot, \cdot]_2\).
"⇐" Let us show that \((V, \beta, \rho, \mu)\) is a representation of \((g, \ [, \ ]_g^1, \ [, \ ]_g^2, \phi_g)\). For \(x \in g\) and \(v \in V\), we have \((\phi_g \oplus \beta)([x, v])_1 = [\phi_g(x), \beta(v)]_1\), therefore \(\beta(\rho(x)v) = \rho(\phi_g(x))\beta(v)\). Likewise \((\phi_g \oplus \beta)([v, x])_2 = [\beta(v), \phi_g(x)]_2\), thus \(\beta(\mu(x)v) = \mu(\phi_g(x))\beta(v)\). On the other hand, we have

\[
\begin{align*}
\rho([x, y]_g^1)v &= [[x, y]_g^1, v]_1 = [x, y]_1, v_1 \\
&= [[x, v]_1, y_1] + [[v, y]_1, x_1] \\
&= (\rho(x)\rho(y) - \rho(y)\rho(x))v.
\end{align*}
\]

In the same way, we show that \(\mu([x, y]_g^1)v = (\mu(x)\mu(y) - \mu(y)\mu(x))v\). Finally, we have

\[
\begin{align*}
\rho([x, y]_g^2)v + \mu([x, y]_g^1)v &= [[x, y]_g^2, v]_1 + [[x, y]_g^1, v_1] \\
&= -[[v, x]_2, y_1] - [[y, v]_2, x_1] \\
&= -\rho(y)(\mu(x)v) + \rho(x)(\mu(y)v) \\
&= -\mu(y)(\rho(x)v) + \mu(x)(\rho(y)v).
\end{align*}
\]

The result is established.

\[\square\]

**Lemma 1.** Let \((g, \ [, \ ]_g^1, \ [, \ ]_g^2, \phi_g)\) be a CE-Lie algebra. Let \((V, \rho, \mu)\) be a representation of the compatible Lie algebra \((g, \ [, \ ]_g^1, \ [, \ ]_g^2)\). For \(\beta \in \text{gl}(V)\), the triple \((V^*, \beta^*, \rho^*, \mu^*)\) is a representation of \((g, \ [, \ ]_g^1, \ [, \ ]_g^2, \phi_g)\) if and only if \(\beta\) satisfies \(\beta(\rho(\phi_g(x))v) = \rho(x)(\beta(v))\) and \(\beta(\mu(\phi_g(x))v) = \mu(x)(\beta(v))\). We say that \(\beta\) dually represents the CE-Lie algebra \((g, \ [, \ ]_g^1, \ [, \ ]_g^2, \phi_g)\) on \((V, \rho, \mu)\).

**Proof.** First of all, for \(x \in g\) and \(u, v \in V\),

\[
\begin{align*}
< \beta^*(\rho^*(x)v), u > &=< \rho^*(\phi_g(x))(\beta^*(v)), u >,
\end{align*}
\]

which is equivalent to \(< v, \rho(x)(\beta(u)) > =< v, \beta(\rho(\phi(x))u) >\). Similarly,

\[
\begin{align*}
< \beta^*(\mu^*(x)v), u > &=< \mu^*(\phi_g(x))(\beta^*(v)), u >
\end{align*}
\]

is equivalent to \(< v, \mu(x)(\beta(u)) > =< v, \beta(\mu(\phi(x))u) >\).

"⇒" Obvious.

"⇐" Let us show that \((V^*, \rho^*)\) and \((V^*, \mu^*)\) are representations of Lie algebras \((g, \ [, \ ]_g^1)\) and \((g, \ [, \ ]_g^2)\), respectively. For \(\eta \in V^*\) and \(v \in V\), we have

\[
\begin{align*}
< \rho^*([,]_g^1)\eta, v > &=< \eta, \rho([x, y]_g^1)v > \\
&= -< \eta, (\rho(x)\rho(y) - \rho(y)\rho(x))v > \\
&= -< (\rho^*(x)\rho^*(y) - \rho^*(y)\rho^*(x))\eta, v >.
\end{align*}
\]

In the same way, \((V^*, \mu^*)\) is a Lie representation. On the other hand

\[
\begin{align*}
< (\rho^*([,]_g^2) + \mu^*([,]_g^1))\eta, v > &= -< \eta, (\rho([x, y]_g^2) + \mu([x, y]_g^1))v > \\
&= -< \eta, ([\rho(x), \mu(y)]_{\text{gl}(V)} + [\mu(x), \rho(y)]_{\text{gl}(V)})v > \\
&= -< ([\mu^*(y), \rho^*(x)]_{\text{gl}(V)} + [\rho^*(y), \mu^*(x)]_{\text{gl}(V)})\eta, v >
\end{align*}
\]
Let us recall the following definitions.

Definition 5 ([II]). A matched pair of Lie algebras is a quadruple denoted by \((g,\,[\,\,\,],h,\,[\,\,\,])\) and \((h,\,[\,\,\,],g,\,[\,\,\,])\) respectively, and \(\rho,\mu\) or simply \((g,h,\rho,\mu)\) where \((g,\,[\,\,\,])\) and \((h,\,[\,\,\,])\) are Lie algebras such that, for all \(x,y\in g\) and \(u,v\in h\),

1) \((g,\mu)\) and \((h,\rho)\) are representations of \((h,\,[\,\,\,])\) and \((g,\,[\,\,\,])\) respectively,
2) \(\rho(x)[u,v]_h - [\rho(x)u,v]_h - [u,\rho(x)v]_h + \rho(\mu(u)x)v - \rho(\mu(v)x)u = 0\),
3) \(\mu(u)[x,y]_g - [\mu(u)x,y]_g - [x,\mu(u)y]_g + \mu(\rho(x)u)y - \mu(\rho(y)u)x = 0\).

Definition 7 ([II]). A matched pair of Endo-Lie algebras is a quadruple denoted by \(((g,\,[\,\,\,]),\phi_g),(h,\,[\,\,\,],\phi_h),(\rho,\mu)\) or simply \(((g,\phi_g),(h,\phi_h),(\rho,\mu)\) where \((g,\,[\,\,\,])\) and \((h,\,[\,\,\,])\) are Endo-Lie algebras such that:

a) \((g,\phi_g,\mu)\) and \((h,\phi_h,\rho)\) are representations of Endo-Lie algebras \((h,\phi_h)\) and \((g,\phi_g)\) respectively,
b) \((g,h,\rho,\mu)\) is a matched pair of Lie algebras.

Definition 8. Let \((g,\,[\,\,\,],\phi_g)\) and \((h,\,[\,\,\,],\phi_h)\) be two CE-Lie algebras. We say that \(((g,\,[\,\,\,]),\phi_g),(h,\,[\,\,\,],\phi_h),(\rho_g,\mu_g,\rho_h,\mu_h)\) or simply \(((g,\phi_g),(h,\phi_h),(\rho_g,\mu_g,\rho_h,\mu_h)\) is a matched pair of CE-Lie algebras \(g\) and \(h\) if, for all \(k_1, k_2 \in \mathbb{K}\),

\(((g,\phi_g),(h,\phi_h),(\rho_g,\mu_g,\rho_h,\mu_h)\) is a matched pair of Endo-Lie algebras.

From Proposition 2, the result follows.

Corollary 1. A linear operator \(\beta\) on \(g\) dually represents the CE-Lie algebra \((g,\,[\,\,\,],\phi_g)\) on \((g,\rho,g,\mu)\) if and only if, for all \(x,y\in g\),

\[
\beta(\phi_g(x),y)_{g_1} = [x,\beta(y)]_{g_2}, \beta(\phi_g(x),y)_{g_2} = [x,\beta(y)]_{g_2}.
\]

Corollary 2. Let \((g,\,[\,\,\,],\phi_g)\) be a CE-Lie algebra. Let \((V,\rho,\mu)\) be a representation of \(g\) and \(\beta \in gl(V)\). Then we have the semi-product CE-Lie algebra \(g \ltimes \rho,\mu^* \ V^*\) with respect to \(\phi_g \oplus \beta^*\).

The notion of a matched pair of two CE-Lie algebras \(g\) and \(h\) allows us to define a CE-Lie algebra structure on the semi-direct product \(g \oplus h\). In the following, \(\rho,\rho_g,\mu_g: g \to gl(h)\) and \(\mu,\rho_h,\mu_h: h \to gl(g)\) denote linear maps. Let us recall the following definitions.
Remark 2. The above definition can be written, for all \( x \in g, v \in h \), as

a) \( \phi_g(\rho_h(v)x) = \rho_g(\phi_h(v))\phi_g(x), \phi_g(\mu_h(v)x) = \mu_g(\phi_h(v))\phi_g(x) \),

\( \phi_h(\rho_g(x)v) = \rho_g(\phi_g(x))\phi_h(v), \phi_h(\mu_g(x)v) = \mu_g(\phi_g(x))\phi_h(v) \),

b) \((g,h,\rho_g,\mu_g,\rho_h,\mu_h)\) is a matched pair of compatible Lie algebras.

Proposition 4. Under the conditions of the definition above, we have that \(((g,\phi_g),(h,\phi_h),\rho_g,\mu_g,\rho_h,\mu_h)\) is a matched pair of CE-Lie algebras if and only if the following conditions are satisfied:

(i) \((h,\phi_h,\rho_g,\mu_g)\) and \((g,\phi_g,\rho_h,\mu_h)\) are representations of CE-Lie algebras \((g,[\cdot],[\cdot]_1,\phi_g)\) and \((h,[\cdot],[\cdot]_1,\phi_h)\), respectively,

(ii) \(((g,\phi_g),(h,\phi_h),\rho_g,\mu_h)\) is a matched pair of Endo-Lie algebras \(g\) and \(h\), likewise, \(((g,\phi_g),(h,\phi_h),\mu_g,\mu_h)\) is matched pair of Endo-Lie algebras \((g,[\cdot],[\cdot]_2,\phi_g)\) and \((h,[\cdot],[\cdot]_2,\phi_h)\),

(iii) for all \( x,y \in g \) and \( u,v \in h \),

\[
\begin{align*}
\alpha) & \quad \mu_g(x)[u,v]_{h1} - [u,\mu_g(x)v]_{h1} - [\mu_g(x)u,v]_{h1} - \mu_g(\rho_h(v)x)u \\
& \quad + \mu_g(\rho_h(u)x)v + \rho_g(x)[u,v]_{h2} - [u,\rho_g(x)v]_{h2} - [u,\rho_g(x)u]_{h2} \\
& \quad + \rho_g(\mu_h(u)x)v - \rho_g(\mu_h(v)x)u = 0,

\beta) & \quad \rho_h(u)[x,y]_{g2} - [x,\rho_h(u)y]_{g2} - [\rho_h(u)x,y]_{g2} - \rho_h(\mu_g(y)x)u \\
& \quad + \rho_h(\mu_g(x)u)y + \mu_h(u)[x,y]_{g1} - [x,\mu_h(u)y]_{g1} - [y,\mu_h(u)x]_{g1} \\
& \quad + \mu_h(\mu_g(x)u)y - \mu_h(\rho_g(y)u)x = 0.
\end{align*}
\]

Proof. \( \Rightarrow \) Let us check Proposition 2. We have, for all \( k_1, k_2 \in \mathbb{K} \),

\( ((g,k_1[\cdot],[\cdot]_1 + k_2[\cdot],[\cdot]_2,\phi_g),(h,k_1[\cdot],[\cdot]_1 + k_2[\cdot],[\cdot]_2,\phi_h),k_1\rho_g + k_2\mu_g, k_1\rho_h + k_2\mu_h) \)

is a matched pair for Endo-Lie algebras. By Definition 6, \((h,\phi_h,\rho_g,\mu_g)\) is a representation of the Endo-Lie algebra \((g,k_1[\cdot],[\cdot]_1 + k_2[\cdot],[\cdot]_2,\phi_g)\), thus, for \( k_1 = 1 \) and \( k_2 = 0 \), \((h,\phi_h,\rho_g)\) is a representation of the Endo-Lie algebra \((g,[\cdot],[\cdot]_1,\phi_g)\). Likewise, \((h,\phi_h,\mu_g)\) is a representation of \((g,[\cdot],[\cdot]_2,\phi_g)\). On the other hand, for \( k_1 = k_2 = 1 \), \((g,[\cdot],[\cdot]_1 + [\cdot],[\cdot]_2,\phi_g),(h,[\cdot],[\cdot]_1 + [\cdot],[\cdot]_2,\phi_h),\rho_g + \mu_g, \rho_h + \mu_h\) is a matched pair of Endo-Lie algebras. As \((h,\phi_h,\rho_g + \mu_g)\) is a representation of \((g,[\cdot],[\cdot]_1 + [\cdot],[\cdot]_2,\phi_g)\), we deduce that

\[
(\rho_g + \mu_g)\{[x,y]_{g1} + [x,y]_{g2}\} = ([\rho_g(x) + \mu_g(x)],[\rho_g(y) + \mu_g(y)])_{gl(h)}.
\]

After simplifying, we obtain

\[
\rho_g([x,y]_{g2}) + \mu_g([x,y]_{g1}) = [\rho_g(x),\mu_g(y)]_{gl(h)} + [\mu_g(x),\rho_g(y)]_{gl(h)}.
\]

By Proposition 2 and the above, we deduce that \((h,\phi_h,\rho_g,\mu_g)\) is a representation of a CE-Lie algebra \((g,[\cdot],[\cdot]_1,[\cdot],[\cdot]_2,\phi_g)\). In this way, we show that \((g,\phi_g,\rho_h,\mu_h)\) is a representation of \((h,[\cdot],[\cdot]_1,[\cdot],[\cdot]_2,\phi_h)\). Condition (i) is established.

For the condition (ii), \(((g,[\cdot],[\cdot]_1,\phi_g),(h,[\cdot],[\cdot]_1,\phi_h),\rho_g,\rho_h)\) and \(((g,[\cdot],[\cdot]_2,\phi_g),(h,[\cdot],[\cdot]_2,\phi_h),\mu_g,\mu_h)\) are matched pairs, they correspond, respectively, to the cases \((k_1,k_2) = (1,0)\) and \((k_1,k_2) = (0,1)\). For (iii), we have that \(((g,[\cdot],[\cdot]_1 + [\cdot],[\cdot]_2,\phi_g),(h,[\cdot],[\cdot]_1 + [\cdot],[\cdot]_2,\phi_h),\rho_g + \mu_g, \rho_h + \mu_h\) is
a matched pair of Endo-Lie algebras, thus, by Definitions 5 and 7,

\[(\rho_g(x) + \mu_g(x))(\{u, v\}|_{h1} + \{u, v\}|_{h2}) - (\rho_g(x) + \mu_g(x))u, v|_{h1} - [\rho_g(x) + \mu_g(x)]u, v|_{h1}
- (\rho_g(x) + \mu_g(x))u|_{h2} - [u, (\rho_g(x) + \mu_g(x))v|_{h1} - [u, (\rho_g(x) + \mu_g(x))v|_{h2}
+ (\rho_g + \mu_g)((\rho_h(u) + \mu_h(u))x)v - (\rho_g + \mu_g)((\rho_h(v) + \mu_h(v))x)u = 0.

By developing the above calculation and since ((g, φ_g), (h, φ_h), ρ_g, ρ_h) is a matched pair of (g, [\cdot, \cdot]_{g1}, φ_g) and (h, [\cdot, \cdot]_{h1}, φ_h) and ((g, φ_g), (h, φ_h), μ_g, μ_h) is a matched pair of Endo-Lie algebras (g, [\cdot, \cdot]_{g2}, φ_g) and (h, [\cdot, \cdot]_{h2}, φ_h), we have

\[\rho_g(x)(\{u, v\}|_{h1}) - [\rho_g(x)u, v|_{h1} - [u, \rho_g(x)v|_{h1} + \rho_g(\rho_h(u)x)v
- \rho_g(\rho_h(v)x)u = 0,
\]
\[\mu_g(x)(\{u, v\}|_{h2}) - [\mu_g(x)u, v|_{h2} - [u, \mu_g(x)v|_{h2} + \mu_g(\mu_h(u)x)v
- \mu_g(\mu_h(v)x)u = 0.
\]

Therefore

\[\mu_g(x)(\{u, v\}|_{h1}) - [\mu_g(x)u, v|_{h1} - [u, \mu_g(x)v|_{h1} + \mu_g(\rho_h(u)x)v
- \mu_g(\rho_h(v)x)u = 0.
\]

In the same way, we show that

\[\mu_h(u)(\{x, y\}|_{g1}) - [\mu_h(u)x, y|_{g1} - [x, \mu_h(u)y|_{g1} + \mu_h(\rho_g(x)u)y
- \mu_h(\rho_g(y)u)x + \rho_h(\mu_g(x)y)u - \rho_h(\mu_g(y)x)u = 0.
\]

“implies” By (i), condition a) of Remark 2 is verified. For b), we check Definition 5. It is obvious that 1) of this definition is satisfied. For condition 2), we develop the following sum:

\[(k_1\rho_g + k_2\mu_g)(x)(k_1[u, v]|_{h1} + k_2[u, v]|_{h2}) - k_1[u, k_1\rho_g(x)v + k_2\mu_g(x)v]|_{h1}
- k_2[u, k_1\rho_g(x)v + k_2\mu_g(x)v]|_{h2} + k_1[v, k_1\rho_g(x)u + k_2\mu_g(x)u]|_{h1}
+ k_2[v, k_1\rho_g(x)u + k_2\mu_g(x)u]|_{h2} - (k_1\rho_g + k_2\mu_g)(k_1\rho_h(x) + k_2\mu_h(x))u
+ (k_1\rho_g + k_2\mu_g)(k_1\rho_h(u)x + k_2\mu_h(u)x)v.
\]

The components of k^2_1, k^2_2 and k_1k_2 are, respectively,

- \(\rho_g(x)|_{h1} - [u, \rho_g(x)v]|_{h1} + [v, \rho_g(x)u]|_{h1} - \rho_g(\mu_h(\rho_g(x)v)u + \rho_g(\mu_h(u)x)v
- \rho_g(\rho_h(v)u)x + \rho_g(\rho_h(u)x)v
- \mu_g(x)|_{h2} - \rho_g(\rho_h(v)u)|_{h2} - \rho_g(\mu_h(v)x)u + \mu_g(\mu_h(u)x)v
+ \mu_g(x)|_{h1} - \rho_g(\rho_h(v)u)|_{h1} - \rho_g(\rho_h(u)x)v + \rho_g(\rho_h(u)x)v.
\]

By (ii), the components of k^2_1 and k^2_2 are zero. By (iii), the component of k_1k_2 is zero. For condition 3), we proceed in the same way.
For compatible Lie algebras \((g, [,], g_1, [.,]_2)\) and \((h, [,], h_1, [.,]_2)\) and for linear maps \(\rho_g, \mu_g : g \to gl(h), \rho_h, \mu_h : h \to gl(g)\), define two bilinear operations on semi-direct product \(g \oplus h\), by
\[
[x + u, y + v]_1 = [x, y]_g + \rho_g(x)v - \rho_g(y)u + [u, v]_h1 + \rho_h(u)y - \rho_h(v)x,
\]
\[
[x + u, y + v]_2 = [x, y]_g + \rho_g(x)v - \rho_g(y)u + [u, v]_h2 + \mu_h(u)y - \mu_h(v)x.
\]
By \[3\], \((g \oplus h, [,], [.,]_1, [.,]_2)\) is a compatible Lie algebra if and only if the sextuple \(((g, [,], g_1, [.,]_2), (h, [,], h_1, [.,]_2), \rho_g, \mu_g, \rho_h, \mu_h)\) is a matched pair. We denote the resulting compatible Lie algebra \((g \oplus h, [,], [.,]_1, [.,]_2)\) by \(g \bowtie \mu_h h\), or simply \(g \bowtie h\).

**Theorem 1.** Under the conditions of Definition 8 and assuming that \((g, h, \rho_g, \mu_g, \rho_h, \mu_h)\) is a matched pair of compatible Lie algebras, \(g \bowtie h\) is a CE-Lie algebra, with respect to \(\phi_g \oplus \phi_h\), if and only if \(((g, \phi_g), (h, \phi_h), \rho_g, \mu_g, \rho_h, \mu_h)\) is a matched pair of CE-Lie algebras.

**Proof.** Let \(x, y \in g\) and \(u, v \in h\).

\[\implies\] We have
\[
0 = (\phi_g \oplus \phi_h)([u, x])_1 - [(\phi_g \oplus \phi_h)(u), (\phi_g \oplus \phi_h)(x)]_1
\]
\[= (\phi_g \oplus \phi_h)(\rho_h(u)x - \rho_h(x)u) - \rho_h(\phi_h(x))\phi_g(x) + \rho_g(\phi_g(x))\phi_h(u)
\]
\[= (\phi_g(\rho_h(u)x) - \rho_h(\phi_h(u))\phi_g(x)) + (\rho_g(\phi_g(x))\phi_h(u) - \phi_h(\rho_g(x)u)) \in g \oplus h.
\]
We deduce \(\phi_g(\rho_h(u)x) = \rho_h(\phi_h(u))\phi_g(x)\) and \(\phi_h(\rho_g(x)u) = \rho_g(\phi_g(x))\phi_h(u)\). We proceed in the same way for the other conditions. From Remark 2 we have the result.

\[\iff\] We show that \(\phi_g \oplus \phi_h\) is a compatible Lie algebra endomorphism:
\[
(\phi_g \oplus \phi_h)([x + u, y + v]_1)
\]
\[= (\phi_g \oplus \phi_h)([x, y]_g + \rho_g(x)v - \rho_g(y)u + [u, v]_h1 + \rho_h(u)y - \rho_h(v)x)
\]
\[= \phi_g([x, y]_g + \rho_h(u)y - \rho_h(v)x) + \phi_h([u, v]_h1 + \rho_g(x)v - \rho_g(y)u)
\]
\[= [\phi_g(x), \phi_g(y)]_g + \phi_h(\rho_h(u)y) - \phi_h(\rho_h(v)x) + [\phi_h(u), \phi_h(v)]_h1 + \phi_h(\rho_g(x)v) - \rho_h(\phi_h(u))\phi_g(x)
\]
\[= [\phi_g(x), \phi_g(y)]_g + \phi_h(\rho_h(u)y) - \rho_h(\phi_h(v)x) + [\phi_h(u), \phi_h(v)]_h1 + \phi_h(\rho_g(x)v) - \rho_h(\phi_h(u))\phi_g(x)
\]
\[= [\phi_g(x) + \phi_h(u), \phi_g(y) + \phi_h(v)]_1 = [(\phi_g \oplus \phi_h)(x + u), (\phi_g \oplus \phi_h)(y + v)]_1.
\]
Similarly, we see that
\[
(\phi_g \oplus \phi_h)([x + u, y + v]_2) = [(\phi_g \oplus \phi_h)(x + u), (\phi_g \oplus \phi_h)(y + v)]_2.
\]
The result is established. \(\square\)

**Remark 3.** If we set \([,]_h1 = [.,]_2 = 0\) \((h\) is abelian) and if \(\rho_h = \mu_h = 0\), we find Proposition 3. Any linear space can be seen as an Endo-Lie algebra compatible with the representation \((V, id_V, 0, 0)\).
4. Manin triples of CE-Lie algebras

We recall a result concerning compatible Lie algebras. We will need it in the rest of this article.

**Theorem 2 (⪿).** Let \( (g,[,],\phi) \) be a compatible Lie algebra equipped with two linear maps \( \Delta_1, \Delta_2 : g \to g \otimes g \). Suppose that \( \Delta_1, \Delta_2 \) induce a compatible Lie algebra structure on \( g \). Then the following conditions are equivalent:

(i) \( (g,[,],\phi,\Delta_1,\Delta_2) \) is a compatible Lie bialgebra,

(ii) \( (g \oplus g^*,g,g^*) \) is a standard Manin triple of compatible Lie algebras,

(iii) \( (g,g^*,\text{ad}_1^*,\text{ad}_2^*,\text{Ad}_1^*,\text{Ad}_2^*) \) is a matched pair of compatible Lie algebras.

In this section, we assume that \( g \) is finite-dimensional.

**Definition 9.** Let \( (g,[,],\phi) \) be an CE-Lie algebra and \( \Omega \) a nondegenerate symmetric bilinear form on \( g \). Then \( \Omega \) is called invariant if for all \( x,y,z \in g \) and for all \( k_1, k_2 \in \mathbb{K} \), \( \Omega(k_1[x,y]_1 + k_2[x,y]_2, z) = \Omega(x, k_1[y,z]_1 + k_2[y,z]_2) \). In this case, \( (g,[,],\phi,\Omega) \) is called a quadratic CE-Lie algebra and is denoted by \( ((g,[,],\phi),\Omega) \) or simply by \( (g,\phi,\Omega) \).

**Remark 4.** 1) Note that, for all \( k_1, k_2 \in \mathbb{K}, x,y \in g \), the following equations (a) and (b) are equivalent:

\[
(a) : \Omega(k_1[x,y]_1 + k_2[x,y]_2, z) = \Omega(x, k_1[y,z]_1 + k_2[y,z]_2),
\]

\[
(b) : \Omega([x,y]_{g_1}, z) = \Omega(x, [y,z]_{g_1}) \quad \text{and} \quad \Omega([x,y]_{g_2}, z) = \Omega(x, [y,z]_{g_2}).
\]

2) If we take in the definition above \( \phi = \text{id}_g \), we find the definition of quadratic compatible Lie algebra denoted by \( (g,\Omega) \).

Let \( (g,[,],\phi) \) be a compatible Lie algebra. Suppose that there is a compatible Lie algebra structure \( (g^*,[,],\phi^*) \) on the dual \( g^* \) and a compatible Lie algebra structure on the semi-direct product \( g \oplus g^* \) which contains both \( (g,[,],\phi) \) and \( (g^*,[,],\phi^*) \) as sub-algebras and for which the natural scalar product on \( g \oplus g^* \), \( < x + \eta, y + \xi > = < x, \xi > + < y, \eta > \), for all \( x,y \in g \eta, \xi \in g^* \), is invariant. The resulting algebra is denoted by \( g \bowtie g^* \) and the triple \( (g \bowtie g^*,g,g^*) \) is called a (standard) Manin triple of compatible algebras. This Lie algebra structure on \( g \oplus g^* \) comes from a matched pair. Let \( \text{Ad}_1 \) and \( \text{Ad}_2 \) be the adjoint representations with respect to \( [,]_{g_1} \) and \( [,]_{g_2} \), respectively. By Theorem 2, \( (g \bowtie g^*,g,g^*) \) is a standard Manin triple if and only if \( (g,\text{ad}_1^*,\text{ad}_2^*,\text{Ad}_1^*,\text{Ad}_2^*) \) is a matched pair of compatible Lie algebras. Let \( \hat{\phi} : g \to g \) denote the adjoint linear transformation of \( \phi \) under the nondegenerate bilinear form \( \Omega : \Omega(\phi(x),y) = \Omega(x,\hat{\phi}(y)) \), for all \( x,y \in g \).
Proposition 5. Let \((g, \phi, \Omega)\) be a quadratic CE-Lie algebra. Then \(\widehat{\phi}\) dually represents the CE-Lie algebra \((g, \phi)\) on the representation \((g, ad_1, ad_2)\). In other words, \((g^*, \widehat{\phi}^*, ad_1^*, ad_2^*)\) is a representation of the CE-Lie algebra \(g\). Furthermore, \((g, \phi, ad_1, ad_2)\) and \((g^*, \widehat{\phi}^*, ad_1^*, ad_2^*)\) are equivalent. Conversely, let \((g, \phi)\) be a CE-Lie algebra and let \(\psi \in gl(g)\) dually represent \((g, \phi)\) on \((g, ad_1, ad_2)\). If \((g^*, \psi^*, ad_1^*, ad_2^*)\) is equivalent to \((g, \phi, ad_1, ad_2)\), then there exists a nondegenerate invariant bilinear form \(\Omega\) on \(g\) such that \(\widehat{\phi} = \psi\).

Proof. For \(x, y, z \in g\), we have

\[
0 = \Omega([\phi(x), \phi(y)]_{g1}, z) - \Omega([x, y]_{g1}, \phi(z)) = \Omega(x, \widehat{\phi}([\phi(y), z]_{g1}) - \Omega([x, y]_{g1}, \widehat{\phi}(z)) = \Omega(x, \widehat{\phi}([\phi(y), z]_{g1}) - \Omega(x, [y, \widehat{\phi}(z)]_{g1}).
\]

We deduce that \(\widehat{\phi}([\phi(y), z]_{g1}) = [y, \widehat{\phi}(z)]_{g1}\). In the same way, \(\widehat{\phi}([\phi(y), z]_{g2}) = [y, \widehat{\phi}(z)]_{g2}\). \((g^*, \widehat{\phi}^*, ad_1^*, ad_2^*)\) is a representation of \((g, [\cdot, \cdot]_{g1}, [\cdot, \cdot]_{g2}, \phi)\). Define a linear map \(\varphi : g \to g^*\) by \(\varphi(x), y > = \Omega(x, y)\), for all \(x, y \in g\). Then \(\varphi\) is a linear isomorphism. Moreover, for \(x, y, z \in g\) we have

\[
\varphi(ad_{1,x}y), z > = \Omega(y, [z, x]_{g1}) = < ad_{1,x}^*\varphi(y), z >.
\]

We deduce that \(\varphi(ad_{1,x}y) = ad_{1,x}^*\varphi(y)\). Likewise, \(\varphi(ad_{2,x}y) = ad_{2,x}^*\varphi(y)\).

On the other hand

\[
\varphi(x), y > = \Omega(\phi(x), y) = \Omega(\widehat{\phi}(y)) = < \varphi(x), \widehat{\phi}(y) > = < \widehat{\phi}^*(\phi(x)), y >.
\]

We deduce that \(\varphi(\phi(x)) = \widehat{\phi}^*(\varphi(x))\). Thus \((g, \phi, ad_1, ad_2)\) is equivalent to \((g^*, \widehat{\phi}^*, ad_1^*, ad_2^*)\) as a representation of \((g, [\cdot, \cdot]_{g1}, [\cdot, \cdot]_{g2}, \phi)\). Conversely, suppose that \(\varphi : g \to g^*\) is a linear isomorphism giving the equivalence between \((g, \phi, ad_1, ad_2)\) and \((g^*, \psi^*, ad_1^*, ad_2^*)\). Define a bilinear form \(\Omega\) on \(g\) by posing \(\Omega(x, y) = < \varphi(x), y >\) for all \(x, y \in g\). Then by a similar argument as above, we show that \(\Omega\) is a nondegenerate invariant bilinear form on \(g\) such that \(\widehat{\phi} = \psi\). The proof is complete.

Definition 10. Let \((g, \phi)\) be a CE-Lie algebra. We assume that \((g^*, \psi^*)\) is also a CE-Lie algebra. A Manin triple of CE-Lie algebras is a triple \((g \bowtie g^*, g, g^*)\) of compatible Lie algebras such that \(g \bowtie g^*\) is quadratic with respect to the natural scalar product and \((g \bowtie g^*, \phi \oplus \psi)\) is a CE-Lie algebra. We denote this Manin triple by \(((g \bowtie g^*, \phi \oplus \psi^*), (g, \phi), (g^*, \psi^*))\).

Lemma 2. Let \((g \bowtie g^*, \phi \oplus \psi^*, <;>)\) be a quadratic CE-Lie algebra.
a) The adjoint $\hat{\phi} \oplus \hat{\psi}$ of $\phi \oplus \psi^*$ with respect to the natural scalar product $\langle , \rangle$ is $\psi \oplus \phi^*$. Further, $\psi \oplus \phi^*$ dually represents the CE-Lie algebra $(g \bowtie g^*, \phi \oplus \psi^*)$ on the adjoint representation.

b) $\psi$ dually represents the CE-Lie algebra $(g, \phi)$ on the adjoint representation $(g, \text{ad}_1, \text{ad}_2)$.

c) $\phi^*$ dually represents the CE-Lie algebra $(g^*, \psi^*)$ on the adjoint representation $(g, \text{Ad}_1, \text{Ad}_2)$.

**Proof.** For $a$), if $x, y \in g$ and $\eta, \xi \in g^*$, then we have
\[
\langle (\phi \oplus \psi^*)(x + \eta), y + \xi \rangle = \langle \phi(x) + \psi^*(\eta), y + \xi \rangle \\
= \langle \phi(x), y + \xi \rangle + \langle y, \psi^*(\eta) \rangle \\
= \langle \phi(x), y \rangle + \langle y, \psi^*(\eta) \rangle \\
= \langle x + \eta, (\psi \oplus \phi^*)(y + \xi) \rangle.
\]

By Proposition 5, for the quadratic CE-Lie algebra $(g \bowtie g^*, \phi \oplus \psi^*, \langle , \rangle)$, the linear map $\phi \oplus \psi^* = \psi \oplus \phi^*$ dually represents $(g \bowtie g^*, \phi \oplus \psi^*)$ on the adjoint representation. For $b$), by Corollary 1, we have, for all $x, y \in g$ and $\eta, \xi \in g^*$,
\[
(\psi \oplus \phi^*)[(\phi \oplus \psi^*)(x + \eta), y + \xi]_1 = [x + \eta, (\psi \oplus \phi^*)(y + \xi)]_1, \\
(\psi \oplus \phi^*)[(\phi \oplus \psi^*)(x + \eta), y + \xi]_2 = [x + \eta, (\psi \oplus \phi^*)(y + \xi)]_2.
\]

Now taking $\eta = \xi = 0$ in the above equations, we have the equalities $\psi[\phi(x), y]_g = [x, \psi(y)]_g$ and $\psi[\phi(x), y]_{g2} = [x, \psi(y)]_{g2}$. For $c$), the result is obtained for $x = y = 0$.

**Theorem 3.** Let $(g, \phi)$ be a CE-Lie algebra. Suppose that there is an CE-Lie algebra structure $(g^*, \psi^*)$ on its dual space $g^*$. Then there is a Manin triple of CE-Lie algebras $((g \bowtie g^*, \phi \oplus \psi^*), (g, \phi), (g^*, \psi^*))$ if and only if $((g, \phi), (g^*, \psi^*), \text{ad}_1^*, \text{ad}_2^*, \text{Ad}_1^*, \text{Ad}_2^*)$ is a matched pair of CE-Lie algebras.

**Proof.** “$\Longrightarrow$” Suppose there is a Manin triple structure of CE-Lie algebras $((g \bowtie g^*, \phi \oplus \psi^*), (g, \phi), (g^*, \psi^*))$. By Definition 10, $(g \bowtie g^*, g, g^*)$ is a Manin triple of compatible Lie algebras. By Theorem 2, $(g, g^*, \text{ad}_1^*, \text{ad}_2^*, \text{Ad}_1^*, \text{Ad}_2^*)$ is a matched pair of compatible Lie algebras for which the compatible Lie algebra on $g \oplus g^*$ is the compatible Lie algebra $g \bowtie g^*$. Since the homomorphism of $g \bowtie g^*$ is $\phi \oplus \psi^*$, by Lemma 2, $(g^*, \psi^*, \text{ad}_1^*, \text{ad}_2^*)$ and $(g, \phi, \text{Ad}_1^*, \text{Ad}_2^*)$ are representations of the CE-Lie algebras $(g, \phi)$ and $(g^*, \psi^*)$, respectively.

“$\Longleftarrow$” If $((g, \phi), (g^*, \psi^*), \text{ad}_1^*, \text{ad}_2^*, \text{Ad}_1^*, \text{Ad}_2^*)$ is a matched pair of CE-Lie algebras, then $(g, g^*, \text{ad}_1^*, \text{ad}_2^*, \text{Ad}_1^*, \text{Ad}_2^*)$ is a matched pair of compatible Lie algebras. Hence, by Theorem 2, the natural scalar product $\langle , \rangle$ is invariant on $g \oplus g^*$. By Theorem 1, the matched pair of CE-Lie algebras also equips the compatible Lie algebra $g \bowtie g^*$ with the endomorphism $\phi \oplus \psi^*$, giving us a quadratic of the CE-Lie algebra.
Remark 5. If we take $\phi = id_g$ and $\psi^* = id_{g^*}$, we find the theorem analogous to Theorem 3 concerning compatible Lie algebras [3].

5. Compatible Endo-Lie bialgebras

We recall some definitions and give the corresponding definitions in the context of compatible algebras.

5.1. The case of any dimension.

Definition 11 (II). A linear space $g$ with a linear map $\Delta : g \to g \otimes g$ is called a Lie coalgebra if $\Delta$ is coantisymmetric, in the sense that $\Delta = -^{\tau} \Delta$ for the flip map $\tau : g \otimes g \to g \otimes g$, and satisfies the co-Jacobian identity

$$(id + \sigma + \sigma^2)(id \otimes \Delta)\Delta = 0,$$

where $\sigma(x \otimes y \otimes z) = z \otimes x \otimes y$, for $x, y, z \in g$ and $id = id_g$.

Definition 12 (II). A Lie bialgebra is a pair $((g, [, ]_g), \Delta)$, where $(g, [, ]_g)$ is a Lie algebra, $(g, \Delta)$ is a Lie coalgebra such that, for all $x, y, z \in g$,

$$\Delta[x, y]_g = (ad_x \otimes id + id \otimes ad_x)\Delta y - (ad_y \otimes id + id \otimes ad_y)\Delta x.$$  \hspace{1cm} (3)

Definition 13. Let $(g, [, ]_{g1}, [, ]_{g2})$ be a compatible Lie algebra. A compatible Lie bialgebra structure on $(g, [, ]_{g1}, [, ]_{g2})$ is a pair of linear maps $\Delta_1, \Delta_2 : g \to g \otimes g$ such that, for all $k_1, k_2 \in \mathbb{K}$, $(g, k_1[, ]_{g1} + k_2[, ]_{g2}, k_1\Delta_1 + k_2\Delta_2)$ is a Lie bialgebra. We denote it by $((g, [, ]_{g1}, [, ]_{g2}), \Delta_1, \Delta_2)$ or simply $(g, \Delta_1, \Delta_2)$.

Proposition 6. Under the assumptions of the definition above, the triple $((g, [, ]_1, [, ]_2), \Delta_1, \Delta_2)$ is a compatible Lie bialgebra if and only if, for all $x, y \in g$, the following conditions are satisfied:

(a) $((g, [, ]_1), \Delta_1)$ and $((g, [, ]_2), \Delta_2)$ are Lie bialgebras,
(b) $\Delta_1([x, y]_{g1}) = (ad_{1,x} \otimes id + id \otimes ad_{1,x})\Delta_1(y) - (ad_{1,y} \otimes id + id \otimes ad_{1,y})\Delta_1(x)$,
(c) $\Delta_2([x, y]_{g2}) = (ad_{2,x} \otimes id + id \otimes ad_{2,x})\Delta_2(y) - (ad_{2,y} \otimes id + id \otimes ad_{2,y})\Delta_1(x)$.

Proof. “$\Rightarrow$” Let $k_1, k_2 \in \mathbb{K}$. Then $((g, [, ]_1), \Delta_1)$ and $((g, [, ]_2), \Delta_2)$ are Lie bialgebras, they correspond to the cases $(k_1, k_2) = (1, 0)$ and $(k_1, k_2) = (0, 1)$, respectively. On the other hand, we have

$$(id + \sigma + \sigma^2)(id \otimes (k_1\Delta_1 + k_2\Delta_2))(k_1\Delta_1 + k_2\Delta_2)$$

$$= k_1^2 (id + \sigma + \sigma^2)(id \otimes \Delta_1)\Delta_1 + k_2^2 (id + \sigma + \sigma^2)(id \otimes \Delta_2)\Delta_2$$

$$+ k_1 k_2 ((id + \sigma + \sigma^2)(id \otimes \Delta_1)\Delta_2 + (id + \sigma + \sigma^2)(id \otimes \Delta_2)\Delta_1)$$

$$= k_1 k_2 ((id + \sigma + \sigma^2)(id \otimes \Delta_1)\Delta_2 + (id + \sigma + \sigma^2)(id \otimes \Delta_2)\Delta_1).$$
By (a), we have the result (b). For (c), we develop the sum

\[(k_1 \Delta_1 + k_2 \Delta_2)(k_1[x, y]_{g_1} + k_2[x, y]_{g_2})\]
\[-((k_1 ad_{1,x} + k_2 ad_{2,x}) \otimes id + id \otimes (k_1 ad_{1,x} + k_2 ad_{2,x}))(k_1 \Delta_1 y + k_2 \Delta_2 y)\]
\[+((k_1 ad_{1,y} + k_2 ad_{2,y}) \otimes id + id \otimes (k_1 ad_{1,y} + k_1 k_2 ad_{2,y}))(k_1 \Delta_1 x + k_2 \Delta_2 x).\]

The components of \(k_1^2, k_2^2\) and \(k_1 k_2\) are, respectively,

- \(\Delta_1([x, y]_{g_1}) - (ad_{1,x} \otimes id + id \otimes ad_{1,x}) \Delta_1 y + (ad_{1,y} \otimes id + id \otimes ad_{1,y}) \Delta_1 x\)
- \(\Delta_2([x, y]_{g_2}) - (ad_{2,x} \otimes id + id \otimes ad_{2,x}) \Delta_2 y + (ad_{2,y} \otimes id + id \otimes ad_{2,y}) \Delta_2 x\)
- \(\Delta_1([x, y]_{g_1}) + \Delta_2([x, y]_{g_2})\)

\[= (ad_{1,x} \otimes id + id \otimes ad_{1,x}) \Delta_2 y - (ad_{2,x} \otimes id + id \otimes ad_{2,x}) \Delta_1 y\]
\[+ (ad_{1,y} \otimes id + id \otimes ad_{1,y}) \Delta_2 x + (ad_{2,y} \otimes id + id \otimes ad_{2,y}) \Delta_1 x.\]

By equation (3), we have the result (c).

\[\Longleftrightarrow\] It suffices to show that \((g, k_1 \Delta_1 + k_2 \Delta_2)\) is a Lie coalgebra and that the equation (3) holds. By (a) and (b), it is obvious that co-antisymmetry and co-Jacobian identity are satisfied. From the calculation we have just made, it is clear that equation (3) is true.

**Definition 14** ([II]). An Endo-Lie coalgebra is a Lie coalgebra \((g, \Delta)\) together with a Lie coalgebra endomorphism \(\psi \in gl(g)\) such that \((\psi \otimes \psi) \Delta = \Delta \psi\). It is denoted by \((g, \Delta, \psi)\).

**Remark 6.** If \(g\) is finite-dimensional, then \(\psi\) is a Lie coalgebra endomorphism of \((g, \Delta)\) if and only if \(\psi^*\) is a Lie algebra endomorphism of \(g^*\). Indeed, \((g, \Delta)\) is a Lie coalgebra if and only if \((g^*, \Delta^*)\) is a Lie algebra [4]. On the other hand, for \(\eta, \xi \in g^*\) and \(x \in g\), we have

\[< \psi^*(\eta \otimes \xi), x > = < \eta \otimes \xi, (\Delta \psi)x > = < \eta \otimes \xi, (\psi \otimes \psi) \Delta \sigma > = < \psi^*(\eta), \psi^*(\xi) \sigma, x > .\]

**Definition 15.** A quadruple \((g, \Delta_1, \Delta_2, \psi)\) is a compatible Endo-Lie coalgebra if, for all \(k_1, k_2 \in \mathbb{K}\), \(((g, [\cdot, \cdot]_{g_1} + k_1 [\cdot, \cdot]_{g_2}), k_1 \Delta_1 + k_2 \Delta_2, \psi)\) is an Endo-Lie coalgebra.

**Proposition 7.** \(((g, [\cdot, \cdot]_{g_1} + [\cdot, \cdot]_{g_2}), \Delta_1, \Delta_2, \psi)\) is a CE-Lie coalgebra if and only if the following conditions are satisfied:

1. \(((g, [\cdot, \cdot]_{g_1}), \Delta_1, \psi)\) and \(((g, [\cdot, \cdot]_{g_2}), \Delta_2, \psi)\) are Endo-Lie coalgebras,
2. \((id + \sigma + \sigma^2)(id \otimes \Delta_1)\Delta_2 + (id + \sigma + \sigma^2)(id \otimes \Delta_2)\Delta_1 = 0.\)

**Proof.** The proof is obvious.

**Definition 16.** The quadruple \(((g, [\cdot, \cdot]_{g_1} + [\cdot, \cdot]_{g_2}, \phi), \Delta_1, \Delta_2, \psi)\), denoted simply by \(((g, \phi), \Delta_1, \Delta_2, \psi)\), is a CE-Lie bialgebra if

1. \((g, \Delta_1, \Delta_2)\) is a compatible Lie bialgebra,
(b) \((g, \phi)\) is a CE-Lie algebra,
(c) \((g, \Delta_1, \Delta_2, \psi)\) is a CE-Lie coalgebra,
(d) \((id \otimes \phi)\Delta_1 = (\psi \otimes id)\Delta_1 \phi, (id \otimes \phi)\Delta_2 = (\psi \otimes id)\Delta_2 \phi, \psi[\phi(x), y]_{g1} = [x, \psi(y)]_{g1}\) and \(\psi[\phi(x), y]_{g2} = [x, \psi(y)]_{g2}\) \(\forall x, y \in g\).

5.2. Finite dimension case. Under our assumption of a finite dimension, \((g, \Delta)\) is a Lie coalgebra if and only if \((g^*, \Delta^*)\) is a Lie algebra [4]. As a consequence of this assumption, the compatible Lie coalgebra structure \((g, \Delta_1, \Delta_2)\) is equivalent to the compatible Lie algebra structure \((g^*, \Delta^*_1, \Delta^*_2)\).

By Theorem 2, \((g, [\cdot, \cdot]_{g1}, [\cdot, \cdot]_{g2}, \Delta_1, \Delta_2)\) is a compatible Lie bialgebra if and only if \((g, g^*, ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)\) is a matched pair of compatible Lie algebras, where \(Ad_1\) and \(Ad_2\) are the adjoint representations concerning \(\Delta_1^*\) and \(\Delta_2^*\), respectively.

**Theorem 4.** Let \((g, [\cdot, \cdot]_{g1}, [\cdot, \cdot]_{g2}, \phi)\) be CE-Lie algebras. Suppose that there is a CE-Lie algebra \((g^*, [\cdot, \cdot]_{g1}, [\cdot, \cdot]_{g2}, \psi^*)\) on the linear dual \(g^*\) of \(g\). Let \(\Delta_1, \Delta_2 : g \rightarrow g \otimes g\) denote the linear duals of the multiplications \([\cdot, \cdot]_{g1}\) and \([\cdot, \cdot]_{g2}\), respectively. Then \(((g, \phi), (g^*, \psi^*), ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)\) is a matched pair of CE-Lie algebras if and only if the triple \(((g, \phi), \Delta_1, \Delta_2, \psi)\) is a CE-Lie bialgebra.

**Proof.** By Remark 2, \(((g, \phi), (g^*, \psi^*), ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)\) is a matched pair of CE-Lie algebras if and only if \((g, g^*, ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)\) is a matched pair of compatible Lie algebras \((g, [\cdot, \cdot]_{g1}, [\cdot, \cdot]_{g2})\) and \((g^*, \Delta_1^*, \Delta_2^*)\). For all \(x \in g, \eta \in g^*, \psi^*(ad_{1, x}^* \eta) = ad_{1, \phi(x)}^* \psi^*(\eta), \psi^*(ad_{2, x}^* \eta) = ad_{2, \phi(x)}^* \psi^*(\eta), \phi(Ad_{1, \eta, x}) = Ad_{1, \psi^*(\eta), \phi(x)}, \phi(Ad_{2, \eta, x}) = Ad_{2, \psi^*(\eta), \phi(x)}\).

By Theorem 2, \((g, g^*, ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)\) being a matched pair of compatible Lie algebras is equivalent to \((g, \Delta_1, \Delta_2)\) being a compatible Lie bialgebra. On the other hand, the above conditions are equivalent to

\[
\psi([\phi(x), y]_{g1}) = [x, \psi(y)]_{g1}, \psi([\phi(x), y]_{g2}) = [x, \psi(y)]_{g2},
\]

\[(id \otimes \phi)\Delta_1 = (\psi \otimes id)\Delta_1 \phi, (id \otimes \phi)\Delta_2 = (\psi \otimes id)\Delta_2 \phi.\]

Since conditions (b), (c) of Definition 16, are satisfied, we say that the sextuple \(((g, \phi), (g^*, \psi^*), ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)\) is a matched pair of CE-Lie algebras if and only if \(((g, \phi), \Delta_1, \Delta_2, \psi)\) is a CE-Lie bialgebra. \(\square\)

**Remark 7.** If we take \(\phi = id_g\) and \(\psi^* = id_{g^*}\), we find a theorem analogous to Theorem 4 concerning compatible Lie algebras [5].

Under our assumption of finite dimension, combining Theorem 3 and Theorem 4, we have the following result.

**Theorem 5.** Let \((g, [\cdot, \cdot]_{g1}, g, [\cdot, \cdot]_{g2}, \phi)\) be a CE-Lie algebra. Suppose that there is a CE-Lie algebra \((g^*, [\cdot, \cdot]_{g1}, g, [\cdot, \cdot]_{g2}, \psi^*)\) on the linear dual \(g^*\) of \(g\).
Let $\Delta_1, \Delta_2 : g \to g \otimes g$ denote the linear dual of the multiplications $[,]_{g^*1}$ and $[,]_{g^*2}$, respectively. Then the following statements are equivalent:

(i) $((g, \phi), (g^*, \psi^*))$, $\text{ad}^*_{\Delta_1}, \text{ad}^*_{\Delta_2}, \text{Ad}^*_{\Delta_1}, \text{Ad}^*_{\Delta_2})$ is a matched pair of CE-Lie algebras $(g, [\cdot , [\cdot , g, [\cdot , g^*1, g, [\cdot , l^*_2, \phi])$ and $(g^*, [\cdot , [\cdot , g^*, [\cdot , g^*1, g, [\cdot , l^*_2, \psi^*])$,

(ii) there is a Manin triple of CE-Lie algebras,

(iii) the triple $((g, \phi), \Delta_1, \Delta_2, \psi)$ is a CE-Lie bialgebra.

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