ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 28, Number 1, June 2024
Available online at https://ojs.utlib.ee/index.php/ACUTM

A refinement of a lemma by Aron, Cascales, and Kozhushkina on Asplund operators

Jaagup Kirme

Abstract. We prove a refinement of a lemma by Aron, Cascales, and Kozhushkina on Asplund operators. Refinements of a Bishop–Phelps–Bollobás type theorem for Asplund operators with values in spaces $C_0(L)$ by the same authors, and an extension of this theorem for Asplund operators with values in uniform algebras by Cascales, Guirao, and Kadets, follow.

Throughout this note, X and Y are (real or complex) Banach spaces, S_X and B_X denote, respectively, the unit sphere and the closed unit ball of X, and $\mathcal{L}(X,Y)$ denotes the space of bounded linear operators from X to Y. The symbol w^* stands for the weak* topology on the dual space X^* of X.

The objective of this note is to prove the following refinement of [1, Lemma 2.3] and [4, Lemma 3.5] (in [1], the lemma was proven for real Banach spaces). For the concepts of an *Asplund operator* and 1*-normingness* we refer, respectively, to [4, Section 3] and [1, Section 2].

Lemma 1 (cf. [1, Lemma 2.3] and [4, Lemma 3.5]). *Let $T\colon X \to Y$ be an Asplund operator with $\|T\| = 1$, let $0 < \varepsilon < \sqrt{2}$, and let $x_0 \in S_X$ be such that*

$$\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}.$$

Then, for every 1*-norming set $B \subset B_{Y^*}$, there exist*

(1) *a w^*-open set $U \subset X^*$ such that $U \cap T^*(B) \neq \varnothing$,*
(2) *elements $y^* \in S_{X^*}$ and $u_0 \in S_X$ with $|y^*(u_0)| = 1$ such that*

$$\|x_0 - u_0\| < \varepsilon \quad \textit{and} \quad \sup_{z^* \in U \cap T^*(B)} \|z^* - y^*\| < \varepsilon.$$

Received January 11, 2024.
2020 *Mathematics Subject Classification.* Primary 46B20, 46B22, 46J10; Secondary 47B07, 47B10.
Key words and phrases. Bishop–Phelps–Bollobás, Asplund operator, norm-attaining operator, uniform Banach algebra.
https://doi.org/10.12697/ACUTM.2024.28.10

Recall that a subset C of X^* is said to be *norm-fragmented* in the w^*-topology if, for every non-empty subset A of C and every $\varepsilon > 0$, there exists a non-empty w^*-open subset U of X^* such that $A \cap U \neq \varnothing$ and the norm-diameter $\operatorname{diam}_{\|\cdot\|}(A \cap U) < \varepsilon$ (see [1, Section 2] and [7, §1]).

Proof of Lemma 1. We follow the main idea of the proof of [1, Lemma 2.3]. As in [1, proof of Lemma 2.3], first observe that the adjoint operator T^* maps B_{Y^*} to a w^*-compact subset of X^* which is norm-fragmented in the w^*-topology.

Let $B \subset B_{Y^*}$ be a 1-norming subset. Pick a positive real number $\gamma < \varepsilon$ such that $\|Tx_0\| > 1 - \frac{\gamma^2}{2}$. Since B is 1-norming, there exists $b_0^* \in B$ such that

$$|(T^*b_0^*)(x_0)| = |b_0^*(Tx_0)| > 1 - \frac{\gamma^2}{2}.$$

Now the set

$$U_1 := \left\{ x^* \in X^* : |x^*(x_0)| > 1 - \frac{\gamma^2}{2} \right\}$$

is w^*-open and

$$T^*b_0^* \in U_1 \cap T^*(B) \subset T^*(B_{Y^*}) \subset B_{X^*}.$$

Since the set $T^*(B_{Y^*})$ is norm-fragmented in the w^*-topology and the set $U_1 \cap T^*(B)$ is non-empty, there exists a w^*-open set $U_2 \subset X^*$ such that $(U_1 \cap T^*(B)) \cap U_2 \neq \varnothing$ and

$$\operatorname{diam}_{\|\cdot\|}((U_1 \cap T^*(B)) \cap U_2) \leq \varepsilon - \gamma. \tag{1}$$

Defining $U := U_1 \cap U_2$, the set U is a w^*-open subset of X^*. Fix any $x_0^* \in U \cap T^*(B)$; then $x_0^* \in T^*(B) \subset T^*(B_{Y^*}) \subset B_{X^*}$ and, since $x_0^* \in U_1$, we have

$$|x_0^*(x_0)| > 1 - \frac{\gamma^2}{2}.$$

From [6, Corollary 2.4] it follows that there exist $y^* \in S_{X^*}$ and $u_0 \in S_X$ such that $|y^*(u_0)| = 1$, and

$$\|x_0 - u_0\| < \gamma < \varepsilon \quad \text{and} \quad \|x_0^* - y^*\| < \gamma.$$

Now, for any $z^* \in U \cap T^*(B)$, we have $\|z^* - x_0^*\| \leq \varepsilon - \gamma$ by (1) and therefore

$$\sup_{z^* \in U \cap T^*(B)} \|z^* - y^*\| \leq \sup_{z^* \in U \cap T^*(B)} \|z^* - x_0^*\| + \|x_0^* - y^*\| < \varepsilon.$$

□

Remark 1. The estimate "$\|z^* - y^*\| < \varepsilon$ for every $z^* \in U \cap T^*(B)$" in Lemma 1 refines the estimate "$\|z^* - y^*\| < 3\varepsilon$ for every $z^* \in U \cap T^*(B)$" in [1, Lemma 2.3]. "One ε" of this refinement is achieved by using instead of the original Bishop–Phelps–Bollobás theorem [2, Theorem 1] (or more precisely, its corollary [1, Remark 2.2]) its sharper version [6, Corollary 2.4].

"The other ε" of this refinement is achieved by choosing the b_0^*, U_1, U_2, and y^* in a more precise manner (starting by picking a positive real number γ with $\gamma < \varepsilon$ such that $\|Tx_0\| > 1 - \frac{\gamma^2}{2}$).

Using Lemma 1 instead of [1, Lemma 2.3] in the proof of [1, Theorem 2.4] gives the following refinement of [1, Theorem 2.4] (with the estimate $\|T - S\| < 3\varepsilon$ in [1, Theorem 2.4] refined to $\|T - S\| < \varepsilon$).

Theorem 1 (cf. [1, Theorem 2.4])**.** *Let L be a locally compact Hausdorff space, let $T\colon X \to C_0(L)$ be an Asplund operator with $\|T\| = 1$, let $0 < \varepsilon < \sqrt{2}$, and let $x_0 \in S_X$ be such that*

$$\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}.$$

Then there exist $u_0 \in S_X$ and an Asplund operator $S \in S_{\mathcal{L}(X,C_0(L))}$ such that

$$\|Su_0\| = 1, \quad \|x_0 - u_0\| < \varepsilon, \quad \textit{and} \quad \|T - S\| < \varepsilon.$$

Theorem 1 was partially generalised and refined by Cascalez, Guirao, and Kadets in [4, Theorem 3.6]. The refinement in Lemma 1 when compared to [4, Lemma 3.5] carries on to [4, Theorem 3.6]: the estimate $\|T - \tilde{T}\| < 2\varepsilon$ in [4, Theorem 3.6] is refined to $\|T - \tilde{T}\| < \varepsilon$ in the following theorem.

Theorem 2 (cf. [4, Theorem 3.6])**.** *Let $\mathfrak{A}$ be a uniform algebra, let $T\colon X \to \mathfrak{A}$ be an Asplund operator with $\|T\| = 1$, let $0 < \varepsilon < \sqrt{2}$, and let $x_0 \in S_X$ be such that*

$$\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}.$$

Then there exist $u_0 \in S_X$ and an Asplund operator $\tilde{T} \in S_{\mathfrak{L}(X,\mathfrak{A})}$ such that

$$\|\tilde{T}u_0\| = 1, \quad \|x_0 - u_0\| < \varepsilon, \quad \textit{and} \quad \|T - \tilde{T}\| < \varepsilon.$$

Proof. Pick a positive real number γ with $\gamma < \varepsilon$ satisfying $\|Tx_0\| > 1 - \frac{\gamma^2}{2}$, and let a positive real number ε' satisfy $4\varepsilon' < \varepsilon$ and $2\varepsilon' < \varepsilon - \gamma$. The result is obtained by following the proof of [4, Theorem 3.6] using this ε' and applying Lemma 1 with γ in the place of ε instead of [4, Lemma 3.5] as in the proof of [4, Theorem 3.6]. □

Remark 2. The estimates achieved in Theorems 1 and 2 are in some sense optimal: see the remark in [2].

Remark 3. Theorem 1 was proven in [3, Theorem 1.2] for the real case by a different argument (using a characterisation of Asplund operators via Fréchet differentiability properties of convex functions [3, Theorem 3.4]). In fact, both Theorems 1 and 2 are contained in [5, Theorem 3.4]. Indeed, every Asplund operator $T \in \mathcal{L}(X, Y)$ is Γ-flat for every $\Gamma \subset B_{Y^*}$ by [5, Example A], every uniform algebra has simple ACK structure by [5, Corollary 4.6] (we refer to [5, Definitions 2.8 and 3.1] for the concepts of Γ-flatness and ACK structure), and it is straightforward to verify that also the space $C_0(L)$ has simple ACK structure. To see this, defining $\Gamma := \{\delta_s \colon s \in L\}$ where $\delta_s \colon C_0(L) \ni f \mapsto f(s)$ is the point evaluation functional, observe that $\Gamma \subset B_{Y^*}$ is a 1-norming set. Let $\varepsilon > 0$ and let $U \subset \Gamma$ be a non-empty relatively w^*-open subset. Since the mapping $L \ni s \mapsto \delta_s \in \Gamma$ is continuous with respect to the relative w^*-topology on Γ, the original W of U with respect to this mapping is an open subset of L. Picking an arbitrary $s \in W$, Urysohn's lemma produces a function $e \in C_0(L)$ with values in the segment $[0, 1]$ such that $e(s) = 1$ and $e = 0$ outside the set W. Define $V \colon = \{v^* \in U \colon |v^*(e) - 1| < \varepsilon\}$; then V is a non-empty (with $\delta_s \in V$) relatively w^*-open subset of U; thus the original W' of V with respect to the mapping $L \ni s \mapsto \delta_s$ is an open subset of L; hence Urysohn's lemma produces a function $h \in C_0(L)$ with values in the segment $[0, 1]$ such that $h(s) = 1$ and $h = 0$ outside the set W'. Defining $x_1^* := \delta_s$, and $F \colon C_0(L) \ni f \mapsto h \cdot g \in C_0(L)$, we have x_1^* and $F \in \mathcal{L}(C_0(L), C_0(L))$, and the conditions (I)–(III), (IV)', (V), and (VI) in [5, Definition 3.1] hold.

Acknowledgements

Research is supported by the Estonian Research Council grant PRG1901. The author thanks the anonymous referee for the swift review.

References

[1] R. M. Aron, B. Cascales, and O. Kozhushkina, *The Bishop–Phelps–Bollobás theorem and Asplund operators*, Proc. Amer. Math. Soc. **139** (2011), no. 10, 3553–3560.

[2] B. Bollobás, *An extension to the theorem of Bishop and Phelps*, Bull. London Math. Soc. **2** (1970), 181–182.

[3] L. X. Cheng, Q. J. Cheng, K. K. Xu, W. Zhang, and Z. M. Zheng, *A Bishop–Phelps–Bollobás theorem for Asplund operators*, Acta Math. Sin. (Engl. Ser.) **36** (2020), no. 7, 765–782.

[4] B. Cascales, A. J. Guirao, and V. Kadets, *A Bishop–Phelps–Bollobás type theorem for uniform algebras*, Adv. Math. **240** (2013), 370–382.

[5] B. Cascales, A. J. Guirao, V. Kadets, and M. Soloviova, *Γ-flatness and Bishop-Phelps-Bollobás type theorems for operators*, J. Funct. Anal. **274** (2018), no. 3, 863–888.

[6] M. Chica, V. Kadets, M. Martín, S. Moreno-Pulido, and F. Rambla-Barreno, *Bishop–Phelps–Bollobás moduli of a Banach space*, J. Math. Anal. Appl. **412** (2014), no. 2, 697–719.

[7] I. Namioka, *Radon-Nikodým compact spaces and fragmentability*, Mathematika **34** (1987), no. 2, 258–281.

Institute of Mathematics and Statistics, University of Tartu, Narva mnt 18, 51009, Tartu, Estonia

E-mail address: jaagup.kirme@ut.ee