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Equivalent notions in the context of compatible Endo-Lie algebras

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Abstract. In this article, we introduce a notion of compatibility between two Endo-Lie algebras defined on the same linear space. Compatibility means that any linear combination of the two structures always induces a new Endo-Lie algebras structure. In this case of compatibility, we show that the notions of bialgebras, standard Manin triples and matched pairs are equivalent. We find this equivalence for the case of compatible Lie algebras since this is a particular case of compatible Endo-Lie algebras.

1. Introduction

An Endo-Lie algebra [1] is a triple $(g,[,]_g,\phi)$, or simply (g,ϕ), where $(g,[,]_g)$ is a Lie algebra and ϕ is a Lie algebra endomorphism. Two structures of Endo-Lie algebras, $(g,[,]_{g1},\phi)$ and $(g,[,]_{g2},\phi)$, are compatible if, for all $k_1,k_2\in\mathbb{K}$, $(g,k_1[,]_{g1}+k_2[,]_{g2},\phi)$ is still an Endo-Lie algebra, in this case, the compatibility of the two algebras is noted by $(g,[,]_{g1},[,]_{g2},\phi)$ or (g,ϕ) and we call $(g,[,]_{g1},[,]_{g2},\phi)$ a compatible Endo-Lie algebra (for short CE-Lie algebra). If we adopt the notation (g,ϕ), the context will indicate whether the algebra is an Endo-Lie or a CE-Lie algebra. Note that if $\phi=id_g$, then we find the definition of compatible Lie algebra, denoted by $(g,[,]_{g1},[,]_{g2})$ or g if no confusion is to be expected. A CE-Lie algebra $(g,[,]_{g1},[,]_{g2},\phi)$ is therefore a compatible Lie algebra $(g,[,]_{g1},[,]_{g2})$ with a compatible Lie algebra endomorphism ϕ: $\phi([x,y]_{g1})=[\phi(x),\phi(y)]_{g1}$ and $\phi([x,y]_{g2})=[\phi(x),\phi(y)]_{g2}$ for all $x,y\in g$. This notion of compatible Lie algebra was introduced by Golubchik and Sokolov [2] and is characterised by

$$\circlearrowleft_{x,y,z}\left([x,[y,z]_{g1}]_{g2}+[x,[y,z]_{g2}]_{g1}\right)=0,\forall x,y,z\in g. \tag{1}$$

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This paper is structured as follows. In the second section, we illustrate the notion of CE-Lie algebra with an example. In the third section, we study the notion of CE-Lie algebra representation. This notion is characterized by the following result: $(V, \phi_V, \rho_1, \rho_2)$ is a representation of a CE-Lie algebra $(g, [,]_1, [,]_2, \phi_g)$ if and only if (V, ϕ_V, ρ_1) and (V, ϕ_V, ρ_2) are representations of Endo-Lie algebras $(g, [,]_1, \phi_g)$ and $(g, [,]_2, \phi_g)$ respectively, and

$$\rho_1([x, y]_{g2}) + \rho_2([x, y]_{g1}) = [\rho_1(x), \rho_2(y)]_{gl(V)} + [\rho_2(x), \rho_1(y)]_{gl(V)}, \forall x, y \in g.$$

Unlike the case of compatible Lie algebras, if $(V, \phi_V, \rho_1, \rho_2)$ is a representation of a CE-Lie algebra, then the quadruple $(V^*, \phi_V^*, \rho_1^*, \rho_2^*)$ is not in general a representation of $(g, [,]_1, [,]_2, \phi_g)$. We then show that if (V, ρ_1, ρ_2) is a representation of compatible Lie algebras $(g, [,]_{g1}, [,]_{g2})$ and for $\beta \in gl(V)$, the triplet $(V^*, \beta^*, \rho_1^*, \rho_2^*)$ is a representation of a CE-Lie algebra $(g, [,]_1, [,]_2, \phi_g)$ if and only if $\beta(\rho(\phi_g(x)v) = \rho(x)(\beta(v))$ and $\beta(\mu(\phi_g(x)v) = \mu(x)(\beta(v))$. $\forall x \in g, v \in V$. We say that β dually represents $(g, [,]_{g1}, [,]_{g2}, \phi_g)$ on (V, ρ_1, ρ_2).

The notion of matched pairs of two CE-Lie algebras is essential. Proposition 4 characterises this notion. Let two CE-Lie algebras g and h where g is an h-bimodule and h is a g-bimodule via two representations $(g, \phi_g, \rho_h, \mu_h)$ and $(h, \phi_h, \rho_g, \mu_g)$. Theorem 1 shows the following result: if the sextuple $(g, h, \rho_g, \mu_g, \rho_h, \mu_h)$ is a matched pair of compatible Lie algebras $(g, [,]_{g1}, [,]_{g2})$ and $(h, [,]_{h1}, [,]_{h2})$, then $((g, \phi_g), (h, \phi_h), \rho_g, \mu_g, \rho_h, \mu_h)$ is a matched pair if and only if $(g \oplus h, [,]_1, [,]_2, \phi_g \oplus \phi_h)$ is a CE-Lie algebra, where $[,]_1, [,]_2$ are defined by

$$[x + u, y + v]_1 = [x, y]_{g1} + \rho_g(x)v - \rho_g(y)u + [u, v]_{h1} + \rho_h(u)y - \rho_h(v)x,$$
$$[x + u, y + v]_2 = [x, y]_{g2} + \mu_g(x)v - \mu_g(y)u + [u, v]_{h2} + \mu_h(u)y - \mu_h(v)x,$$

for all $x, y \in g$ and $u, v \in h$. The resulting compatible Lie algebra structure on the semi-direct product $g \oplus h$ is denoted by $g \bowtie h$.

In the fourth section, we study the Manin triples. Let $(g, [,]_{g1}, [,]_{g2})$ be a compatible Lie algebra. Suppose that there is a compatible Lie algebra structure $(g^*, [,]_{g*1}, [,]_{g*2})$ on the dual g^* and a compatible Lie algebra structure on the semi-direct product $g \oplus g^*$ which contains both $(g, [,]_{g1}, [,]_{g2})$ and $(g^*, [,]_{g*1}, [,]_{g*2})$ as sub-algebras and for which the natural scalar product on the semi-direct product $g \oplus g^*$:

$$< x + \eta, y + \xi >=< x, \xi > + < y, \eta >, \ \forall x, y \in g \ \eta, \xi \in g^*$$

is invariant. This structure comes from a matched pair of compatible Lie algebras. The triple $(g \bowtie g^*, g, g^*)$ is called a (standard) Manin triple of compatible algebras. We then have $(g \bowtie g^*, g, g^*)$ is a standard Manin triple if and only if $(g, g^*, ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)$ is a matched pair, where ad_i and Ad_i are defined by $ad_{i,x}y = [x, y]_{gi}$ and $Ad_{i,\xi}\eta = [\xi, \eta]_{g^*i}$ for $x, y \in g$, $\xi, \eta \in g^*$ and $i = 1, 2$ (see Theorem 2). We now assume that (g, ϕ) and $(g*, \psi*)$ are

CE-Lie algebras. A few more structures on $(g \bowtie g^*, g, g^*)$, and we get the notion of Manin triple of CE-Lie algebras: $((g \bowtie g^*, \phi \oplus \psi^*), (g, \phi), (g^*, \psi^*))$ is a Manin triple of CE-Lie algebras if $(g \bowtie g^*, g, g^*)$ is a standard Manin triple of compatible Lie algebras g and g^* such that $g \bowtie g^*$ is quadratic concerning the natural scalar product and $(g \bowtie g^*, \phi \oplus \psi^*)$ is a CE-Lie algebra. By Theorem 2 and Lemma 2, $((g \bowtie g^*, \phi \oplus \psi^*), (g, \phi), (g^*, \psi^*))$ is a Manin triple of CE-Lie algebras if and only if $((g, \phi), (g^*, \psi^*), ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)$ is a matched pair of CE-Lie algebras. This result is the subject of Theorem 3.

Finally, in the fifth section, we study, in any dimension, the CE-Lie bialgebra structure. Then we show that in finite dimension, we have the following result: $((g, \phi), (g^*, \psi^*), ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)$ is a matched pair of CE-Lie algebras if and only if $((g, \phi), \Delta_1, \Delta_2, \psi)$ is a CE-Lie bialgebra where Δ_1, Δ_2 denote the linear dual of the multiplications $[,]_{g*1}$ and $[,]_{g*2}$, respectively. This result is the subject of Theorem 4. Combining all these results, we have that the expressions (i) $((g, \phi), (g^*, \psi^*), ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)$ is a matched pair of CE-Lie algebras, (ii) there is a structure of Manin triple of CE-Lie algebras $((g \bowtie g^*, \phi \oplus \psi^*), (g, \phi), (g^*, \psi^*))$ and (iii) the triple $((g, \phi), \Delta_1, \Delta_2, \psi)$ is a CE-Lie bialgebra, are equivalent. Note that if we take $\phi = id_g$ and $\psi^* = id_{g^*}$, the results obtained are those obtained in the case of compatible Lie algebras, see [5].

2. Compatibility through an example

Definition 1. Let $(g, [,]_{g1}, \phi)$ and $(g, [,]_{g2}, \phi)$ be two Endo-Lie algebras over a field $\mathbb{K}$. They are called compatible if for any $k_1, k_2 \in \mathbb{K}$, the following bilinear operation

$$[x, y] = k_1[x, y]_{g1} + k_2[x, y]_{g2}, \quad \forall x, y \in g, \tag{2}$$

defines an Endo-Lie algebra structure on g. We denote the two compatible Endo-Lie algebras by $(g, [,]_{g1}, [,]_{g2}, \phi)$ and call $(g, [,]_{g1}, [,]_{g2}, \phi)$ a compatible Endo-Lie algebra.

Let $(g, [,]_{g1}, [,]_{g2}, \phi)$ be a compatible Endo-Lie algebra. We denote the Endo-Lie algebra defined by equation (2) by $(g, k_1[,]_{g1} + k_2[,]_{g2}, \phi)$ for any $k_1, k_2 \in \mathbb{K}$. A compatible Endo-Lie sub-algebra of $(g, [,]_{g1}, [,]_{g2}, \phi)$ is a subspace of g, wich is an Endo-Lie sub-algebra of $(g, k_1[,]_{g1} + k_2[,]_{g2}, \phi)$, for any $k_1, k_2 \in \mathbb{K}$.

Example 1. Consider two Lie operations $[,]_1$ and $[,]_2$ on a three dimensional vector space g over $\mathbb{R}$. We want to determine, if they exist, the ϕ maps for which $(g, [,]_{g1}, \phi)$, $(g, [,]_{g2}, \phi)$ are two Endo-Lie algebras and $(g, [,]_1, [,]_2, \phi)$ is a CE-Lie algebra. Let $\{x, y, z\}$ be a basis of g:

$$[x, y]_{g1} = z, \quad [y, x]_{g1} = -z,$$
$$[x, y]_{g2} = z, \quad [y, x]_{g2} = -z,$$

$$[z,x]_{g2}=y,\quad [x,z]_{g2}=-y,$$
$$[y,z]_{g2}=x,\quad [z,y]_{g2}=-x.$$

Let us put $$Mat[\phi,\{x,y,z\}]=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix},$$
where $\phi(x)$, $\phi(y)$ and $\phi(z)$ are the column vectors ($a_{ij}\in\mathbb{R}$, i is the ith row and j is the jth column).

Using equation (1), we show that $(g,[,]_1,[,]_2)$ is a compatible Lie algebra. Then ϕ is an algebra endomorphism of $(g,[,]_{g1},[,]_{g2})$ if and only if

$$\begin{cases}\phi(z)=[\phi(x),\phi(y)]_{g1},\\ \phi(z)=[\phi(x),\phi(y)]_{g2},\\ -\phi(y)=[\phi(x),\phi(z)]_{g2},\\ \phi(x)=[\phi(y),\phi(z)]_{g2}.\end{cases}\iff\begin{cases} & a_{13}=a_{23}=a_{31}=a_{32}=0,\\ (1) & a_{33}=a_{11}a_{22}-a_{12}a_{21},\\ (2) & a_{22}=a_{11}a_{33},\\ (3) & a_{11}=a_{22}a_{33},\\ (4) & a_{12}=-a_{21}a_{33},\\ (5) & a_{21}=-a_{12}a_{33}.\end{cases}$$

Let us calculate (2) $\times\, a_{11}$ and (4) $\times\, a_{21}$, then use (1) to find $a_{33}(a_{11}^2+a_{21}^2-1)=0$.

Case 1. If $a_{33}=0$, we have $a_{11}=a_{12}=a_{22}=a_{21}=0$, hence ϕ is zero.

Case 2. If $a_{11}^2+a_{21}^2=1$, then by equalities (4) and (5), we have $a_{12}(1-a_{33}^2)=0$.

Case 2.1. If $a_{33}=\pm1$, we have

$$Mat[\phi,\{x,y,z\}]=\begin{pmatrix}\alpha & -\beta & 0\\ \beta & \alpha & 0\\ 0 & 0 & 1\end{pmatrix},\ Mat[\phi,\{x,y,z\}]=\begin{pmatrix}\alpha & \beta & 0\\ \beta & -\alpha & 0\\ 0 & 0 & -1\end{pmatrix},$$
$$\alpha^2+\beta^2=1.$$

Case 2.2. If $a_{12}=0$, then $a_{11}^2=1$. Possible solutions are

$$Mat[\phi,\{x,y,z\}]=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\ Mat[\phi,\{x,y,z\}]=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix},$$
$$Mat[\phi,\{x,y,z\}]=\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix},\ Mat[\phi,\{x,y,z\}]=\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Remark 1. Case 2.2 is a sub-case of the case 2.1. The solutions are
$$\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\ \begin{pmatrix}\alpha & -\beta & 0\\ \beta & \alpha & 0\\ 0 & 0 & 1\end{pmatrix}\text{ and }\begin{pmatrix}\alpha & \beta & 0\\ \beta & -\alpha & 0\\ 0 & 0 & -1\end{pmatrix},\ \alpha^2+\beta^2=1.$$
In this example, we have a trivial Lie-compatible algebra (in the context of CE-Lie algebras), a compatible Lie algebra and CE-Lie algebras.

3. Matched pair of CE-Lie algebras

Let us recall the following definitions.

Definition 2 ([3]). We call a representation of a compatible Lie algebra $(g, [,]_{g1}, [,]_{g2})$, a triple (V, ρ, μ) where V is a vector space and $\rho, \mu : g \to gl(V)$ are linear maps such that for any $k_1, k_2 \in \mathbb{K}$, $(V, k_1\rho + k_2\mu)$ is a representation of the Lie algebra $(g, k_1[,]_{g1} + k_2[,]_{g2})$.

Definition 3 ([1]). A representation of an Endo-Lie algebra $(g, [,]_g, \phi_g)$ is a triple (V, ϕ_V, ρ), where (V, ρ) is a representation of the Lie algebra $(g, [,]_g)$ and $\phi_V \in gl(V)$, such that, for all $x \in g$, $v \in V$, $\phi_V(\rho(x)v) = \rho(\phi_g(x))\phi_V(v)$.

Definition 4. A representation of CE-Lie algebra $(g, [,]_{g1}, [,]_{g2}, \phi_g)$ on a vector space V is a quadruple $(V, \phi_V, \rho_1, \rho_2)$ such that $(V, \phi_V, k_1\rho_1 + k_2\rho_2)$ is a representation of the Endo-Lie algebra $(g, k_1[,]_{g1} + k_2[,]_{g2}, \phi_g)$, for all $k_1, k_2 \in \mathbb{K}$.

The above definitions lead to the following results.

Proposition 1. *Let (g, ϕ_g) be a CE-Lie algebra. Then (V, ϕ_V, ρ, μ) is a representation of (g, ϕ_g) if and only if the following conditions are satisfied:*

a) (V, ρ, μ) is a representation of compatible Lie algebras g,

b) for all $x \in g, v \in V$,

$$\phi_V(\rho(x)v) = \rho(\phi_g(x))\phi_V(v),$$
$$\phi_V(\mu(x)v) = \mu(\phi_g(x))\phi_V(v).$$

Proof. The proof is obvious. $\square$

Proposition 2. *$(V, \phi_V, \rho_1, \rho_2)$ is a representation of $(g, [,]_{g1}, [,]_{g2}, \phi_g)$ if and only if, for all $x, y \in g$, the following conditions are satisfied:*

1) the triplets (V, ϕ_V, ρ_1) and (V, ϕ_V, ρ_2) are representations of Endo-Lie algebras $(g, [,]_{g1}, \phi_g)$ and $(g, [,]_{g2}, \phi_g)$, respectively;

2) $\rho_1([x, y]_{g2}) + \rho_2([x, y]_{g1}) = [\rho_1(x), \rho_2(y)]_{gl(V)} + [\rho_2(x), \rho_1(y)]_{gl(V)}$.

Proof. "$\Longrightarrow$" (V, ϕ_V, ρ_1) and (V, ϕ_V, ρ_2) are representations of $(g, [,]_{g1}, \phi_g)$ and $(g, [,]_{g2}, \phi_g)$, respectively, they correspond to the cases $(k_1, k_2) = (1, 0)$ and $(k_1, k_2) = (0, 1)$. Hence the result in 1) holds. For $(k_1, k_2) = (1, 1)$ we have that $(V, \phi_V, \rho_1 + \rho_2)$ is a representation of $(g, [,]_{g1} + [,]_{g2}, \phi_g)$. By Definition 2, we have

$$\begin{aligned}(\rho_1 + \rho_2)([x, y]_{g1} + [x, y]_{g2}) = {} & (\rho_1(x) + \rho_2(x))(\rho_1(y) + \rho_2(y)) \\ & - (\rho_1(y) + \rho_2(y))(\rho_1(x) + \rho_2(x)).\end{aligned}$$

This is equivalent to

$$\begin{aligned}&\rho_1([x, y]_{g1}) + \rho_1([x, y]_{g2}) + \rho_2([x, y]_{g1}) + \rho_2([x, y]_{g2}) \\ &\qquad = \rho_1(x)\rho_1(y) + \rho_1(x)\rho_2(y) + \rho_2(x)\rho_1(y) + \rho_2(x)\rho_2(y)\end{aligned}$$

$$-\rho_1(y)\rho_1(x)-\rho_1(y)\rho_2(x)-\rho_2(y)\rho_1(x)-\rho_1(y)\rho_2(x).$$

After simplification, we obtain the result 2).

"$\Longleftarrow$" Let us show that $(V,\phi_V,k_1\rho_1+k_2\rho_2)$ is a representation of Endo-Lie algebra $(g,k_1[,]_{g1}+k_2[,]_{g2},\phi_g)$, for all k_1, $k_2\in\mathbb{K}$. To do this, let us check Definition 2. If $x,y\in g$ and $v\in V$, then

$$\begin{aligned}\phi_V(k_1\rho_1(x)v+k_2\rho_2(x)v)&=k_1\phi_V(\rho_1(x)v)+k_2(\rho_2(x)v)\\&=k_1\rho_1(\phi_g(x))\phi_V(v)+k_2\rho_2(\phi_g(x))\phi_V(v)\\&=(k_1\rho_1+k_2\rho_2)(\phi_g(x))\phi_V(v).\end{aligned}$$

In addition, we have

$$\begin{aligned}(k_1\rho_1+k_2\rho_2)(k_1[x,y]_{g1}+k_2[x,y]_{g2})&=k_1^2\rho_1([x,y]_{g1})+k_2^2\rho_2([x,y]_{g2})\\&\quad+k_1k_2(\rho_1([x,y]_{g2})+\rho_2([x,y]_{g1})).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&(k_1\rho_1(x)+k_2\rho_2(x))(k_1\rho_1(y)+k_2\rho_2(y))\\&\qquad-(k_1\rho_1(y)+k_2\rho_2(y))(k_1\rho_1(x)+k_2\rho_2(x))\\&=\ k_1^2(\rho_1(x)\rho_1(y)-\rho_1(y)\rho_1(x))+k_2^2(\rho_2(x)\rho_2(y)-\rho_2(y)\rho_2(x))\\&\qquad+k_1k_2(\rho_1(x)\rho_2(y)-\rho_2(y)\rho_1(x)+\rho_2(x)\rho_1(y)-\rho_1(y)\rho_2(x)).\end{aligned}$$

By hypothesis, the two members on the left are equal. □

Example 2. Let $(g,[,]_{g1},[,]_{g2},\phi_g)$ be a CE-Lie algebra. Denote by ad_1 the adjoint representation with respect to $[,]_{g1}$ and by ad_2 that with respect to $[,]_{g2}$. Then (g,ϕ_g,ad_1,ad_2) is naturally a representation of g, called an adjoint representation. Indeed, check the conditions of the previous proposition.

Proof. Condition 1) is obvious. For condition 2), using equation (1), we have for all $x,y,z\in g$,

$$\begin{aligned}ad_{1,[x,y]_{g2}}z+ad_{2,[x,y]_{g1}}z&=[[x,y]_{g2},z]_{g1}+[[x,y]_{g1},z]_{g2}\\&=[[x,z]_{g2},y]_{g1}-[[y,z]_{g2},x]_{g1}+[[x,z]_{g1},y]_{g2}-[[y,z]_{g1},x]_{g2}\\&=(ad_{1,x}ad_{2,y}-ad_{2,y}ad_{1,x})(z)+(ad_{2,x}ad_{1,y}-ad_{1,y}ad_{2,x})(z).\end{aligned}$$

Thus (g,ϕ_g,ad_1,ad_2) is indeed a representation of g. □

Two representations $(V_1,\alpha_1,\rho_1,\mu_1)$ and $(V_2,\alpha_2,\rho_2,\mu_2)$ of a CE-Lie algebra $(g,[,]_{g1},[,]_{g2},\phi)$ are called equivalent if there exists a linear isomorphism $\psi:V_1\to V_2$ such that, for all $x\in g$ and $v\in V_1$,

$$\psi(\rho_1(x)v)=\rho_2(x)\psi(v),\psi(\mu_1(x)v)=\mu_2(x)\psi(v)\ \textit{and}\ (\psi\circ\alpha_1)(v)=(\alpha_2\circ\psi)(v)$$

Proposition 3. *Let $(g,[,]_{g1},[,]_{g2},\phi_g)$ be a CE-Lie algebra, (V,ρ,μ) a representation of the compatible Lie algebra $(g,[,]_{g1},[,]_{g2})$ and β a linear operator on V. Then (V,β,ρ,μ) is a representation of $(g,[,]_{g1},[,]_{g2},\phi_g)$ if* *and only if* $((g \oplus V, [,]_1, [,]_2), \phi_g \oplus \beta)$ *is a CE-Lie algebra, where* $[,]_1$ *and* $[,]_2$ *are defined by*

$$[x+u, y+v]_1 = [x,y]_{g1} + \rho(x)v - \rho(y)u,$$
$$[x+u, y+v]_2 = [x,y]_{g2} + \mu(x)v - \mu(y)u.$$

In this case, it is called the semi-product of g *and* V *and is denoted by* $g \ltimes_{\rho,\mu} V$.

Proof. "$\Longrightarrow$" By the Jacobi identity and the fact that (V, ρ_1) is a representation of the Lie algebra $(g, [,]_{g1})$, we have

$$\begin{aligned}
\circlearrowleft_{x+u,y+v,z+w}&[x+u,[y+v,z+w]_1]_1 = \\
&- \rho([x,y]_{g1})(w) + \rho(x)\circ\rho(y)w - \rho(y)\circ\rho(x)w \\
&- \rho([z,x]_{g1})(v) + \rho(z)\circ\rho(x)v - \rho(x)\circ\rho(z)v \\
&- \rho([y,z]_{g1})(u) + \rho(y)\circ\rho(z)u - \rho(z)\circ\rho(y)u \\
&+ [x,[y,z]_{g1}]_{g1} + [z,[x,y]_{g1}]_{g1} + [y,[z,x]_{g1}]_{g1} = 0.
\end{aligned}$$

Similarly, we have $\circlearrowleft_{x+u,y+v,z+w} [x+u,[y+v,z+w]_2]_2 = 0$.

On the other hand, by equation (1) and condition 2) of Proposition 2, we have

$$\begin{aligned}
\circlearrowleft_{x+u,y+v,z+w}& ([x+u,[y+v,z+w]_{g1}]_{g2} + [x+u,[y+v,z+w]_{g2}]_{g1}) \\
&= - \rho([z,x]_{g2})v - \mu([z,x]_{g1})v + \rho(z)(\mu(x)v) - \rho(x)(\mu(z)v) \\
&\quad + \mu(z)(\rho(x)v) - \mu(x)(\rho(z)v) \\
&\quad - \rho([y,z]_{g2})u - \mu([y,z]_{g1})u + \rho(y)(\mu(z)u) - \rho(z)(\mu(y)u) \\
&\quad + \mu(y)(\rho(z)u) - \mu(z)(\rho(y)u) \\
&\quad - \rho([x,y]_{g2})w - \mu([x,y]_{g1})w + \rho(x)(\mu(y)w) - \rho(y)(\mu(x)w) \\
&\quad + \mu(x)(\rho(y)w) - \mu(y)(\rho(x)w) \\
&\quad + [x,[y,z]_{g1}]_{g2} + [z,[x,y]_{g1}]_{g2} + [y,[z,x]_{g1}]_{g2} \\
&\quad + [x,[y,z]_{g2}]_{g1} + [z,[x,y]_{g2}]_{g1} + [y,[z,x]_{g2}]_{g1} = 0.
\end{aligned}$$

We conclude that $(g \oplus V, [,]_1, [,]_2)$ is a compatible Lie algebra.

Now let us show that $\phi_g \oplus \beta$ is an endomorphism of $g \oplus V$. Indeed,

$$\begin{aligned}
(\phi_g \oplus \beta)([x+u, y+v]_1) &= (\phi_g \oplus \beta)([x,y]_{g1} + \rho(x)v - \rho(y)u) \\
&= \phi_g([x,y]_{g1}) + \beta(\rho(x)v) - \beta(\rho(y)u) \\
&= [\phi_g(x), \phi_g(y)]_1 + \rho(\phi_g(x))\beta(v) - \rho(\phi_g(y))\beta(u) \\
&= [\phi_g(x) + \beta(u), \phi_g(y) + \beta(v)]_1 \\
&= [(\phi_g \oplus \beta)(x+u), (\phi_g \oplus \beta)(y+v)]_1.
\end{aligned}$$

In the same way, we proceed for $[,]_2$.

"$\Longleftarrow$" Let us show that (V, β, ρ, μ) is a representation of $(g, [,]_{g1}, [,]_{g2}, \phi_g)$. For $x \in g$ and $v \in V$, we have $(\phi_g \oplus \beta)([x, v]_1) = [\phi_g(x), \beta(v)]_1$, therefore $\beta(\rho(x)v) = \rho(\phi_g(x))\beta(v)$. Likewise $(\phi_g \oplus \beta)([v, x]_2) = [\beta(v), \phi_g(x)]_2$, thus $\beta(\mu(x)v) = \mu(\phi_g(x))\beta(v)$. On the other hand, we have

$$\begin{aligned}\rho([x,y]_{g1})v &= [[x,y]_{g1}, v]_1 = [[x,y]_1, v]_1\\ &= [[x,v]_1, y]_1 + [[v,y]_1, x]_1\\ &= (\rho(x)\rho(y) - \rho(y)\rho(x))v.\end{aligned}$$

In the same way, we show that $\mu([x, y]_{g1})v = (\mu(x)\mu(y)-\mu(y)\mu(x))v$. Finally, we have

$$\begin{aligned}\rho([x,y]_{g2})v + \mu([x,y]_{g1})v &= [[x,y]_{g2}, v]_1 + [[x,y]_1, v]_2\\ &= -[[v,x]_2, y]_1 - [[y,v]_{g2}, x]_1\\ &\quad - [[v,x]_1, y]_2 - [[y,v]_{g1}, x]_2\\ &= -\rho(y)(\mu(x)v) + \rho(x)(\mu(y)v)\\ &\quad - \mu(y)(\rho(x)v) + \mu(x)(\rho(y)v).\end{aligned}$$

The result is established. □

Lemma 1. *Let $(g, [,]_{g1}, [,]_{g2}, \phi_g)$ be a CE-Lie algebra. Let (V, ρ, μ) be a representation of the compatible Lie algebra $(g, [,]_{g1}, [,]_{g2})$. For $\beta \in gl(V)$, the triple $(V^*, \beta^*, \rho^*, \mu^*)$ is a representation of $(g, [,]_{g1}, [,]_{g2}, \phi_g)$ if and only if β satisfies $\beta(\rho(\phi_g(x)v) = \rho(x)(\beta(v))$ and $\beta(\mu(\phi_g(x)v) = \mu(x)(\beta(v))$. We say that β dually represents the CE-Lie algebra $(g, [,]_{g1}, [,]_{g2}, \phi_g)$ on (V, ρ, μ).*

Proof. First of all, for $x \in g$ and $u, v \in V$,

$$< \beta^*(\rho^*(x)v), u >=< \rho^*(\phi_g(x))(\beta^*(v)), u >,$$

which is equivalent to $< v, \rho(x)(\beta(u)) >=< v, \beta(\rho(\phi(x))u) >$. Similarly,

$$< \beta^*(\mu^*(x)v), u >=< \mu^*(\phi_g(x))(\beta^*(v)), u >$$

is equivalent to $< v, \mu(x)(\beta(u)) >=< v, \beta(\mu(\phi(x))u) >$.

"$\Longrightarrow$" Obvious.

"$\Longleftarrow$" Let us show that (V^*, ρ^*) and (V^*, μ^*) are representations of Lie algebras $(g, [,]_{g1})$ and $(g, [,]_{g2})$, respectively. For $\eta \in V^*$ and $v \in V$, we have

$$\begin{aligned}< \rho^*([x,y]_{g1})\eta, v > &= - < \eta, \rho([x,y]_{g1})v >\\ &= - < \eta, (\rho(x)\rho(y) - \rho(y)\rho(x))v >\\ &=< (\rho^*(x)\rho^*(y) - \rho^*(y)\rho^*(x))\eta, v > .\end{aligned}$$

In the same way, (V^*, μ^*) is a Lie representation. On the other hand

$$\begin{aligned}< (\rho^*([x,y]_{g2}) + \mu^*([x,y]_{g1}))\eta, v > &= - < \eta, (\rho([x,y]_{g2}) + \mu([x,y]_{g1}))v >\\ &= - < \eta, ([\rho(x), \mu(y)]_{gl(V)} + [\mu(x), \rho(y)]_{gl(V)})v >\\ &= - < ([\mu^*(y), \rho^*(x)]_{gl(V)} + [\rho^*(y), \mu^*(x)]_{gl(V)})\eta, v >\end{aligned}$$

$$= \ < ([\rho^*(x), \mu^*(y)]_{gl(V)} + [\mu^*(x), \rho^*(y)]_{gl(V)})\eta, v > .$$

From Proposition 2, the result follows. □

Corollary 1. *A linear operator β on g dually represents the CE-Lie algebra $(g, [,]_{g1}, [,]_{g2}, \phi_g)$ on (g, ad_1, ad_2) if and only if, for all $x, y \in g$,*

$$\beta[\phi_g(x), y]_{g1} = [x, \beta(y)]_{g1}, \ \beta[\phi_g(x), y]_{g2} = [x, \beta(y)]_{g2}.$$

Corollary 2. *Let $(g, [,]_{g1}, [,]_{g2}, \phi_g)$ be a CE-Lie algebra. Let (V, ρ, μ) be a representation of g and $\beta \in gl(V)$. Then we have the semi-product CE-Lie algebra $g \ltimes_{\rho^*,\mu^*} V^*$ with respect to $\phi_g \oplus \beta^*$.*

The notion of a matched pair of two CE-Lie algebras g and h allows us to define a CE-Lie algebra structure on the semi-direct product $g \oplus h$. In the following, $\rho, \rho_g, \mu_g : g \to gl(h)$ and $\mu, \rho_h, \mu_h : h \to gl(g)$ denote linear maps. Let us recall the following definitions.

Definition 5 ([1])**.** A matched pair of Lie algebras is a quadruple denoted by $((g, [,]_g), (h, [,]_h), \rho, \mu)$ or simply (g, h, ρ, μ) where $(g, [,]_g)$ and $(h, [,]_h)$ are Lie algebras such that, for all $x, y \in g$ and $u, v \in h$,

1) (g, μ) and (h, ρ) are representations of $(h, [,]_h)$ and $(g, [,]_g)$, respectively,
2) $\rho(x)[u, v]_h - [\rho(x)u, v]_h - [u, \rho(x)v]_h + \rho(\mu(u)x)v - \rho(\mu(v)x)u = 0$,
3) $\mu(u)[x, y]_g - [\mu(u)x, y]_g - [x, \mu(u)y]_g + \mu(\rho(x)u)y - \mu(\rho(y)u)x = 0$.

Definition 6 ([1])**.** Let $(g, [,]_{g1}, [,]_{g2})$ and $(h, [,]_{h1}, [,]_{h2})$ be two compatible Lie algebras. We call $((g, [,]_{g1}, [,]_{g2}), (h, [,]_{h1}, [,]_{h2}), \rho_g, \mu_g, \rho_h, \mu_h)$ or simply $(g, h, \rho_g, \mu_g, \rho_h, \mu_h)$ a matched pair of compatible Lie algebras g and h if, for all $k_1, k_2 \in \mathbb{K}$,

$$((g, k_1[,]_{g1} + k_2[,]_{g2}), (h, k_1[,]_{h1} + k_2[,]_{h2}), k_1\rho_g + k_2\mu_g, k_1\rho_h + k_2\mu_h)$$

is a matched pair of Lie algebras.

Definition 7 ([1])**.** A matched pair of Endo-Lie algebras is a quadruple denoted by $((g, [,]_g, \phi_g), (h, [,]_h, \phi_h), \rho, \mu)$, or simply $((g, \phi_g), (h, \phi_h), \rho, \mu)$, where $(g, [,]_g, \phi_g)$ and $(h, [,]_h, \phi_h)$ are Endo-Lie algebras such that:

a) (g, ϕ_g, μ) and (h, ϕ_h, ρ) are representations of Endo-Lie algebras (h, ϕ_h) and (g, ϕ_g), respectively,
b) (g, h, ρ, μ) is a matched pair of Lie algebras.

Definition 8. Let $(g, [,]_{g1}, [,]_{g2}, \phi_g)$ and $(h, [,]_{h1}, [,]_{h2}, \phi_h)$ be two CE-Lie algebras. We say that $((g, [,]_{g1}, [,]_{g2}, \phi_g), (h, [,]_{h1}, [,]_{h2}, \phi_h), \rho_g, \mu_g, \rho_h, \mu_h)$ or simply $((g, \phi_g), (h, \phi_h), \rho_g, \mu_g, \rho_h, \mu_h)$ is a matched pair of CE-Lie algebras g and h if, for all $k_1, k_2 \in \mathbb{K}$,

$$((g, k_1[,]_{g1} + k_2[,]_{g2}, \phi_g), (h, k_1[,]_{h1} + k_2[,]_{h2}, \phi_h), k_1\rho_g + k_2\mu_g, k_1\rho_h + k_2\mu_h)$$

is a matched pair of Endo-Lie algebras.

Remark 2. The above definition can be written, for all $x \in g,\ v \in h$, as

a) $\phi_g(\rho_h(v)x) = \rho_h(\phi_h(v))\phi_g(x)$, $\phi_g(\mu_h(v)x) = \mu_h(\phi_h(v))\phi_g(x)$,
$\phi_h(\rho_g(x)v) = \rho_g(\phi_g(x))\phi_h(v)$, $\phi_h(\mu_g(x)v) = \mu_g(\phi_g(x))\phi_h(v)$,

b) $(g, h, \rho_g, \mu_g, \rho_h, \mu_h)$ is a matched pair of compatible Lie algebras.

Proposition 4. *Under the conditions of the definition above, we have that* $((g, \phi_g), (h, \phi_h), \rho_g, \mu_g, \rho_h, \mu_h)$ *is a matched pair of CE-Lie algebras if and only if the following conditions are satisfied:*

(i) $(h, \phi_h, \rho_g, \mu_g)$ *and* $(g, \phi_g, \rho_h, \mu_h)$ *are representations of CE-Lie algebras* $(g, [,]_{g1}, [,]_{g2}, \phi_g)$ *and* $(h, [,]_{h1}, [,]_{h2}, \phi_h)$*, respectively,*

(ii) $((g, \phi_g), (h, \phi_h), \rho_g, \rho_h)$ *is a matched pair of Endo-Lie algebras* g *and* h*, likewise,* $((g, \phi_g), (h, \phi_h), \mu_g, \mu_h)$ *is matched pair of Endo-Lie algebras* $(g, [,]_{g2}, \phi_g)$ *and* $(h, [,]_{h2}, \phi_h)$*,*

(iii) *for all* $x, y \in g$ *and* $u, v \in h$*,*

a) $\mu_g(x)[u, v]_{h1} - [u, \mu_g(x)v]_{h1} - [\mu_g(x)u, v]_{h1} - \mu_g(\rho_h(v)x)u$
$+\mu_g(\rho_h(u)x)v + \rho_g(x)[u, v]_{h2} - [u, \rho_g(x)v]_{h2} + [v, \rho_g(x)u]_{h2}$
$+\rho_g(\mu_h(u)x)v - \rho_g(\mu_h(v)x)u = 0,$

b) $\rho_h(u)[x, y]_{g2} - [x, \rho_h(u)y]_{g2} - [\rho_h(u)x, y]_{g2} - \rho_h(\mu_g(y)u)x$
$+\rho_h(\mu_g(x)u)y + \mu_h(u)[x, y]_{g1} - [x, \mu_h(u)y]_{g1} + [y, \mu_h(u)x]_{g1}$
$+\mu_h(\rho_g(x)u)y - \mu_h(\rho_g(y)u)x = 0.$

Proof. "$\Longrightarrow$" Let us check Proposition 2. We have, for all $k_1, k_2 \in \mathbb{K}$ $((g, k_1[,]_{g1} + k_2[,]_{g2}, \phi_g), (h, k_1[,]_{h1} + k_2[,]_{h2}, \phi_h), k_1\rho_g + k_2\mu_g, k_1\rho_h + k_2\mu_h)$ is a matched pair for Endo-Lie algebras. By Definition 6, $(h, \phi_h, k_1\rho_g + k_2\mu_g)$ is a representation of the Endo-Lie algebra $(g, k_1[,]_{g1} + k_2[,]_{g2}, \phi_g)$, thus, for $k_1 = 1$ and $k_2 = 0$, (h, ϕ_h, ρ_g) is a representation of the Endo-Lie algebra $(g, [,]_{g1}, \phi_g)$. Likewise, (h, ϕ_h, μ_g) is a representation of $(g, [,]_{g2}, \phi_g)$. On the other hand, for $k_1 = k_2 = 1$, $((g, [,]_{g1} + [,]_{g2}, \phi_g), (h, [,]_{h1} + [,]_{h2}, \phi_h), \rho_g + \mu_g, \rho_h + \mu_h)$ is a matched pair of Endo-Lie algebras. As $(h, \phi_h, \rho_g + \mu_g)$ is a representation of $(g, [,]_{g1} + [,]_{g2}, \phi_g)$, we deduce that

$$(\rho_g + \mu_g)([x, y]_{g1} + [x, y]_{g2}) = [(\rho_g(x) + \mu_g(x)), (\rho_g(y) + \mu_g(y))]_{gl(h)}.$$

After simplifying, we obtain

$$\rho_g([x, y]_{g2}) + \mu_g([x, y]_{g1}) = [\rho_g(x), \mu_g(y)]_{gl(h)} + [\mu_g(x), \rho_g(y)]_{gl(h)}.$$

By Proposition 2 and from the above, we deduce that $(h, \phi_h, \rho_g, \mu_g)$ is a representation of a CE-Lie algebra $(g, [,]_{g1}, [,]_{g2}, \phi_g)$. In this way, we show that $(g, \phi_g, \rho_h, \mu_h)$ is a representation of $(h, [,]_{h1}, [,]_{h2}, \phi_h)$. Condition (i) is established. For the condition (ii), $((g, [,]_{g1}, \phi_g), (h, [,]_{h1}, \phi_h), \rho_g, \rho_h)$ and $((g, [,]_{g2}, \phi_g), (h, [,]_{h2}, \phi_h), \mu_g, \mu_h)$ are matched pairs, they correspond, respectively, to the cases $(k_1, k_2) = (1, 0)$ and $(k_1, k_2) = (0, 1)$. For (iii), we have that $((g, [,]_{g1} + [,]_{g2}, \phi_g), (h, [,]_{h1} + [,]_{h2}, \phi_h), \rho_g + \mu_g, \rho_h + \mu_h)$ is a matched pair of Endo-Lie algebras, thus, by Definitions 5 and 7,

$$\begin{aligned}&(\rho_g(x)+\mu_g(x))([u,v]_{h1}+[u,v]_{h2})-[(\rho_g(x)+\mu_g(x))u,v]_{h1}\\&-[(\rho_g(x)+\mu_g(x))u,v]_{h2}-[u,(\rho_g(x)+\mu_g(x))v]_{h1}-[u,(\rho_g(x)+\mu_g(x))v]_{h2}\\&+(\rho_g+\mu_g)((\rho_h(u)+\mu_h(u))x)v-(\rho_g+\mu_g)((\rho_h(v)+\mu_h(v))x)u=0.\end{aligned}$$

By developing the above calculation and since $((g,\phi_g),(h,\phi_h),\rho_g,\rho_h)$ is a matched pair of $(g,[,]_{g1},\phi_g)$ and $(h,[,]_{h1},\phi_h)$ and $((g,\phi_g),(h,\phi_h),\mu_g,\mu_h)$ is a matched pair of Endo-Lie algebras $(g,[,]_{g2},\phi_g)$ and $(h,[,]_{h2},\phi_h)$, we have

$$\begin{aligned}&\rho_g(x)([u,v]_{h1})-[\rho_g(x)u,v]_{h1}-[u,\rho_g(x)v]_{h1}+\rho_g(\rho_h(u)x)v\\&-\rho_g(\rho_h(v)x)u=0,\\&\mu_g(x)([u,v]_{h2})-[\mu_g(x)u,v]_{h2}-[u,\mu_g(x)v]_{h2}+\mu_g(\mu_h(u)x)v\\&-\mu_g(\mu_h(v)x)u=0.\end{aligned}$$

Therefore

$$\begin{aligned}&\mu_g(x)([u,v]_{h1})-[\mu_g(x)u,v]_{h1}-[u,\mu_g(x)v]_{h1}+\mu_g(\rho_h(u)x)v\\&-\mu_g(\rho_h(v)x)u+\rho_g(x)([u,v]_{h2})-[\rho_g(x)u,v]_{h2}-[u,\rho_g(x)v]_{h2}\\&+\rho_g(\mu_h(u)x)v-\rho_g(\mu_h(v)x)u=0.\end{aligned}$$

In the same way, we show that

$$\begin{aligned}&\mu_h(u)([x,y]_{g1})-[\mu_h(u)x,y]_{g1}-[x,\mu_h(u)y]_{g1}+\mu_h(\rho_g(x)u)y\\&-\mu_h(\rho_g(y)u)x+\rho_h(u)([x,y]_{g2})-[\rho_h(u)x,y]_{g2}-[x,\rho_h(u)y]_{g2}\\&+\rho_h(\mu_g(x)u)y-\rho_h(\mu_g(y)u)x=0.\end{aligned}$$

"$\Longleftarrow$" By (i), condition $a)$ of Remark 2 is verified. For $b)$, we check Definition 5. It is obvious that 1) of this definition is satisfied. For condition 2), we develop the following sum:

$$\begin{aligned}&(k_1\rho_g+k_2\mu_g)(x)(k_1[u,v]_{h1}+k_2[u,v]_{h2})-k_1[u,k_1\rho_g(x)v+k_2\mu_g(x)v]_{h1}\\&-k_2[u,k_1\rho_g(x)v+k_2\mu_g(x)v]_{h2}+k_1[v,k_1\rho_g(x)u+k_2\mu_g(x)u]_{h1}\\&+k_2[v,k_1\rho_g(x)u+k_2\mu_g(x)u]_{h2}-(k_1\rho_g+k_2\mu_g)(k_1\rho_h(v)x+k_2\mu_h(v)x)u\\&+(k_1\rho_g+k_2\mu_g)(k_1\rho_h(u)x+k_2\mu_h(u)x)v.\end{aligned}$$

The components of k_1^2, k_2^2 and k_1k_2 are, respectively,

- $\rho_g(x)[u,v]_{h1}-[u,\rho_g(x)v]_{h1}+[v,\rho_g(x)u]_{h1}-\rho_g(\mu_h(v)x)u+\rho_g(\mu_h(u)x)v$
- $\mu_g(x)[u,v]_{h2}-[u,\mu_g(x)v]_{h2}+[v,\mu_g(x)u]_{h2}-\mu_g(\mu_h(v)x)u)+\mu_g(\mu_h(u)x)v$
- $\rho_g(x)[u,v]_{h2}-[u,\rho(x)v]_{h2}+[v,\rho(x)u]_{h2}-\rho_g(\mu_h(v)x)u+\rho_g(\mu_h(u)x)v$

$+\mu_g(x)[u,v]_{h1}-[u,\rho_g(x)v]_{h1}+[v,\mu_g(x)u]_{h1}-\mu_g(\rho_h(v)x)u+\mu_g(\rho_h(u)x)v.$

By (ii), the components of k_1^2 and k_2^2 are zero. By (iii), the component of k_1k_2 is zero. For condition 3), we proceed in the same way. $\square$

For compatible Lie algebras $(g, [,]_{g1}, [,]_{g2})$ and $(h, [,]_{h1}, [,]_{h2})$ and for linear maps $\rho_g, \mu_g : g \to gl(h)$, $\rho_h, \mu_h : h \to gl(g)$, define two bilinear operations on semi-direct product $g \oplus h$, by

$$[x+u, y+v]_1 = [x,y]_{g1} + \rho_g(x)v - \rho_g(y)u + [u,v]_{h1} + \rho_h(u)y - \rho_h(v)x,$$
$$[x+u, y+v]_2 = [x,y]_{g2} + \mu_g(x)v - \mu_g(y)u + [u,v]_{h2} + \mu_h(u)y - \mu_h(v)x.$$

By [5], $(g \oplus h, [,]_1, [,]_2)$ is a compatible Lie algebra if and only if the sextuple $((g, [,]_{g1}, [,]_{g2}), (h, [,]_{h1}, [,]_{h2}), \rho_g, \mu_g, \rho_h, \mu_h)$ is a matched pair. We denote the resulting compatible Lie algebra $(g \oplus h, [,]_1, [,]_2)$ by $g \bowtie^{\rho_h,\mu_h}_{\rho_g,\mu_g} h$, or simply $g \bowtie h$.

Theorem 1. *Under the conditions of Definition* 8 *and assuming that* $(g, h, \rho_g, \mu_g, \rho_h, \mu_h)$ *is a matched pair of compatible Lie algebras,* $g \bowtie h$ *is a CE-Lie algebra, with respect to* $\phi_g \oplus \phi_h$*, if and only if* $((g, \phi_g), (h, \phi_h), \rho_g, \mu_g, \rho_h, \mu_h)$ *is a matched pair of CE-Lie algebras.*

Proof. Let $x, y \in g$ and $u, v \in h$.
"$\Longrightarrow$" We have

$$\begin{aligned}0 &= (\phi_g \oplus \phi_h)([u,x]_1) - [(\phi_g \oplus \phi_h)(u), (\phi_g \oplus \phi_h)(x)]_1 \\ &= (\phi_g \oplus \phi_h)(\rho_h(u)x - \rho_g(x)u) - \rho_h(\phi_h(u))\phi_g(x) + \rho_g(\phi_g(x))\phi_h(u) \\ &= (\phi_g(\rho_h(u)x) - \rho_h(\phi_h(u))\phi_g(x)) + (\rho_g(\phi_g(x))\phi_h(u) - \phi_h(\rho_g(x)u)) \in g \oplus h.\end{aligned}$$

We deduce $\phi_g(\rho_h(u)x) = \rho_h(\phi_h(u))\phi(x)$ and $\phi_h(\rho_g(x)u) = \rho_g(\phi_g(x))\phi_h(u)$. We proceed in the same way for the other conditions. From Remark 2 we have the result.

"$\Longleftarrow$" We show that $\phi_g \oplus \phi_h$ is a compatible Lie algebra endomorphism:

$$\begin{aligned}&(\phi_g \oplus \phi_h)([x+u, y+v]_1) \\ &\quad= (\phi_g \oplus \phi_h)([x,y]_{g1} + \rho_g(x)v - \rho_g(y)u + [u,v]_{h1} + \rho_h(u)y - \rho_h(v)x) \\ &\quad= \phi_g([x,y]_{g1} + \rho_h(u)y - \rho_h(v)x) + \phi_h([u,v]_{h1} + \rho_g(x)v - \rho_g(y)u) \\ &\quad= [\phi_g(x), \phi_g(y)]_{g1} + \phi_g(\rho_h(u)y) - \phi_g(\rho_h(v)x) + [\phi_h(u), \phi_h(v)]_{h1} \\ &\qquad + \phi_h(\rho_g(x)v) - \phi_h(\rho_g(y)u) \\ &\quad= [\phi_g(x), \phi_g(y)]_{g1} + \rho_h(\phi_h(u))\phi_g(y) - \rho_h(\phi_h(v))\phi_g(x) + [\phi_h(u), \phi_h(v)]_{h1} \\ &\qquad + \rho_g(\phi_g(x))\phi_h(v) - \rho_g(\phi_g(y))\phi_h(u) \\ &\quad= [\phi_g(x) + \phi_h(u), \phi_g(y) + \phi_h(v)]_1 = [(\phi_g \oplus \phi_h)(x+u), (\phi_g \oplus \phi_h)(y+v)]_1.\end{aligned}$$

Similarly, we see that

$$(\phi_g \oplus \phi_h)([x+u, y+v]_2) = [(\phi_g \oplus \phi_h)(x+u), (\phi_g \oplus \phi_h)(y+v)]_2.$$

The result is established. □

Remark 3. If we set $[,]_{h1} = [,]_{h2} = 0$ (h is abelian) and if $\rho_h = \mu_h = 0$, we find Proposition 3. Any linear space can be seen as an Endo-Lie algebra compatible with the representation $(V, id_V, 0, 0)$.

4. Manin triples of CE-Lie algebras

We recall a result concerning compatible Lie algebras. We will need it in the rest of this article.

Theorem 2 ([5]). *Let $(g, [,]_{g1}, [,]_{g2})$ be a compatible Lie algebra equipped with two linear maps $\Delta_1, \Delta_2 : g \to g \otimes g$. Suppose that Δ_1, Δ_2 induce a compatible Lie algebra structure on g^*. Then the following conditions are equivalent:*

(i) $(g, [,]_{g1}, [,]_{g2}, \Delta_1, \Delta_2)$ *is a compatible Lie bialgebra,*
(ii) $(g \oplus g^*, g, g^*)$ *is a standard Manin triple of compatible Lie algebras,*
(iii) $(g, g^*, ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)$ *is a matched pair of compatible Lie algebras.*

In this section, we assume that g is finite-dimensional.

Definition 9. Let $(g, [,]_1, [,]_2, \phi)$ be an CE-Lie algebra and Ω a nondegenerate symmetric bilinear form on g. Then Ω is called invariant if for all $x, y, z \in g$ and for all $k_1, k_2 \in \mathbb{K}$, $\Omega(k_1[x, y]_1 + k_2[x, y]_2, z) = \Omega(x, k_1[y, z]_1 + k_2[y, z]_2)$. In this case, $(g, [,]_1, [,]_2, \phi)$ is called a quadratic CE-Lie algebra and is denoted by $((g, [,]_1, [,]_2, \phi), \Omega)$ or simply by (g, ϕ, Ω).

Remark 4. 1) Note that, for all $k_1, k_2 \in \mathbb{K}, x, y \in g$, the following equations (a) and (b) are equivalent:

$$(a): \ \Omega(k_1[x, y]_1 + k_2[x, y]_2, z) = \Omega(x, k_1[y, z]_1 + k_2[y, z]_2),$$
$$(b): \ \Omega([x, y]_{g1}, z) = \Omega(x, [y, z]_{g1}) \text{ and } \Omega([x, y]_{g2}, z) = \Omega(x, [y, z]_{g2}).$$

2) If we take in the definition above $\phi = id_g$, we find the definition of quadratic compatible Lie algebra denoted by (g, Ω).

Let $(g, [,]_{g1}, [,]_{g2})$ be a compatible Lie algebra. Suppose that there is a compatible Lie algebra structure $(g^*, [,]_{g*1}, [,]_{g*2})$ on the dual g^* and a compatible Lie algebra structure on the semi-direct product $g \oplus g^*$ which contains both $(g, [,]_{g1}, [,]_{g2})$ and $(g^*, [,]_{g*1}, [,]_{g*2})$ as sub-algebras and for which the natural scalar product on $g \oplus g^*$, $< x + \eta, y + \xi >=< x, \xi > + < y, \eta >$, for all $x, y \in g$ $\eta, \xi \in g^*$, is invariant. The resulting algebra is denoted by $g \bowtie g^*$ and the triple $(g \bowtie g^*, g, g^*)$ is called a (standard) Manin triple of compatible algebras. This Lie algebra structure on $g \oplus g^*$ comes from a matched pair. Let Ad_1 and Ad_2 be the adjoint representations with respect to $[,]_{g*1}$ and $[,]_{g*2}$, respectively. By Theorem 2, $(g \bowtie g^*, g, g^*)$ is a standard Manin triple if and only if $(g, g^*, ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)$ is a matched pair of compatible Lie algebras. Let $\widehat{\phi} : g \to g$ denote the adjoint linear transformation of ϕ under the nondegenerate bilinear form $\Omega : \Omega(\phi(x), y) = \Omega(x, \widehat{\phi}(y))$, for all $x, y \in g$.

Proposition 5. *Let (g, ϕ, Ω) be a quadratic CE-Lie algebra. Then $\widehat{\phi}$ dually represents the CE-Lie algebra (g, ϕ) on the representation (g, ad_1, ad_2). In other words, $(g^*, \widehat{\phi}^*, ad_1^*, ad_2^*)$ is a representation of the CE-Lie algebra g. Furthermore, (g, ϕ, ad_1, ad_2) and $(g^*, \widehat{\phi}^*, ad_1^*, ad_2^*)$ are equivalent. Conversely, let (g, ϕ) be a CE-Lie algebra and let $\psi \in gl(g)$ dually represent (g, ϕ) on (g, ad_1, ad_2). If $(g^*, \psi^*, ad_1^*, ad_2^*)$ is equivalent to (g, ϕ, ad_1, ad_2), then there exists a nondegenerate invariant bilinear form Ω on g such that $\widehat{\phi} = \psi$.*

Proof. For $x, y, z \in g$, we have

$$\begin{aligned}
0 &= \Omega([\phi(x), \phi(y)]_{g1}, z) - \Omega(\phi([x, y]_{g1}), z) \\
&= \Omega(\phi(x), [\phi(y), z]_{g1}) - \Omega([x, y]_{g1}, \widehat{\phi}(z)) \\
&= \Omega(x, \widehat{\phi}([\phi(y), z]_{g1}) - \Omega(x, [y, \widehat{\phi}(z)]_{g1}).
\end{aligned}$$

We deduce that $\widehat{\phi}([\phi(y), z]_{g1}) = [y, \widehat{\phi}(z)]_{g1}$. In the same way, $\widehat{\phi}([\phi(y), z]_{g2}) = [y, \widehat{\phi}(z)]_{g2}$. $(g^*, \widehat{\phi}^*, ad_1^*, ad_2^*)$ is a representation of $(g, [,]_{g1}, [,]_{g2}, \phi)$. Define a linear map $\varphi : g \to g^*$ by $< \varphi(x), y >= \Omega(x, y)$, for all $x, y \in g$. Then φ is a linear isomorphism. Moreover, for $x, y, z \in g$ we have

$$< \varphi(ad_{1,x}y), z >= \Omega(y, [z, x]_{g1}) =< ad_{1,x}^*\varphi(y), z > .$$

We deduce that $\varphi(ad_{1,x}y) = ad_{1,x}^*\varphi(y)$. Likewise, $\varphi(ad_{2,x}y) = ad_{2,x}^*\varphi(y)$.
On the other hand

$$\begin{aligned}
< \varphi(\phi(x)), y > &= \Omega(\phi(x), y) = \Omega(x, \widehat{\phi}(y)) \\
&=< \varphi(x), \widehat{\phi}(y) >=< \widehat{\phi}^*(\phi(x)), y > .
\end{aligned}$$

We deduce that $\varphi(\phi(x)) = \widehat{\phi}^*(\varphi(x))$. Thus (g, ϕ, ad_1, ad_2) is equivalent to $(g^*, \widehat{\phi}^*, ad_1^*, ad_2^*)$ as a representation of $(g, [,]_{g1}, [,]_{g2}, \phi)$. Conversely, suppose that $\varphi : g \to g^*$ is a linear isomorphism giving the equivalence between (g, ϕ, ad_1, ad_2) and $(g^*, \psi^*, ad_1^*, ad_2^*)$. Define a bilinear form Ω on g by posing $\Omega(x, y) =< \varphi(x), y >$ for all $x, y \in g$. Then by a similar argument as above, we show that Ω is a nondegenerate invariant bilinear form on g such that $\widehat{\phi} = \psi$. The proof is complete □

Definition 10. Let (g, ϕ) be a CE-Lie algebra. We assume that (g^*, ψ^*) is also an CE-Lie algebra. A Manin triple of CE-Lie algebras is a triple $(g \bowtie g^*, g, g^*)$ of compatible Lie algebras such that $g \bowtie g^*$ is quadratic with respect to the natural scalar product and $(g \bowtie g^*, \phi \oplus \psi^*)$ is a CE-Lie algebra. We denote this Manin triple by $((g \bowtie g^*, \phi \oplus \psi^*), (g, \phi), (g^*, \psi^*))$.

Lemma 2. *Let $(g \bowtie g^*, \phi \oplus \psi^*, <; >)$ be a quadratic CE-Lie algebra.*

a) *The adjoint* $\widehat{\phi \oplus \psi^*}$ *of* $\phi \oplus \psi^*$ *with respect to the natural scalar product* $<,>$ *is* $\psi \oplus \phi^*$. *Further,* $\psi \oplus \phi^*$ *dually represents the CE-Lie algebra* $(g \bowtie g^*, \phi \oplus \psi^*)$ *on the adjoint representation.*
b) ψ *dually represents the CE-Lie algebra* (g, ϕ) *on the adjoint representation* (g, ad_1, ad_2).
c) ϕ^* *dually represents the CE-Lie algebra* (g^*, ψ^*) *on the adjoint representation* (g, Ad_1, Ad_2).

Proof. For a), if $x, y \in g$ and $\eta, \xi \in g^*$, then we have

$$\begin{aligned} < (\phi \oplus \psi^*)(x + \eta), y + \xi > &=< \phi(x) + \psi^*(\eta), y + \xi > \\ &=< \phi(x), \xi > + < y, \psi^*(\eta) > \\ &=< x, \phi^*(\xi) > + < \psi(y), \eta > \\ &=< x + \eta, (\psi \oplus \phi^*)(y + \xi) > . \end{aligned}$$

By Proposition 5, for the quadratic CE-Lie algebra $(g \bowtie g^*, \phi \oplus \psi^*, <,>)$, the linear map $\widehat{\phi \oplus \psi^*} = \psi \oplus \phi^*$ dually represents of $(g \bowtie g^*, \phi \oplus \psi^*)$ on the adjoint representation. For b), by Corollary 1, we have, for all $x, y \in g$ and $\eta, \xi \in g^*$,

$$\begin{aligned} (\psi \oplus \phi^*)[(\phi \oplus \psi^*)(x + \eta), y + \xi]_1 &= [x + \eta, (\psi \oplus \phi^*)(y + \xi)]_1, \\ (\psi \oplus \phi^*)[(\phi \oplus \psi^*)(x + \eta), y + \xi]_2 &= [x + \eta, (\psi \oplus \phi^*)(y + \xi)]_2. \end{aligned}$$

Now taking $\eta = \xi = 0$ in the above equations, we have the equalities $\psi[\phi(x), y]_{g1} = [x, \psi(y)]_{g1}$ and $\psi[\phi(x), y]_{g2} = [x, \psi(y)]_{g2}$. For c), the result is obtained for $x = y = 0$. □

Theorem 3. *Let* (g, ϕ) *be a CE-Lie algebra. Suppose that there is an CE-Lie algebra structure* (g^*, ψ^*) *on its dual space* g^*. *Then there is a Manin triple of CE-Lie algebras* $((g \bowtie g^*, \phi \oplus \psi^*), (g, \phi), (g^*, \psi^*))$ *if and only if* $((g, \phi), (g^*, \psi^*), ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)$ *is a matched pair of CE-Lie algebras.*

Proof. "$\Longrightarrow$" Suppose there is a Manin triple structure of CE-Lie algebras $((g \bowtie g^*, \phi \oplus \psi^*), (g, \phi), (g^*, \psi^*))$. By Definition 10, $(g \bowtie g^*, g, g^*)$ is a Manin triple of compatible Lie algebras. By Theorem 2, $(g, g^*, ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)$ is a matched pair of compatible Lie algebras for which the compatible Lie algebra on $g \oplus g^*$ is the compatible Lie algebra $g \bowtie g^*$. Since the homomorphism on $g \bowtie g^*$ is $\phi \oplus \psi^*$, by Lemma 2, $(g^*, \psi^*, ad_1^*, ad_2^*)$ and $(g, \phi, Ad_1^*, Ad_2^*)$ are representations of the CE-Lie algebras (g, ϕ) and (g^*, ψ^*), respectively.

"$\Longleftarrow$" If $((g, \phi), (g^*, \psi^*), ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)$ is a matched pair of CE-Lie algebras, then $(g, g^*, ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)$ is a matched pair of compatible Lie algebras. Hence, by Theorem 2, the natural scalar product $<,>$ is invariant on $g \oplus g^*$. By Theorem 1, the matched pair of CE-Lie algebras also equips the compatible Lie algebra $g \bowtie g^*$ with the endomorphism $\phi \oplus \psi^*$, giving us a quadratic of the CE-Lie algebra. □

Remark 5. If we take $\phi = id_g$ and $\psi^* = id_{g*}$, we find the theorem analogous to Theorem 3 concerning compatible Lie algebras [5].

5. Compatible Endo-Lie bialgebras

We recall some definitions and give the corresponding definitions in the context of compatible algebras.

5.1. The case of any dimension.

Definition 11 ([1]). A linear space g with a linear map $\Delta : g \to g \otimes g$ is called a Lie coalgebra if Δ is coantisymmetric, in the sense that $\Delta = -\tau\Delta$ for the flip map $\tau : g \otimes g \to g \otimes g$, and satisfies the co-Jacobian identity

$$(id + \sigma + \sigma^2)(id \otimes \Delta)\Delta = 0,$$

where $\sigma(x \otimes y \otimes z) = z \otimes x \otimes y$, for $x, y, z \in g$ and $id = id_g$.

Definition 12 ([1]). A Lie bialgebra is a pair $((g, [,]_g), \Delta)$, where $(g, [,]_g)$ is a Lie algebra, (g, Δ) is a Lie coalgebra such that, for all $x, y \in g$,

$$\Delta[x, y]_g = (ad_x \otimes id + id \otimes ad_x)\Delta y - (ad_y \otimes id + id \otimes ad_y)\Delta x. \quad (3)$$

Definition 13. Let $(g, [,]_{g1}, [,]_{g2})$ be a compatible Lie algebra. A compatible Lie bialgebra structure on $(g, [,]_{g1}, [,]_{g2})$ is a pair of linear maps $\Delta_1, \Delta_2 : g \to g \otimes g$ such that, for all $k_1, k_2 \in \mathbb{K}$, $(g, k_1[,]_{g1} + k_2[,]_{g2}, k_1\Delta_1 + k_2\Delta_2)$ is a Lie bialgebra. We denote it by $((g, [,]_{g1}, [,]_{g2}), \Delta_1, \Delta_2)$ or simply (g, Δ_1, Δ_2).

Proposition 6. *Under the assumptions of the definition above, the triple* $((g, [,]_1, [,]_2), \Delta_1, \Delta_2)$ *is a compatible Lie bialgebra if and only if, for all* $x, y \in g$, *the following conditions are satisfied:*

(a) $((g, [,]_1), \Delta_1)$ *and* $((g, [,]_2), \Delta_2)$ *are Lie bialgebras,*

(b) $(id + \sigma + \sigma^2)(id \otimes \Delta_1)\Delta_2 + (id + \sigma + \sigma^2)(id \otimes \Delta_2)\Delta_1 = 0,$

(c) $\Delta_1([x, y]_{g2}) + \Delta_2([x, y]_{g1})$

$$\begin{aligned} &= (ad_{1,x} \otimes id + id \otimes ad_{1,x})\Delta_2 y - (ad_{1,y} \otimes id + id \otimes ad_{1,y})\Delta_2 x \\ &\quad + (ad_{2,x} \otimes id + id \otimes ad_{2,x})\Delta_1 y - (ad_{2,y} \otimes id + id \otimes ad_{2,y})\Delta_1 x. \end{aligned}$$

Proof. "$\Longrightarrow$" Let $k_1, k_2 \in \mathbb{K}$. Then $((g, [,]_1), \Delta_1)$ and $((g, [,]_2), \Delta_2)$ are Lie bialgebras, they correspond to the cases $(k_1, k_2) = (1, 0)$ and $(k_1, k_2) = (0, 1)$, respectively. On the other hand, we have

$$\begin{aligned} &(id + \sigma + \sigma^2)(id \otimes (k_1\Delta_1 + k_2\Delta_2))(k_1\Delta_1 + k_2\Delta_2) \\ &\quad = k_1^2(id + \sigma + \sigma^2)(id \otimes \Delta_1)\Delta_1 + k_2^2(id + \sigma + \sigma^2)(id \otimes \Delta_2)\Delta_2 \\ &\qquad + k_1k_2((id + \sigma + \sigma^2)(id \otimes \Delta_1)\Delta_2 + (id + \sigma + \sigma^2)(id \otimes \Delta_2)\Delta_1) \\ &\quad = k_1k_2((id + \sigma + \sigma^2)(id \otimes \Delta_1)\Delta_2 + (id + \sigma + \sigma^2)(id \otimes \Delta_2)\Delta_1). \end{aligned}$$

By (a), we have the result (b). For (c), we develop the sum

$$(k_1\Delta_1 + k_2\Delta_2)(k_1[x, y]_{g1} + k_2[x, y]_{g2})$$
$$-((k_1 ad_{1,x} + k_2 ad_{2,x}) \otimes id + id \otimes (k_1 ad_{1,x} + k_1k_2 ad_{2,x}))(k_1\Delta_1 y + k_2\Delta_2 y)$$
$$+((k_1 ad_{1,y} + k_2 ad_{2,y}) \otimes id + id \otimes (k_1 ad_{1,y} + k_1k_2 ad_{2,y}))(k_1\Delta_1 x + k_2\Delta_2 x).$$

The components of k_1^2, k_2^2 and k_1k_2 are, respectively,

- $\Delta_1([x, y]_{g1}) - (ad_{1,x} \otimes id + id \otimes ad_{1,x})\Delta_1 y + (ad_{1,y} \otimes id + id \otimes ad_{1,y})\Delta_1 x$
- $\Delta_2([x, y]_{g2}) - (ad_{2,x} \otimes id + id \otimes ad_{2,x})\Delta_2 y + (ad_{2,y} \otimes id + id \otimes ad_{2,y})\Delta_2 x$
- $\Delta_1([x, y]_{g2}) + \Delta_2([x, y]_{g1})$
  $- (ad_{1,x} \otimes id + id \otimes ad_{1,x})\Delta_2 y - (ad_{2,x} \otimes id + id \otimes ad_{2,x})\Delta_1 y$
  $+ (ad_{1,y} \otimes id + id \otimes ad_{1,y})\Delta_2 x + (ad_{2,y} \otimes id + id \otimes ad_{2,y})\Delta_1 x.$

By equation (3), we have the result (c).

"$\Longleftarrow$" It suffices to show that $(g, k_1\Delta_1 + k_2\Delta_2)$ is a Lie coalgebra and that the equation (3) holds. By (a) and (b), it is obvious that co-antisymmetry and co-Jacobian identity are satisfied. From the calculation we have just made, it is clear that equation (3) is true. □

Definition 14 ([1])**.** An Endo-Lie coalgebra is a Lie coalgebra (g, Δ) together with a Lie coalgebra endomorphism $\psi \in gl(g)$ such that $(\psi\otimes\psi)\Delta = \Delta\psi$. It is denoted by (g, Δ, ψ).

Remark 6. If g is finite-dimensional, then ψ is a Lie coalgebra endomorphism of (g, Δ) if and only if ψ^* is a Lie algebra endomorphism of g^*. Indeed, (g, Δ) is a Lie coalgebra if and only if (g^*, Δ^*) is a Lie algebra [4]. On the other hand, for $\eta, \xi \in g^*$ and $x \in g$, we have

$$\begin{aligned} < \psi^*([\eta, \xi]_g), x >=< \eta \otimes \xi, (\Delta\psi)x > &=< \eta \otimes \xi, (\psi \otimes \psi)\Delta x > \\ &=< [\psi^*(\eta), \psi^*(\xi)]_{g*}, x > . \end{aligned}$$

Definition 15. A quadruple $(g, \Delta_1, \Delta_2, \psi)$ is a compatible Endo-Lie coalgebra if, for all $k_1, k_2 \in \mathbb{K}$, $((g, k_1[,]_{g1} + k_2[,]_{g2}), k_1\Delta_1 + k_2\Delta_2, \psi)$ is an Endo-Lie coalgebra.

Proposition 7. *$((g, [,]_{g1}, [,]_{g2}), \Delta_1, \Delta_2, \psi)$ is a CE-Lie coalgebra if and only if the following conditions are satisfied:*

a) $((g, [,]_{g1}), \Delta_1, \psi)$ *and* $((g, [,]_{g2}), \Delta_2, \psi)$ *are Endo-Lie coalgebras,*
b) $(id + \sigma + \sigma^2)(id \otimes \Delta_1)\Delta_2 + (id + \sigma + \sigma^2)(id \otimes \Delta_2)\Delta_1 = 0.$

Proof. The proof is obvious. □

Definition 16. The quadruple $((g, [,]_{g1}, [,]_{g2}, \phi), \Delta_1, \Delta_2, \psi)$, denoted simply by $((g, \phi), \Delta_1, \Delta_2, \psi)$, is a CE-Lie bialgebra if

(a) (g, Δ_1, Δ_2) is a compatible Lie bialgebra,

(b) (g, ϕ) is a CE-Lie algebra,

(c) $(g, \Delta_1, \Delta_2, \psi)$ is a CE-Lie coalgebra,

(d) $(id \otimes \phi)\Delta_1 = (\psi \otimes id)\Delta_1\phi$, $(id \otimes \phi)\Delta_2 = (\psi \otimes id)\Delta_2\phi$,
$\psi[\phi(x), y]_{g1} = [x, \psi(y)]_{g1}$, and $\psi[\phi(x), y]_{g2} = [x, \psi(y)]_{g2} \quad \forall x, y \in g$.

5.2. Finite dimension case. Under our assumption of a finite dimension, (g, Δ) is a Lie coalgebra if and only if (g^*, Δ^*) is a Lie algebra [4]. As a consequence of this assumption, the compatible Lie coalgebra structure (g, Δ_1, Δ_2) is equivalent to the compatible Lie algebra structure $(g^*, \Delta_1^*, \Delta_2^*)$. By Theorem 2, $(g, [,]_{g1}, [,]_{g2}, \Delta_1, \Delta_2)$ is a compatible Lie bialgebra if and only if $(g, g^*, ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)$ is a matched pair of compatible Lie algebras, where Ad_1 and Ad_2 are the adjoint representations concerning Δ_1^* and Δ_2^*, respectively.

Theorem 4. *Let $(g, [,]_{g1}, [,]_{g2}, \phi)$ be CE-Lie algebras. Suppose that there is a CE-Lie algebra $(g^*, [,]_{g*1}, [,]_{g*2}, \psi^*)$ on the linear dual g^* of g. Let $\Delta_1, \Delta_2 : g \to g \otimes g$ denote the linear duals of the multiplications $[,]_{g*1}$ and $[,]_{g*2}$, respectively. Then $((g, \phi), (g^*, \psi^*), ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)$ is a matched pair of CE-Lie algebras if and only if the triple $((g, \phi), \Delta_1, \Delta_2, \psi)$ is a CE-Lie bialgebra.*

Proof. By Remark 2, $((g, \phi), (g^*, \psi^*), ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)$ is a matched pair of CE-Lie algebras if and only if $(g, g^*, ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)$ is a matched pair of compatible Lie algebras $(g, [,]_{g1}, [,]_{g2})$ and $(g^*, \Delta_1^*, \Delta_2^*)$. For all $x \in g, \eta \in g^*$,

$$\psi^*(ad_{1,x}^*\eta) = ad_{1,\phi(x)}^*\psi^*(\eta),\ \psi^*(ad_{2,x}^*\eta) = ad_{2,\phi(x)}^*\psi^*(\eta),$$
$$\phi(Ad_{1,\eta}x) = Ad_{1,\psi^*(\eta)}\phi(x),\ \phi(Ad_{2,\eta}x) = Ad_{2,\psi^*(\eta)}\phi(x).$$

By Theorem 2, $(g, g^*, ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)$ being a matched pair of compatible Lie algebras is equivalent to (g, Δ_1, Δ_2) being a compatible Lie bialgebra. On the other hand, the above conditions are equivalent to

$$\psi([\phi(x), y]_{g1} = [x, \psi(y)]_{g1},\ \psi([\phi(x), y]_{g2} = [x, \psi(y)]_{g2},$$
$$(id \otimes \phi)\Delta_1 = (\psi \otimes id)\Delta_1\phi,\ (id \otimes \phi)\Delta_2 = (\psi \otimes id)\Delta_2\phi.$$

Since conditions (b), (c) of Definition 16, are satisfied, we say that the sextuple $((g, \phi), (g^*, \psi^*), ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)$ is a matched pair of CE-Lie algebras if and only if $((g, \phi), \Delta_1, \Delta_2, \psi)$ is a CE-Lie bialgebra. □

Remark 7. If we take $\phi = id_g$ and $\psi^* = id_{g*}$, we find a theorem analogous to Theorem 4 concerning compatible Lie algebras [5].

Under our assumption of finite dimension, combining Theorem 3 and Theorem 4, we have the following result.

Theorem 5. *Let $(g, [,]_{g1}, g, [,]_{g2}, \phi)$ be a CE-Lie algebra. Suppose that there is a CE-Lie algebra $(g^*, [,]_{g*1}, g, [,]_{g*2}, \psi^*)$ on the linear dual g^* of g.*

Let $\Delta_1, \Delta_2 : g \to g \otimes g$ *denote the linear dual of the multiplications* $[,]_{g*1}$ *and* $[,]_{g*2}$*, respectively. Then the following statements are equivalent:*

(*i*) $((g,\phi),(g^*,\psi^*), ad_1^*, ad_2^*, Ad_1^*, Ad_2^*)$ *is a matched pair of CE-Lie algebras* $(g,[,]_{g1}, g,[,]_{g2},\phi)$ *and* $(g^*,[,]_{g*1}, g,[,]_{g*2},\psi^*)$,

(*ii*) *there is a Manin triple of CE-Lie algebras,*

(*iii*) *the triple* $((g,\phi),\Delta_1,\Delta_2,\psi)$ *is a CE-Lie bialgebra.*

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