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Structure of BiHom-pre-Poisson algebras

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Abstract. In the current research paper, we define and investigate the structure of a BiHom-pre-Poisson algebra. This algebraic structure is defined by two products "$\wedge$", "$\diamond$" and two linear maps f, g on A. In particular, $(A, \wedge, f, g)$ is a BiHom-Zinbiel algebra and $(A, \diamond, f, g)$ is a BiHom-pre-Lie algebra. Additionally two compatibility conditions between $\wedge$ and $\diamond$ are verified. Our first main results are devoted to demonstrating that if A is a BiHom-pre-Lie algebra, then a tensorial algebra of A has a structure of a BiHom-pre-Poisson algebra. Furthermore, we prove that any BiHom-Poisson algebra together with a Rota–Baxter operator defines a BiHom-pre-Poisson algebra. Finally, we define the structure of a dual BiHom-pre-Poisson algebra and we demonstrate that an averaging operator on a BiHom-Poisson algebra gives rise to a dual BiHom-pre-Poisson algebra.

1. Introduction

The origin of Hom-algebra structures dates back to the physics literature of 1990's, with regard to quantum deformation of some algebras of vector fields, which satisfy a modified Jacobi identity involving an algebra morphism (such algebras were called Hom-Lie algebras, see [7], [8]). Other Hom-algebraic structures have been introduced afterwards like a Hom-Poisson algebra, a Hom-pre-Poisson algebra etc.

Recall that a Poisson algebra $(A, \mu, \{-,-\})$ consists of a commutative associative algebra (A, μ) together with a Lie algebra $(A, \{-,-\})$, satisfying a compatibility condition called the Leibniz rule:

$$\{x, \mu(y,z)\} = \mu(\{x,y\}, z) + \mu(y, \{x,z\}), \text{ for all } x, y, z \in A.$$

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Moreover, a Poisson algebra is widely invested in a panoply of many fields in mathematics and physics. In particular, in mathematics Poisson algebras play an intrinsic role in quantum groups (see [4]) and deformation of a commutative associative algebra (see [5]). Furthermore, a definition of a Hom-Poisson algebra was elaborated by Laurent-Gengous and Teles in [10]. In this respect, a Hom-Poisson algebra $(A, \mu, \{-,-\}, f)$ consists of a commutative Hom-associative algebra (A, μ, f) together with a Hom-Lie algebra $(A, \{-,-\}, f)$ satisfying a compatibility condition called the Hom-Leibniz equation:

$$\big\{f(x), \mu(y,z)\big\} = \mu\big(\{x,y\}, f(z)\big) + \mu\big(f(y), \{x,z\}\big), \text{ for all } x,y,z \in A.$$

It is to be noted that, a pre-Poisson algebra was introduced by Aguiar in [1]. This algebra $(A, \circ, *)$ corresponds to both a Zinbiel algebra $(A, *)$ and a pre-Lie algebra $(A, \circ)$ satisfying two compatibility conditions: for all $x, y, z \in A$,

$$(x \circ y - y \circ x) * z = x \circ (y * z) - y * (x \circ z),$$
$$(x * y + y * x) \circ z = x * (y \circ z) + y * (x \circ z).$$

Basically, a Zinbiel algebra was introduced by Loday (see [13]). This algebra is also known as a pre-commutative algebra. Besides, a product of a Zinbiel algebra corresponds to a binary law whose symmetrization is a commutative associative product. Addionally, a pre-Lie algebra arose as early as in 1896 in the work of Cayley (see [2]) and it has been addressed by Livernet and Chapoton in [3]. Furthermore, a product of a pre-Lie algebra stands for a binary law whose antisymmetrization is a Lie bracket. They have connections with some other concepts such as Rota–Baxter operator, an averaging operator etc.

In [12], Liu, Makhlouf and Song modified the structure of a pre-Poisson algebra using a morphism to define a structure of a Hom-pre-Poisson algebra. They examined its relationships with a Hom-Poisson algebra.

Recently, BiHom-type algebras have been further developed in mathematics and mathematical physics. In [6], Graziani, Makhlouf, Menini and Panaite introduced the structure of BiHom-algebras $\big(A, m, f, g\big)$ such that a product m is modified by two morphisms f and g. In particular, the structure of BiHom-Lie algebras is provided by a vector space A, a bilinear map $[-,-]$ and two morphisms f and g satisfying the following conditions: for all elements $x, y, z \in A$,

(1) $f\big([x,y]\big) = \big[f(x), f(y)\big]$ and $g\big([x,y]\big) = \big[g(x), g(y)\big]$,
(2) $\big[g(x), f(y)\big] = -\big[g(y), f(x)\big]$,
(3) $\circlearrowleft_{x,y,z} \big[g^2(x), \big[g(y), f(z)\big]\big] = 0$.

These conditions (1), (2) and (3) are called multiplicativity, BiHom-skew symmetry and the BiHom-Jacobi identity, respectively. In particular, when the linear maps f and g are the same, a structure of a BiHom-Lie algebra reduces to a Hom-Lie algebra.

The basic objective of the current paper is to define the structure of a BiHom-pre-Poisson algebra $\big(A, \wedge, \diamond, f, g\big)$ such that $\big(A, \wedge, f, g\big)$ is a BiHom-Zinbiel algebra and $\big(A, \diamond, f, g\big)$ is a BiHom-pre-Lie algebra satisfying two compatibility conditions between $\wedge$ and $\diamond$. On the one hand, we demonstrate that if $\big(A, \diamond, f, g\big)$ is a BiHom-pre-Lie algebra, then a tensorial algebra that is denoted by $T(A)$ has the structure of a BiHom-pre-Poisson algebra $\big(T(A), \curlywedge, \Diamond, f, g\big)$. On the other hand, we prove that when $\big(A, \mu, \{-,-\}, f, g\big)$ is a BiHom-Poisson algebra and $\beta : A \longrightarrow A$ is a Rota–Baxter-operator on A, then $\big(A, \wedge, \diamond, f, g\big)$ is a BiHom-pre-Poisson algebra. Moreover, we equally aim to demonstrate that if $\big(A, \mu, [-,-], f, g\big)$ is a BiHom-Poisson algebra and $\alpha : A \longrightarrow A$ is an averaging operator on A, then we can define the structure of a dual BiHom-pre-Poisson algebra that is indicated by $\big(A, \bullet, \{-,-\}, f, g\big)$.

This paper is laid out as follows. In Section 2, we recall some definitions of other algebras in case of BiHom-type which will be used in the next sections. In Section 3, we introduce the structure of a BiHom-pre-Poisson algebra and specify its relationship to a BiHom-Poisson algebra. Moreover, we build up a BiHom-pre-Poisson algebra associated with a BiHom-pre-Lie algebra. In addition, we use a Rota–Baxter operator on a BiHom-Poisson algebra in order to build up a BiHom-pre-Poisson algebra. In Section 4, we illustrate a dual BiHom-pre-Poisson algebra through a combination of a BiHom-permutative algebra and a BiHom-Leibniz algebra. Finally, we corroborate that an averaging operator on a BiHom-Poisson algebra gives rise to a dual BiHom-pre-Poisson algebra.

Throughout this paper, $\mathbb{K}$ denotes a field of characteristic zero. All algebraic structures are left-handed versions. Furthermore, for the composition of maps f, g, φ and ψ, we write either $f \circ g$, $f \circ \alpha$, $f \circ \beta$, $g \circ \alpha$ and $g \circ \beta$ or simply fg, $f\alpha$, $f\beta$, $g\alpha$ and $g\beta$.

2. Preliminaries: definitions and properties

This section involves basic definitions in the case of a BiHom-type algebra which will be used in the next sections. It also displays pertinent examples illustrating this specific algebra type. Our main references are [6, 11].

2.1. BiHom-pre-Lie algebras.

Definition 1. The term *BiHom-pre-Lie algebra* stands for a quadruple $\big(A, \diamond, f, g\big)$ involving a vector space A, a bilinear product $\diamond : A \times A \longrightarrow A$ and two commuting morphisms $f, g : A \longrightarrow A$ such that $f(x\diamond y)=f(x)\diamond f(y)$ and $g(x\diamond y)=g(x)\diamond g(y)$, satisfying the following condition: for all $x, y, z \in A$,

$$fg(x)\diamond\big(f(y)\diamond z\big)-\big(g(x)\diamond f(y)\big)\diamond g(z) = fg(y)\diamond\big(f(x)\diamond z\big)-\big(g(y)\diamond f(x)\big)\diamond g(z).$$

Example 1. Let A be a 2-dimensional vector space and $\mathcal{B} = \{e_1, e_2\}$ be a basis of A. On A, we define the following nonzero product by

$$e_2 \diamond e_1 = e_1, \quad e_2 \diamond e_2 = e_2 - e_1.$$

Consider the linear maps $f,\, g : A \longrightarrow A$ defined on the basis elements by

$$f(e_1) = e_1,\ f(e_2) = e_2 - e_1,\ g(e_1) = -e_1 \text{ and } g(e_2) = e_2.$$

The quadruple $\big(A, \diamond, f, g\big)$ is a BiHom-pre-Lie algebra.

Definition 2. Let $\big(A, \diamond, f, g\big)$ and $\big(A', \diamond', \varphi, \psi\big)$ be two BiHom-pre-Lie algebras. A linear map $\alpha : A \longrightarrow A'$ is said to be a *morphism of BiHom-pre-Lie algebras* if $\alpha(x \diamond y) = \alpha(x) \diamond' \alpha(y)$ for all $x, y \in A$, as well as $\alpha f = \varphi\alpha$ and $\alpha g = \psi\alpha$.

Proposition 1. *Let $(A, \diamond)$ be a pre-Lie algebra and let $f,\, g : A \longrightarrow A$ be two commuting morphisms of a pre-Lie algebra. Then $\big(A, \diamond_{(f,g)}, f, g\big)$ is a BiHom-pre-Lie algebra obtained by composition, where $x \diamond_{(f,g)} y = f(x) \diamond g(y)$ for all $x, y \in A$.*

Remark 1. Let $(A, \diamond)$ be a pre-Lie algebra and let $f,\, g : A \longrightarrow A$ be two commuting morphisms of a pre-Lie algebra. Assume that $(A', \diamond')$ is another pre-Lie algebra and $\varphi,\, \psi : A' \longrightarrow A'$ are two commuting morphisms of a pre-Lie algebra satisfying $\alpha \circ f = \varphi \circ \alpha$ and $\alpha \circ g = \psi \circ \alpha$. Then $\alpha : \big(A, \diamond_{(f,g)}, f, g\big) \longrightarrow \big(A', \diamond'_{(\varphi,\psi)}, \varphi, \psi\big)$ is a morphism of BiHom-pre-Lie algebras.

Definition 3. The term *BiHom-Lie algebra* stands for a quadruple $\big(A, [-,-], f, g\big)$ involving a vector space A, a bilinear map $[-,-] : A \times A \longrightarrow A$ and two linear maps $f,\, g : A \longrightarrow A$ satisfying the following conditions: for any $x, y, z \in A$,

$$[g(x), f(y)] = -[g(y), f(x)] \qquad \text{(BiHom-skew-symmetry)}, \tag{1}$$

$$\circlearrowleft_{x,y,z} [g^2(x), [g(y), f(z)]] = 0 \qquad \text{(BiHom-Jacobi-identity)}. \tag{2}$$

Definition 4. Let $\big(A, [-,-], f, g\big)$ and $\big(A', [-,-]', \varphi, \psi\big)$ be two BiHom-Lie algebras. A linear map $\alpha : A \longrightarrow A'$ is said to be a *morphism of BiHom-Lie algebras* if $\alpha([x,y]) = [\alpha(x), \alpha(y)]'$ for all $x, y \in A$, as well as $\alpha f = \varphi\alpha$ and $\alpha g = \psi\alpha$.

Proposition 2. *Let $(A, \diamond, f, g)$ be a BiHom-pre-Lie algebra such that f and g are bijective. Define a new operation on A by*

$$[x, y] = x \diamond y - f^{-1}g(y) \diamond fg^{-1}(x),\ \forall x, y \in A.$$

Then $\big(A, [-,-], f, g\big)$ is a BiHom-Lie algebra.

Definition 5. A *right BiHom-pre-Lie algebra* is a quadruple $\big(A, \circ, f, g\big)$ involving a vector space A, a bilinear product $\circ : A \times A \longrightarrow A$ and two commuting morphisms $f,\, g : A \longrightarrow A$ such that, for all $x, y \in A$, $f(x \circ y) = f(x) \circ f(y)$ and $g(x \circ y) = g(x) \circ g(y)$, satisfying the following condition: for all $x, y, z \in A$,

$$f(x)\circ\big(g(y)\circ f(z)\big)-\big(g(x)\circ g(y)\big)\circ fg(z)=$$
$$f(x)\circ\big(g(z)\circ f(y)\big)-\big(g(x)\circ g(z)\big)\circ fg(y).$$

Remark 2. If $\big(A,\diamond,f,g\big)$ is a left BiHom-pre-Lie algebra and we consider a new product $x\circ y=y\diamond x$, for all $x,y\in A$, then $\big(A,\circ,g,f\big)$ is a right BiHom-pre-Lie algebra.

2.2. BiHom-Zinbiel algebras.

Definition 6. A *BiHom-Zinbiel algebra* is a quadruple $\big(A,\wedge,f,g\big)$ involving a vector space A, a bilinear product $\wedge: A\times A\longrightarrow A$ and two commuting morphisms $f,\, g: A\longrightarrow A$ such that for all $x,y\in A$, $f(x\wedge y)=f(x)\wedge f(y)$ and $g(x\wedge y)=g(x)\wedge g(y)$, satisfying the following conditions: for all $x,y,z\in A$,

$$fg(x)\wedge\big(f(y)\wedge z\big)=\big(g(y)\wedge f(x)\big)\wedge g(z)+\big(g(x)\wedge f(y)\big)\wedge g(z),\qquad(3)$$

$$fg(x)\wedge\big(g(z)\wedge f(y)\big)=g^2(z)\wedge\big(f(x)\wedge f(y)\big).\qquad(4)$$

Example 2. Let A be a 2-dimensional vector space and $\mathcal{B}=\{e_1,e_2\}$ be a basis of A. On A, we define the following products:

$$e_1\wedge e_1=e_2,\quad e_1\wedge e_2=e_2\wedge e_1=0,\quad e_2\wedge e_2=0.$$

Consider the linear maps $f,\, g: A\longrightarrow A$ defined on the basis elements by

$$f(e_1)=-e_1,\ f(e_2)=e_2,\ g(e_1)=e_2 \text{ and } g(e_2)=-e_2.$$

The quadruple $\big(A,\wedge,f,g\big)$ is then a BiHom-Zinbiel algebra.

Definition 7. Let $\big(A,\wedge,f,g\big)$ and let $\big(A',\wedge',\varphi,\psi\big)$ be two BiHom-Zinbiel algebras. A linear map $\alpha: A\longrightarrow A'$ is said to be a *morphism of BiHom-Zinbiel algebras* if $\alpha(x\wedge y)=\alpha(x)\wedge'\alpha(y)$, for all $x,y\in A$, as well as $\alpha f=\varphi\alpha$ and $\alpha g=\psi\alpha$.

Remark 3. An immediate consequence of (3) is that

$$fg(x)\wedge\big(f(y)\wedge z\big)=fg(y)\wedge\big(f(x)\wedge z\big),\quad\text{for all } x,y,z\in A.$$

Remark 4. In a BiHom-Zinbiel algebra $\big(A,\wedge,f,g\big)$ such that f and g are bijective, the relation (4) is a consequence of (3).

Proposition 3. *Let $\big(A,\wedge\big)$ be a Zinbiel algebra and let $f,\, g: A\longrightarrow A$ be two commuting morphisms of a Zinbiel algebra. Then $\big(A,\wedge_{(f,g)},f,g\big)$ is a BiHom-Zinbiel algebra obtained by composition, where $x\wedge_{(f,g)}y=f(x)\wedge g(y)$ for all $x,y\in A$.*

Proof. We depart from the assumption that f and g are multiplicative. Furthermore, for any $x,y,z\in A$, we compute

$$fg(x)\wedge_{(f,g)}\big(f(y)\wedge_{(f,g)}z\big)$$
$$=fg(x)\wedge_{(f,g)}\big(f^2(y)\wedge g(z)\big)$$

$$= f^2g(x) \wedge \big(f^2g(y) \wedge g^2(z)\big).$$

According to the definition of Zinbiel algebra, we obtain

$$\begin{aligned}
& f^2g(x) \wedge \big(f^2g(y) \wedge g^2(z)\big) \\
= & \big(f^2g(y) \wedge f^2g(x)\big) \wedge g^2(z) + \big(f^2g(x) \wedge f^2g(y)\big) \wedge g^2(z) \\
= & f\big(fg(y) \wedge fg(x)\big) \wedge g^2(z) + f\big(fg(x) \wedge fg(y)\big) \wedge g^2(z) \\
= & \big(fg(y) \wedge fg(x)\big) \wedge_{(f,g)} g(z) + \big(fg(x) \wedge fg(y)\big) \wedge_{(f,g)} g(z) \\
= & \big(g(y) \wedge_{(f,g)} f(x)\big) \wedge_{(f,g)} g(z) + \big(g(x) \wedge_{(f,g)} f(y)\big) \wedge_{(f,g)} g(z).
\end{aligned}$$

Hence the condition (3) holds. Moreover, by a direct calculation, we obtain the condition (4). We infer that $\big(A, \wedge_{(f,g)}, f, g\big)$ is a BiHom-Zinbiel algebra. □

Remark 5. Let $(A, \wedge)$ be a Zinbiel algebra and let $f,\ g:\ A \longrightarrow A$ be two commuting morphisms of a Zinbiel algebra. Assume that $(A', \wedge')$ is another Zinbiel algebra and $\varphi,\ \psi:\ A' \longrightarrow A'$ are two commuting morphisms of a Zinbiel algebra satisfying $\alpha f = \varphi\alpha$ and $\alpha g = \psi\alpha$. Then $\alpha:\ \big(A, \wedge_{(f,g)}, f, g\big) \longrightarrow \big(A', \wedge'_{(\varphi,\psi)}, \varphi, \psi\big)$ is a morphism of a BiHom-Zinbiel algebra.

Definition 8. A *BiHom-associative algebra* is a quadruple $\big(A, \mu, f, g\big)$ involving of a vector space A, a bilinear map $\mu:\ A\times A \longrightarrow A$ and two endomorphisms $f,\ g:\ A \longrightarrow A$ such that for all $x, y \in A$, $f\big(\mu(x,y)\big) = \mu\big(f(x), f(y)\big)$ and $g\big(\mu(x,y)\big) = \mu\big(g(x), g(y)\big)$, satisfying the following condition: for all $x, y, z \in A$,

$$\mu\big(f(x), \mu(y,z)\big) = \mu\big(\mu(x,y), g(z)\big). \tag{5}$$

If, in addition, we have

$$\mu\big(g(x), f(y)\big) = \mu\big(g(y), f(x)\big), \forall x, y \in A, \tag{6}$$

then, from this perspective, $\big(A, \mu, f, g\big)$ is said to be a *BiHom-commutative algebra.*

Definition 9. Let $\big(A, \mu, f, g\big)$ and let $\big(A', \mu', \varphi, \psi\big)$ be two BiHom-associative algebras. A linear map $\alpha:\ A \longrightarrow A'$ is said to be a *morphism of BiHom-associative algebras* if $\alpha\mu = \mu'(\alpha \otimes \alpha)$, as well as $\alpha f = \varphi\alpha$ and $\alpha g = \psi\alpha$.

Proposition 4. *Let $\big(A, \wedge, f, g\big)$ be a BiHom-Zinbiel algebra such that f, g are two commuting bijective linear maps. Define a new operation on A by*

$$\mu(x,y) = x \wedge y + f^{-1}g(y) \wedge fg^{-1}(x), \quad \forall x, y \in A.$$

Then $\big(A, \mu, f, g\big)$ is a BiHom-commutative algebra.

Proof. First, it is clear that $\mu\big(g(x), f(y)\big) = \mu\big(g(y), f(x)\big)$, for all $x, y \in A$. Next, for any $x, y, z \in A$, we compute:

$$\begin{aligned}
&\mu\big(f(x), \mu(y,z)\big)\\
&= \mu\big(f(x), y \wedge z + f^{-1}g(z) \wedge fg^{-1}(y)\big)\\
&= \mu\big(f(x), y \wedge z\big) + \mu\big(f(x), f^{-1}g(z) \wedge fg^{-1}(y)\big)\\
&= \underbrace{f(x) \wedge (y \wedge z)}_{\beta_1} + \underbrace{\big(f^{-1}g(y) \wedge f^{-1}g(z)\big) \wedge f^2g^{-1}(x)}_{\gamma_3}\\
&\quad + \underbrace{f(x) \wedge \big(f^{-1}g(z) \wedge fg^{-1}(y)\big)}_{\alpha_1} + \underbrace{\big(f^{-2}g^2(z) \wedge y\big) \wedge f^2g^{-1}(x)}_{\gamma_2}
\end{aligned}$$

and

$$\begin{aligned}
&\mu\big(\mu(x,y), g(z)\big)\\
&= \mu\big(x \wedge y + f^{-1}g(y) \wedge fg^{-1}(x), g(z)\big)\\
&= \mu\big(x \wedge y, g(z)\big) + \mu\big(f^{-1}g(y) \wedge fg^{-1}(x), g(z)\big)\\
&= \underbrace{(x \wedge y) \wedge g(z)}_{\beta_2} + \underbrace{f^{-1}g^2(z) \wedge \big(fg^{-1}(x) \wedge fg^{-1}(y)\big)}_{\alpha_2}\\
&\quad + \underbrace{\big(f^{-1}g(y) \wedge fg^{-1}(x)\big) \wedge g(z)}_{\beta_3} + \underbrace{f^{-1}g^2(z) \wedge \big(y \wedge f^2g^{-2}(x)\big)}_{\gamma_1}.
\end{aligned}$$

We verify that $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2 + \beta_3$ and $\gamma_1 = \gamma_2 + \gamma_3$. Then we get $\mu\big(f(x), \mu(y,z)\big) = \mu\big(\mu(x,y), g(z)\big)$. We hence infer that (A, μ, f, g) is a BiHom-commutative algebra. □

Definition 10. A *right BiHom-Zinbiel algebra* is a quadruple $\big(A, *, f, g\big)$ involving a vector space A, a bilinear product $* : A \times A \longrightarrow A$ and two commuting morphisms $f, g : A \longrightarrow A$ such that $f(x * y)$=$f(x) * f(y)$ and $g(x * y) = g(x) * g(y)$ satisfying the following conditions: for any $x, y, z \in A$,

$$\big(x * g(y)\big) * fg(z) = f(x) * \big(g(z) * f(y)\big) + f(x) * \big(g(y) * f(z)\big), \quad (7)$$

$$\big(g(y) * f(x)\big) * fg(z) = \big(g(y) * g(z)\big) * f^2(x). \quad (8)$$

Remark 6. Assume that $\big(A, \wedge, f, g\big)$ is a left BiHom-Zinbiel algebra. Define a new product $x * y = y \wedge x$, for all $x, y \in A$. Then $\big(A, *, g, f\big)$ is a right BiHom-Zinbiel algebra.

3. BiHom-pre-Poisson algebras: definitions and results

In this section, we introduce the structure of a BiHom-Poisson algebra, as well as that of a BiHom-pre-Poisson algebra and we provide certain outstanding results.

3.1. BiHom-pre-Poisson algebras.

Definition 11 ([9]). A *BiHom-Poisson algebra* is a quintuple $\big(A,\mu,\{-,-\},f,g\big)$ comprising a vector space A, a bilinear map μ, $\{-,-\}$: $A\times A\longrightarrow A$ and two endomorphisms f, $g: A\longrightarrow A$ such that

(1) $\big(A,\mu,f,g\big)$ is a BiHom associative algebra,
(2) $\big(A,\{-,-\},f,g\big)$ is a BiHom-Lie algebra,
(3) the BiHom-Leibniz algebra identity
$$\big\{fg(x),\mu(y,z)\big\}=\mu\big(\big\{g(x),y\big\},g(z)\big)+\mu\big(g(y),\big\{f(x),z\big\}\big)$$
is satisfied for any $x,y,z\in A$.

Definition 12. Let $\big(A,\mu,\{-,-\},f,g\big)$ and $\big(A',\mu',\{-,-\}',\varphi,\psi\big)$ be two BiHom-Poisson algebras. A linear map $\alpha: A\longrightarrow A'$ is said to be a *morphism of BiHom-Poisson algebras* if $\alpha\mu=\mu'(\alpha\otimes\alpha)$ and $\alpha(\{x,y\})=\{\alpha(x),\alpha(y)\}'$ for all $x,y\in A$, as well as $\alpha f=\varphi\alpha$ and $\alpha g=\psi\alpha$.

Proposition 5. *Let $(A,\mu,\{-,-\})$ be a Poisson algebra and let f, g : $A\longrightarrow A$ be two commuting morphisms of a Poisson algebra. Define new products on A by*
$$\mu_{(f,g)}(y,z)=\mu\big(f(x),g(y)\big)\quad \textit{and}\quad \{x,y\}_{(f,g)}=\big\{f(x),g(y)\big\},\ \forall x,y\in A.$$
Then $\big(A,\mu_{(f,g)},\{-,-\}_{(f,g)},f,g\big)$ is a BiHom-Poisson algebra obtained by composition.

Remark 7. Let $(A,\mu,\{-,-\})$ be a Poisson algebra and let f, $g: A\longrightarrow A$ be two commuting morphisms of a Poisson algebra. Assume that $(A',\mu',\{-,-\}')$ is another Poisson algebra and φ, ψ : $A'\longrightarrow A'$ are two commuting morphisms of a Poisson algebra satisfying $\alpha f=\varphi\alpha$ and $\alpha g=\psi\alpha$. Then, $\alpha:\big(A,\mu_{(f,g)},\{-,-\}_{(f,g)},f,g\big)\longrightarrow\big(A',\mu'_{(\varphi,\psi)},\{-,-\}'_{(\varphi,\psi)},\varphi,\psi\big)$ is a morphism of BiHom-Poisson algebras.

Definition 13. A *BiHom-pre-Poisson algebra* is a quintuple $\big(A,\wedge,\diamond,f,g\big)$ comprising a vector space A, two bilinear maps $\wedge,\diamond: A\times A\longrightarrow A$ and two endomorphisms f, $g: A\longrightarrow A$ such that

(1) $\big(A,\wedge,f,g\big)$ is a BiHom-Zinbiel algebra,
(2) $\big(A,\diamond,f,g\big)$ is a BiHom-pre-Lie algebra,
(3) the compatibility conditions
  (i) $\big(g(x)\diamond f(y)-g(y)\diamond f(x)\big)\wedge g(z)=fg(x)\diamond\big(f(y)\wedge z\big)-fg(y)\wedge\big(f(x)\diamond z\big)$,
  (ii) $\big(g(x)\wedge f(y)+g(y)\wedge f(x)\big)\diamond g(z)=fg(x)\wedge\big(f(y)\diamond z\big)+fg(y)\wedge\big(f(x)\diamond z\big)$,
are satisfied for any $x,y,z\in A$.

Definition 14. Let $\big(A,\wedge,\diamond,f,g\big)$ and $\big(A',\wedge',\diamond',\varphi,\psi\big)$ be two BiHom-pre-Poisson algebras. A linear map $\alpha: A\longrightarrow A'$ is said to be a *morphism* *of BiHom-pre-Poisson algebras* if $\alpha(x \wedge y) = \alpha(x) \wedge' \alpha(y)$ and $\alpha(x \diamond y) = \alpha(x) \diamond' \alpha(y)$ for all $x, y \in A$, as well as $\alpha f = \varphi\alpha$ and $\alpha g = \psi\alpha$.

Proposition 6. *Let $(A, \wedge, \diamond)$ be a pre-Poisson algebra and $f,\, g : A \longrightarrow A$ be two commuting morphisms of a pre-Poisson algebra. Define new products on A by*

$$x \wedge_{(f,g)} y = f(x) \wedge g(y) \quad \textit{and} \quad x \diamond_{(f,g)} y = f(x) \diamond g(y),\ \forall x, y \in A.$$

Then $\big(A, \wedge_{(f,g)}, \diamond_{(f,g)}, f, g\big)$ is a BiHom-pre-Poisson algebra obtained by composition.

Proof. We already know that $\big(A, \wedge_{(f,g)}, f, g\big)$ is a BiHom-Zinbiel algebra and $\big(A, \diamond_{(f,g)}, f, g\big)$ is a BiHom-pre-Lie algebra. For this reason we only need to prove the compatibility conditions.
Let x, y and z in A. Then, we compute

$$\begin{aligned}
&\big(g(x) \diamond_{(f,g)} f(y) - g(y) \diamond_{(f,g)} f(x)\big) \wedge_{(f,g)} g(z)\\
&= \big(fg(x) \diamond gf(y) - fg(y) \diamond gf(x)\big) \wedge_{(f,g)} g(z)\\
&= \big(f^2g(x) \diamond f^2g(y) - f^2g(y) \diamond gf^2(x)\big) \wedge g^2(z)\\
&= f^2g(x) \diamond \big(f^2g(y) \wedge g^2(z)\big) - f^2g(y) \wedge \big(f^2g(x) \diamond g^2(z)\big)\\
&= fg(x) \diamond_{(f,g)} \big(f^2(y) \wedge g(z)\big) - fg(y) \wedge_{(f,g)} \big(f^2(x) \diamond g(z)\big)\\
&= fg(x) \diamond_{(f,g)} \big(f(y) \wedge_{(f,g)} z\big) - fg(y) \wedge_{(f,g)} \big(f(x) \diamond_{(f,g)} z\big).
\end{aligned}$$

Moreover, it is straightforward to prove the following condition:
$\big(g(x) \wedge_{(f,g)} f(y) + g(y) \wedge_{(f,g)} f(x)\big) \diamond_{(f,g)} g(z) = fg(x) \wedge_{(f,g)} \big(f(y) \diamond_{(f,g)} z\big) + fg(y) \wedge_{(f,g)} \big(f(x) \diamond_{(f,g)} z\big)$.
It is therefore obvious that $\big(A, \wedge_{(f,g)}, \diamond_{(f,g)}, f, g\big)$ is a BiHom-pre-Poisson algebra. □

Remark 8. Let $(A, \wedge, \diamond)$ be a pre-Poisson algebra and let $f,\, g : A \longrightarrow A$ be two commuting morphisms of a pre-Poisson algebra. Assume that $(A', \wedge', \diamond')$ is another pre-Poisson algebra and $\varphi,\ \psi : A' \longrightarrow A'$ are two commuting morphisms of a pre-Poisson algebra satisfying $\alpha \circ f = \varphi \circ \alpha$ and $\alpha \circ g = \psi \circ \alpha$. Then, $\alpha : \big(A, \wedge_{(f,g)}, \diamond_{(f,g)}, f, g\big) \longrightarrow \big(A', \wedge'_{(\varphi,\psi)}, \diamond'_{(\varphi,\psi)}, \varphi, \psi\big)$ is a morphism of BiHom-pre-Poisson algebras.

Proposition 7. *Let $\big(A, \wedge, \diamond, f, g\big)$ be a BiHom-pre-Poisson algebra, such that $f,\, g$ are bijective. Define new operations on A by putting, for all $x, y \in A$*

$$\mu(x, y) = x \wedge y + f^{-1}g(y) \wedge fg^{-1}(x) \textit{ and } \{x, y\} = x \diamond y - f^{-1}g(y) \diamond fg^{-1}(x).$$

Then $\big(A, \mu, \{-,-\}, f, g\big)$ is a BiHom-Poisson algebra.

Proof. We already know that $\big(A, \{-,-\}, f, g\big)$ is a BiHom-Lie algebra and $\big(A, \mu, f, g\big)$ is a BiHom-associative algebra, so we only need to demonstrate the BiHom-Leibniz algebra identity.

For any $x, y, z \in A$, we compute:

$$\begin{aligned}
&\big\{fg(x), \mu(y,z)\big\}\\
&= \big\{fg(x), y \wedge z + f^{-1}g(z) \wedge fg^{-1}(y)\big\}\\
&= \big\{fg(x), y \wedge z\big\} + \big\{fg(x), f^{-1}g(z) \wedge fg^{-1}(y)\big\}\\
&= fg(x) \diamond (y \wedge z) - \big(f^{-1}g(y) \wedge f^{-1}g(z)\big) \diamond f^2(x)\\
&\quad + fg(x) \diamond \big(f^{-1}g(z) \wedge fg^{-1}(y)\big) - \big(f^{-2}g^2(z) \wedge y\big) \diamond f^2(x).
\end{aligned}$$

Next, for any $x, y, z \in A$, we compute $\mu\big(\{g(x), y\}, g(z)\big)$ and $\mu\big(g(y), \{f(x), z\}\big)$, as far as the proof of this condition is concerned, we use the compatibility conditions of a BiHom-pre-Poisson algebra. □

Remark 9. In previous definitions, if we consider $f = g$ (resp. $f = g = id$), then we obtain a Hom-pre-Poisson algebra (resp. a pre-Poisson algebra) (see [1, 12]).

3.2. Examples.

3.2.1. A BiHom-pre-Poisson algebra associated with a BiHom-pre-Lie algebra.

Definition 15. Let A be a vector space.

(1) The tensorial algebra is expressed in terms of $T(A) = \bigoplus_{n \geqslant 1} \bigotimes^n A$ and the exchange is indicated by τ defined by
$$\tau(x \otimes y) = y \otimes x, \quad \forall x, y \in A.$$
We extend τ on $T(A)$ by
$\forall X = x_1 \otimes ... \otimes x_p \in \bigotimes^p A$ and $Y = x_{p+1} \otimes ... \otimes x_{p+q} \in \bigotimes^q A$,
$$\begin{aligned}
&\tau_{p,q}\big((x_1 \otimes ... \otimes x_p) \bigotimes (x_{p+1} \otimes ... \otimes x_{p+q})\big)\\
&= (x_{p+1} \otimes ... \otimes x_{p+q}) \bigotimes (x_1 \otimes ... \otimes x_p).
\end{aligned}$$

(2) We say that a permutation $\sigma \in S_{p+q}$ is a (p,q)-*shuffle* if $\sigma(1) < ... < \sigma(p)$ and $\sigma(p+1) < ... < \sigma(p+p)$.
We define the *shuffle product* $Sh_{p,q}$ on $T(A)$ by
$$\begin{aligned}
Sh_{p,q}(X,Y) &= Sh_{p,q}(x_1 \otimes ... \otimes x_p, x_{p+1} \otimes ... \otimes x_{p+q})\\
&= \sum_{\sigma \in sh_{(p,q)}} x_{\sigma^{-1}(1)} \otimes ... \otimes x_{\sigma^{-1}(p+q)},
\end{aligned}$$
for any $X = x_1 \otimes ... \otimes x_p$, $Y = x_{p+1} \otimes ... \otimes x_{p+q} \in T(A)$.

Proposition 8. *Let A be a vector space and let f, g be two commuting linear maps. Define new product on $T(A)$ by*
$\forall\ X = x_1 \otimes ... \otimes x_p, Y = x_{p+1} \otimes ... \otimes x_{p+q} \in T(A)$,

$$X \curlywedge Y = \big(x_1 \otimes ... \otimes x_p\big) \curlywedge \big(x_{p+1} \otimes ... \otimes x_{p+q}\big)$$

$$= \sum_{\sigma \in sh_{(p,q-1)}} h(x_{\sigma^{-1}(1)}) \otimes ... \otimes h(x_{\sigma^{-1}(p+q-1)}) \otimes g(x_{p+q}),$$

where

$$h(x_{\sigma^{-1}(i)}) = \begin{cases} f(x_{\sigma^{-1}(i)}), \ if \ \ \sigma^{-1}(i) \in \{1, ..., p\}, \\ g(x_{\sigma^{-1}(i)}), \ if \ \ \sigma^{-1}(i) \in \{p+1, ..., p+q-1\}. \end{cases}$$

Then $\big(T(A), \curlywedge, f, g\big)$ *is a BiHom-Zinbiel algebra.*

Proposition 9. *Let* $\big(A, \diamond, f, g\big)$ *be a BiHom-pre-Lie algebra such that* f, g *are two commuting bijective morphisms. Define on* $T(A)$ *the following product: for all* $X = x_1 \otimes ... \otimes x_p, Y = x_{p+1} \otimes ... \otimes x_{p+q} \in T(A)$,

$$\begin{aligned} X \Diamond Y &= \big(x_1 \otimes ... \otimes x_p\big) \Diamond \big(x_{p+1} \otimes ... \otimes x_{p+q}\big) \\ &= \sum_{\sigma \in sh_{(p-1,q-1)}} h(x_{\sigma^{-1}(1)}) \otimes ... \otimes h(x_{\sigma^{-1}(p-1)}) \otimes h(x_{\sigma^{-1}(p+1)}) \otimes ... \\ &\quad ... \otimes h(x_{\sigma^{-1}(p+q-1)}) \otimes \big(x_p \diamond x_{p+q}\big) \\ &\quad + \sum_{k=1}^{p+q-2} \sum_{\substack{\sigma \in sh_{(p,q-1)} \\ 1 \leq \sigma^{-1}(k) \leq p \\ p+1 \leq \sigma^{-1}(k+1) < p+q}} h(x_{\sigma^{-1}(1)}) \otimes ... \otimes h(x_{\sigma^{-1}(k-1)}) \otimes \\ &\quad + \big[x_{\sigma^{-1}(k)}, x_{\sigma^{-1}(k+1)}\big] \otimes h(x_{\sigma^{-1}(k+2)}) \otimes ... \\ &\quad ... \otimes h(x_{\sigma^{-1}(p+q-1)}) \otimes g(x_{p+q}), \end{aligned}$$

where

$$h(x_{\sigma^{-1}(i)}) = \begin{cases} f(x_{\sigma^{-1}(i)}), \ if \ \ \sigma^{-1}(i) \in \{1, ..., p\}, \\ g(x_{\sigma^{-1}(i)}), \ if \ \ \sigma^{-1}(i) \in \{p+1, ..., p+q-1\}, \end{cases}$$

and $\big[x, y\big] = x \diamond y - f^{-1}g(y) \diamond fg^{-1}(x), \ \forall x, y \in A$.
Then $\big(T(A), \Diamond, f, g\big)$ *is a BiHom-pre-Lie algebra.*

Proof. Let $X = x_1 \otimes ... \otimes x_p$, $Y = x_{p+1} \otimes ... \otimes x_{p+q}$ and $Z = x_{p+q+1} \otimes ... \otimes x_{p+q+r}$ be three elements of $T(A)$ satisfying the following relation:

$$\begin{aligned} (*) : &fg(X) \Diamond \big(f(Y) \Diamond Z\big) - \big(g(X) \Diamond f(Y)\big) \Diamond g(Z) - fg(Y) \Diamond \big(f(X) \Diamond Z\big) \\ &-\big(g(Y) \Diamond f(X)\big) \Diamond g(Z) = 0. \end{aligned}$$

In this relation, there are 4 types of terms that appear.
(1) In $(*)$, there are terms that appear with two "$\diamond$"

$$fg(x_p) \diamond \big(f(x_{p+q}) \diamond x_{p+q+r}\big), \big(g(x_p) \diamond f(x_{p+q})\big) \diamond g(x_{p+q+r}),$$
$$fg(x_{p+q}) \diamond \big(f(x_p) \diamond x_{p+q+r}\big) \text{ and } \big(g(x_{p+q}) \diamond f(x_p)\big) \diamond g(x_{p+q+r}).$$

These terms appear in the form

$f^2g(x_{\sigma^{-1}(1)}) \otimes ... \otimes f^2g(x_{\sigma^{-1}(p-1)}) \otimes f^2g(x_{\sigma^{-1}(p+1)}) \otimes ... \otimes f^2g(x_{\sigma^{-1}(p+q-1)})$
$\otimes g^2(x_{\sigma^{-1}(p+q+1)}) \otimes ... \otimes g^2(x_{\sigma^{-1}(p+q+r-1)}) \otimes term,$

where σ is a shuffle permutation of $\{1, ..., p+q+r\} \setminus \{p, p+q, p+q+r\}$, in particular, we assume that

$$\begin{aligned} A_\sigma = f^2g(x_{\sigma^{-1}(1)}) \otimes ... \otimes f^2g(x_{\sigma^{-1}(p-1)}) \otimes f^2g(x_{\sigma^{-1}(p+1)}) \otimes ... \\ ... \otimes f^2g(x_{\sigma^{-1}(p+q-1)}) \otimes g^2(x_{\sigma^{-1}(p+q+1)}) \otimes ... \otimes g^2(x_{\sigma^{-1}(p+q+r-1)}). \end{aligned}$$

The contribution of the corresponding terms is $A_\sigma \otimes C$, with

$$\begin{aligned} C = fg(x_p) \diamond \big(f(x_{p+q}) \diamond x_{p+q+r}\big) - \big(g(x_p) \diamond f(x_{p+q})\big) \diamond g(x_{p+q+r}) \\ -fg(x_{p+q}) \diamond \big(f(x_p) \diamond x_{p+q+r}\big) - \big(g(x_{p+q}) \diamond f(x_p)\big) \diamond g(x_{p+q+r}). \end{aligned}$$

Relying on the relation of a Hom-pre-Lie of "$\diamond$", these terms are simplified.
(2) In $(*)$, there are terms that appear with double brackets or "$\diamond$" in brackets:

$$\begin{aligned} &\big[fg(x_p), \big[f(x_{p+q}), x_k\big]\big], \quad \big[g(x_p) \diamond f(x_{p+q}), g(x_k)\big], \ \big[fg(x_{p+q}), \big[f(x_p), x_k\big]\big] \\ &\text{and} \ \ \big[g(x_{p+q}) \diamond f(x_p), g(x_k)\big]. \end{aligned}$$

These terms appear in the form

$$\begin{aligned} &\pm\big(f^2g(x_{\sigma^{-1}(1)}) \otimes ... \otimes f^2g(x_{\sigma^{-1}(p-1)}) \otimes f^2g(x_{\sigma^{-1}(p+1)}) \otimes ... \\ &\quad ... \otimes f^2g(x_{\sigma^{-1}(k-2)})\big) \otimes \text{term} \otimes \big(g^2(x_{\sigma^{-1}(k)}) \otimes ... \otimes g^2(x_{p+q+r})\big) \\ &= \pm A_\sigma \otimes \text{term} \otimes B_\sigma, \end{aligned}$$

where $p+q+1 \leq k < p+q+r$, and σ is in $Sh_{(p-1,q-1,r-1)}$ acting on $\{1, ..., p+q+r-1\} \setminus \{p, p+q, k\}$.
Basically, we observe that

$$A_\sigma = f^2g(x_{\sigma^{-1}(1)}) \otimes ... \otimes f^2g(x_{\sigma^{-1}(p-1)}) \otimes f^2g(x_{\sigma^{-1}(p+1)}) \otimes ... \otimes f^2g(x_{\sigma^{-1}(k-2)}).$$

As a matter of fact, we obtain the terms $A_\sigma \otimes C \otimes B_\sigma$ such that C takes the following form:

$$\begin{aligned} C &= \big[fg(x_p), \big[f(x_{p+q}), x_k\big]\big] - \big[g(x_p) \diamond f(x_{p+q}), g(x_k)\big] \\ &\quad - \big[fg(x_{p+q}), \big[f(x_p), x_k\big]\big] + \big[g(x_{p+q}) \diamond f(x_p), g(x_k)\big] \\ &= \big[fg(x_p), \big[f(x_{p+q}), x_k\big]\big] - \big[fg(x_{p+q}), \big[f(x_p), x_k\big]\big] \\ &\quad - \big[g(x_p) \diamond f(x_{p+q}) - g(x_{p+q}) \diamond f(x_p), g(x_k)\big] \\ &= \big[fg(x_p), \big[f(x_{p+q}), x_k\big]\big] - \big[fg(x_{p+q}), \big[f(x_p), x_k\big]\big] \\ &\quad - \big[\big[g(x_p), f(x_{p+q})\big], g(x_k)\big] \\ &= \big[g^2\big(fg^{-1}(x_p)\big), \big[g\big(fg^{-1}(x_{p+q})\big), f\big(f^{-1}x_k\big)\big]\big] \\ &\quad + \big[g^2\big(fg^{-1}(x_{p+q})\big), \big[g\big(f^{-1}(x_k)\big), f\big(fg^{-1}x_p\big)\big]\big] \end{aligned}$$

$$\begin{aligned}
&- \big[g\big(\big[x_p, fg^{-1}(x_{p+q})\big]\big), f\big(f^{-1}g(x_k)\big)\big]\\
&= \big[g^2\big(fg^{-1}(x_p)\big), \big[g\big(fg^{-1}(x_{p+q})\big), f\big(f^{-1}x_k\big)\big]\big]\\
&\quad + \big[g^2\big(fg^{-1}(x_{p+q})\big), \big[g\big(f^{-1}(x_k)\big), f\big(fg^{-1}x_p\big)\big]\big]\\
&\quad + \big[g^2\big(f^{-1}(x_k)\big), \big[g\big(fg^{-1}(x_p)\big), f\big(fg^{-1}x_{p+q}\big)\big]\big].
\end{aligned}$$

If we assume $X{=}fg^{-1}(x_p)$, $Y{=}fg^{-1}(x_{p+q})$ and $Z{=}f^{-1}(x_k)$, then we obtain

$$C{=}\circlearrowleft_{X,Y,Z} \big[g^2(X), \big[g(Y), f(Z)\big]\big] = 0.$$

Besides, in $(*)$, there are terms that appear with double brackets:

$$\big[fg(x_i), \big[f(x_j), x_k\big]\big], \quad \big[fg(x_j), \big[f(x_i), x_k\big]\big] \;\text{ and }\; \big[\big[g(x_i), f(x_j)\big], g(x_k)\big].$$

These terms appear with $1 \leq i \leq p$, $p+1 \leq j \leq p+q$ and $p+q+1 \leq k < p+q+r$, in the form

$$\begin{aligned}
&\pm \big(f^2g(x_{\sigma^{-1}(1)}) \otimes ...\widehat{i}...\widehat{j}... \otimes f^2g(x_{\sigma^{-1}(k-2)})\big) \otimes term \otimes \big(g^2(x_{\sigma^{-1}(k)}) \otimes ...\\
&... \otimes g^2(x_{p+q+r})\big) = \pm A_\sigma \otimes term \otimes B_\sigma,
\end{aligned}$$

where σ is in $Sh_{(p-1,q-1,r-1)}$ acting on $\{1, ..., p+q+r-1\} \setminus \{i, j, k\}$. Basically, we set

$$A_\sigma = f^2g(x_{\sigma^{-1}(1)}) \otimes ...\widehat{i}...\widehat{j}... \otimes f^2g(x_{\sigma^{-1}(k-2)}).$$

From this perspective, we obtain the terms $A_\sigma \otimes C \otimes B_\sigma$ such that C takes the following form:

$$\begin{aligned}
C &= \big[fg(x_i), \big[f(x_j), x_k\big]\big] - \big[\big[g(x_i, f(x_j)\big], g(x_k)\big] - \big[fg(x_j), \big[f(x_i), x_k\big]\big]\\
&= \big[fg(x_i), \big[f(x_j), x_k\big]\big] - \big[fg(x_j), \big[f(x_i), x_k\big]\big] - \big[\big[g(x_i, f(x_j)\big], g(x_k)\big]\\
&= \big[g^2\big(fg^{-1}(x_i)\big), \big[g\big(fg^{-1}(x_j)\big), f\big(f^{-1}(x_k)\big)\big]\big]\\
&\quad + \big[g^2\big(fg^{-1}(x_j)\big), \big[g\big(f^{-1}(x_k)\big), f\big(fg^{-1}(x_i)\big)\big]\big]\\
&\quad + \big[g^2\big(f^{-1}(x_k)\big), \big[f(x_i)\big), f\big(fg^{-1}(x_j)\big)\big]\big]\\
&= 0.
\end{aligned}$$

(3) In $(*)$, there are terms that appear in the form

$$\begin{aligned}
&... \otimes g\big(\big[f(x_{j'}), x_k\big]\big) \otimes ... \otimes \big[fg(x_i), f^2(x_j)\big] \otimes ... ,\\
&... \otimes \big[fg(x_i), f^2(x_{k'})\big] \otimes ... \otimes g\big(\big[f(x_j), x_k\big]\big) \otimes ... ,\\
&... \otimes g\big(\big[f(x_{i'}), x_k\big]\big) \otimes ... \otimes \big[fg(x_j), f^2(x_i)\big] \otimes ... ,\\
&... \otimes \big[fg(x_j), f^2(x_{k'})\big] \otimes ... \otimes g\big(\big[f(x_i), x_k\big]\big) \otimes ... ,\\
&... \otimes \big[fg(x_i), f^2(x_{j'})\big] \otimes ... \otimes g\big(\big[f(x_j), x_k\big]\big) \otimes ... ,\\
&... \otimes \big[fg(x_{i'}), f(x_{j'})\big] \otimes ... \otimes g\big(\big[f(x_i), x_k\big]\big) \otimes ... .
\end{aligned}$$

More precisely, we have the following.
– In $fg(X)\diamondsuit\big(f(Y)\diamondsuit Z\big)$, $\forall k\in\{p+q+1,...,p+q+r-1\}$, the terms that appear are

$$(1.1)\ :\ ...\otimes g\big([x_{j'},x_k]\big)\otimes...\otimes\big[fg(x_p),f^2(x_{p+q})\big]\otimes...,$$
$$with\ \ p+1\leq j'<p+q,$$
$$(1.2)\ :\ ...\otimes g\big([f(x_{j'}),x_k]\big)\otimes...\otimes\big[fg(x_p),f^2(x_j)\big]\otimes...,$$
$$with\ \ p+1\leq j'<j<p+q,$$
$$(1.3)\ :\ ...\otimes\big[fg(x_p),f^2(x_j)\big]\otimes...\otimes g\big([f(x_{p+q}),x_k]\big)\otimes...,$$
$$with\ p+1\leq j<p+q,$$
$$(1.4)\ :\ ...\otimes g\big([f(x_{j'}),x_k]\big)\otimes...\otimes\big[fg(x_i),f^2(x_{p+q})\big]\otimes...,$$
$$with\ \ p+1\leq j'<p+q\ \ and\ \ 1\leq i<p,$$
$$(1.5)\ :\ ...\otimes g\big([f(x_{j'}),x_k]\big)\otimes...\otimes\big[fg(x_i),f^2(x_j)\big]\otimes...,$$
$$with\ \ 1\leq i<p\ \ and\ \ p+1\leq j'<j<p+q,$$
$$(1.6)\ :\ ...\otimes\big[fg(x_i),f^2(x_j)\big]\otimes...\otimes g\big([f(x_{p+q}),x_k]\big)\otimes...,$$
$$with\ \ 1\leq i<p\ \ and\ \ p+1\leq j<p+q,$$
$$(1.7)\ :\ ...\otimes\big[fg(x_i),f^2(x_{k'})\big]\otimes...\otimes g\big([f(x_{p+q}),x_k]\big)\otimes...,$$
$$with\ \ p+q+1\leq k'<k<p+q+r\ \ and\ \ 1\neq i<p,$$
$$(1.8)\ :\ ...\otimes\big[fg(x_i),f^2(x_{k'})\big]\otimes...\otimes g\big([f(x_j),x_k]\big)\otimes...,$$
$$with\ \ 1\leq i<p,\ p+1\leq j<p+q\ and\ p+q+1\leq k'<k<p+q+r,$$
$$(1.9)\ :\ ...\otimes\big[fg(x_p),f^2(x_{k'})\big]\otimes...\otimes g\big([f(x_{p+q}),x_k]\big)\otimes...,$$
$$with\ p+q+1\leq k'<k<p+q+r,$$
$$(1.10):...\otimes\big[fg(x_p),f^2(x_{k'})\big]\otimes...\otimes g\big([f(x_j),x_k]\big)\otimes...,$$
$$with\ \ p+1\leq j<p+q\ \ and\ \ p+q+1\leq k'<k<p+q+r.$$

– In $\big(g(X)\diamondsuit f(Y)\big)\diamondsuit g(Z)$, $\forall k\in\{p+q+1,...,p+q+r-1\}$, the terms that appear are

$$(2.1)\ :\ ...\otimes\big[fg(x_{i'}),g(x_k)\big]\otimes...\otimes f\big([g(x_p),f(x_j)]\big)\otimes...,$$
$$with\ \ 1\leq i'<p\ \ and\ \ p+1\leq j<p+q,$$
$$(2.2)\ :\ ...\otimes f\big([g(x_i),f(x_j)]\big)\otimes...\otimes\big[fg(x_{i'}),g(x_k)\big]\otimes...,$$
$$with\ \ p+1\leq j<p+q\ \ and\ 1\leq i<i'<p,$$
$$(2.3)\ :\ ...\otimes f\big([g(x_i),f(x_j)]\otimes...\otimes\big[fg(x_p),g(x_k)\big]\otimes...,$$
$$with\ \ 1\leq i<p\ \ and\ \ p+1\leq j<p+q,$$
$$(2.4)\ :\ ...\otimes f\big([g(x_p),f(x_j)]\big)\otimes...\otimes\big[fg(x_{j'}),g(x_k)\big]\otimes...,$$

with $p+1 \leq j < j' < p+q$,

$(2.5)\ :\ ...\otimes f\big([g(x_p), f(x_j)]\big)\otimes ...\otimes [fg(x_{p+q}), g(x_k)]\otimes ...$,

with $p+1 \leq j < p+q$,

$(2.6)\ :\ ...\otimes f\big([g(x_i), f(x_j)]\big)\otimes ...\otimes [fg(x_{j'}), g(x_k)]\otimes ...$,

with $1 \leq i < p,\ p+1 \leq j < j' < p+q$,

$(2.7)\ :\ ...\otimes f\big([g(x_i), f(x_j)]\big)\otimes ...\otimes [fg(x_{p+q}), g(x_k)]\otimes ...$,

with $1 \leq i < p,\ p+1 \leq j < p+q$.

– In $fg(Y)\diamondsuit\big(f(X)\diamondsuit Z\big)$, $\forall k \in \{p+q+1, ..., p+q+r-1\}$, the terms that appear are

$(1.1)'\ :\ ...\otimes g\big([x_{i'}, x_k]\big)\otimes ...\otimes [fg(x_{p+q}), f^2(x_p)]\otimes ...$, *with* $1 \leq i' < p$,

$(1.2)'\ :\ ...\otimes g\big([f(x_{i'}), x_k]\big)\otimes ...\otimes [fg(x_{p+q}), f^2(x_i)]\otimes ...$,

with $1 \leq i' < i < p$,

$(1.3)'\ :\ ...\otimes [fg(x_{p+q}), f^2(x_i)]\otimes ...\otimes g\big([f(x_p), x_k]\big)\otimes ...$, *with* $1 \leq i < p$,

$(1.4)'\ :\ ...\otimes g\big([f(x_{i'}), x_k]\big)\otimes ...\otimes [fg(x_j), f^2(x_p)]\otimes ...$,

with $1 \leq i' < p$ *and* $p+1 \leq j < p+q$,

$(1.5)'\ :\ ...\otimes g\big([f(x_{i'}), x_k]\big)\otimes ...\otimes [fg(x_j), f^2(x_i)]\otimes ...$,

with $p+1 \leq j < p+q$ *and* $1 \leq i' < i < p$,

$(1.6)'\ :\ ...\otimes [fg(x_j), f^2(x_i)]\otimes ...\otimes g\big([f(x_p), x_k]\big)\otimes ...$,

with $p+1 \leq j < p+q$ *and* $1 \leq i < p$,

$(1.7)'\ :\ ...\otimes [fg(x_j), f^2(x_k)]\otimes ...\otimes g\big([f(x_p), x_{k'}]\big)\otimes ...$,

with $p+q+1 \leq k < k' < p+q+r$ *and* $p+1 \neq j < p+q$,

$(1.8)'\ :\ ...\otimes [fg(x_j), f^2(x_k)]\otimes ...\otimes g\big([f(x_i), x_{k'}]\big)\otimes ...$,

with $p+1 \leq j < p+q,\ 1 \leq i < p$ *and* $p+q+1 \leq k < k' < p+q+r$,

$(1.9)'\ :\ ...\otimes [fg(x_{p+q}), f^2(x_k)]\otimes ...\otimes g\big([f(x_p), x_{k'}]\big)\otimes ...$,

with $p+q+1 \leq k < k' < p+q+r$,

$(1.10)' :\ ...\otimes [fg(x_{p+q}), f^2(x_k)]\otimes ...\otimes g\big([f(x_i), x_{k'}]\big)\otimes ...$,

with $1 \leq i < p$, *and* $p+q+1 \leq k < k' < p+q+r$.

– In $\big(g(Y)\diamondsuit f(X)\big)\diamondsuit g(Z)$, $\forall k \in \{p+q+1, ..., p+q+r-1\}$, the terms that appear are

$(2.1)'\ :\ ...\otimes [fg(x_{j'}), g(x_k)]\otimes ...\otimes f\big([g(x_{p+q}), f(x_i)]\big)\otimes ...$,

with $p+1 \leq j' < p+q$ *and* $1 \leq i < p$,

$(2.2)'\ :\ ...\otimes f\big([g(x_j), f(x_i)]\big)\otimes ...\otimes [fg(x_{j'}), g(x_k)]\otimes ...$,

$with\ \ 1 \leq i < p\ \ and\ \ p+1 \leq j < j' < p+q,$

$(2.3)'\ :\ ... \otimes f\big(\big[g(x_j), f(x_i)\big] \otimes ... \otimes \big[fg(x_{p+q}), g(x_k)\big] \otimes ...,$

$with\ \ p+1 \leq j < p+q\ \ and\ \ 1 \leq i < p,$

$(2.4)'\ :\ ... \otimes f\big(\big[g(x_{p+q}), f(x_i)\big]\big) \otimes ... \otimes \big[fg(x_{i'}), g(x_k)\big] \otimes ...,$

$with\ \ 1 \leq i < i' < p,$

$(2.5)'\ :\ ... \otimes f\big(\big[g(x_{p+q}), f(x_i)\big]\big) \otimes ... \otimes \big[fg(x_p), g(x_k)\big] \otimes ...,$

$with\ \ 1 \leq i < p,$

$(2.6)'\ :\ ... \otimes f\big(\big[g(x_j), f(x_i)\big]\big) \otimes ... \otimes \big[fg(x_{i'}), g(x_k)\big] \otimes ...,$

$with\ \ p+1 \leq j < p+q,\ 1 \leq i < i' < p,$

$(2.7)'\ :\ ... \otimes f\big(\big[g(x_j), f(x_i)\big]\big) \otimes ... \otimes \big[fg(x_p), g(x_k)\big] \otimes ...,$

$with\ \ p+1 \leq j < p+q,\ 1 \leq i < p.$

The proof rests on verifying the following equalities:
$(1.2)-(2.4)=0$, $(1.3)-(2.5)=0$, $(1.4)+(2.1)'=0$, $(1.5)=(2.6)+(2.2)'$, $(1.6)-(2.7)=(2.3)'$, $(1.7)-(1.10)'=0$, $(1.8)-(1.8)'=0$, $(1.9)-(1.9)'=0$, $(1.10)-(1.7)'=0$, $(1.2)'-(2.4)'=0$, $(1.3)'-(2.5)'=0$, $(1.4)'+(2.1)=0$, $(1.5)'=(2.6)'-(2.2)$ and $(1.6)'-(2.7)'=(2.3)$.
Moreover, the remaining terms (1.1) and $(1.1)'$ are simplified with terms in 4.

(4) Finally, in $(*)$, there are terms that appear in the form:

$$... \otimes \big[fg(x_i), g(x_k)\big] \otimes ... \otimes g\big(f(x_{p+q}) \diamond x_{p+q+r}\big),$$
$$... \otimes g\big(\big[f(x_j), x_k\big]\big) \otimes ... \otimes \big(fg(x_p) \diamond g(x_{p+q+r})\big),$$
$$... \otimes f\big(\big[g(x_i), f(x_j)\big]\big) \otimes ... \otimes \big(fg(x_{p+q}) \diamond g(x_{p+q+r})\big),$$
$$... \otimes f\big(\big[g(x_j), f(x_i)\big]\big) \otimes ... \otimes \big(fg(x_p) \diamond fg(x_{p+q+r})\big).$$

More precisely, we have the following.

– In $fg(X)\diamondsuit\big(f(Y)\diamondsuit Z\big)$, the terms that appear are

$(1.1)''\ :\ ... \otimes \big[fg(x_i), g(x_k)\big] \otimes ... \otimes g\big(f(x_{p+q}) \diamond x_{p+q+r}\big),$

$with\ p+q+1 \leq k < p+q+r\ \ and\ \ 1 \leq i \leq p,$

$(1.2)''\ :\ ... \otimes \big[fg(x_i), f^2(x_j)\big] \otimes ... \otimes g\big(f(x_{p+q}) \diamond x_{p+q+r}\big),$

$with\ p+1 \leq j < p+q\ \ and\ \ 1 \leq i \leq p,$

$(1.3)''\ :\ ... \otimes g\big(\big[f(x_j), x_k\big]\big) \otimes ... \otimes \big(fg(x_p) \diamond g(x_{p+q+r})\big),$

$with\ p+q+1 \leq k < p+q+r\ \ and\ \ p+1 \leq j \leq p+q.$

– In $\big(g(X)\diamondsuit f(Y)\big)\diamondsuit g(Z)$, the terms that appear are

$(2.1)''\ :\ ... \otimes \big[fg(x_i), f(x_j)\big] \otimes ... \otimes \big(fg(x_{p+q}) \diamond g(x_{p+q+r})\big),$

$with\ p+1 \leq j < p+q\ \ and\ \ 1 \leq i \leq p,$

$(2.2)''$: $... \otimes \big[fg(x_{i'}), g(x_k)\big] \otimes ... \otimes f\big(g(x_p) \diamond f(x_{p+q})\big) \otimes ...,$
with $1 \leq i' < p$ *and* $p+q+1 \leq k < p+q+r,$
$2.3)''$: $... \otimes \big[fg(x_{j'}), g(x_k)\big] \otimes ... \otimes f\big(g(x_p) \diamond f(x_{p+q})\big) \otimes ...,$
with $p+1 \leq j' < p+q$ *and* $p+q+1 \leq k < p+q+r.$

– In $fg(Y)\diamondsuit\big(f(X)\diamondsuit Z\big)$, the terms that appear are

$(3.1)''$: $... \otimes \big[fg(x_j), g(x_k)\big] \otimes ... \otimes g\big(f(x_p) \diamond x_{p+q+r}\big),$
; *with* $p+q+1 \leq k < p+q+r$ *and* $p+1 \leq j \leq p+q,$
$(3.2)''$: $... \otimes \big[fg(x_j), f^2(x_i)\big] \otimes ... \otimes g\big(f(x_p) \diamond x_{p+q+r}\big),$
with $1 \leq i < p$ *and* $p+1 \leq j \leq p+q,$
$(3.3)''$: $... \otimes g\big(\big[f(x_i), x_k\big]\big) \otimes ... \otimes \big(fg(x_{p+q}) \diamond g(x_{p+q+r})\big),$
with $p+q+1 \leq k < p+q+r$ *and* $1 \leq i \leq p.$

– In $\big(g(Y)\diamondsuit f(X)\big)\diamondsuit g(Z)$, the terms that appear are

$(4.1)''$: $... \otimes \big[fg(x_j), f(x_i)\big] \otimes ... \otimes \big(fg(x_p) \diamond g(x_{p+q+r})\big),$
with $1 \leq i < p$ *and* $p+1 \leq j \leq p+q,$
$(4.2)''$: $... \otimes \big[fg(x_{j'}), g(x_k)\big] \otimes ... \otimes f\big(g(x_{p+q}) \diamond f(x_p)\big) \otimes ...,$
with $p+1 \leq j' < p$ *and* $p+q+1 \leq k < p+q+r,$
$(4.3)''$: $... \otimes \big[fg(x_{i'}), g(x_k)\big] \otimes ... \otimes f\big(g(x_{p+q}) \diamond f(x_p)\big) \otimes ...,$
with $1 \leq i' < p$ *and* $p+q+1 \leq k < p+q+r.$

It is clear that
$(1.1)''-(3.3)''=0$, $(1.2)''-(2.1)''=0$, $(1.3)''-(3.1)''=0$, $(3.2)''-(4.1)''=0$, $(1.1)-(2.3)''+(4.2)''=0$ and $(1.1)'-(4.3)''+(2.2)''=0$,
which completes the proof. □

Theorem 1. *Let* $\big(A, \diamond, f, g\big)$ *be a BiHom-pre-Lie algebra and let* $f, g : A \longrightarrow A$ *be two commuting morphisms of a pre-Lie algebra. Then* $\big(T(A), \curlywedge, \diamondsuit, f, g\big)$ *is a BiHom-pre-Poisson algebra, where, for all* $X = x_1 \otimes ... \otimes x_p$, $Y = x_{p+1} \otimes ... \otimes x_{p+q} \in T(A)$*,*

$$\begin{aligned} X \curlywedge Y &= \big(x_1 \otimes ... \otimes x_p\big) \curlywedge \big(x_{p+1} \otimes ... \otimes x_{p+q}\big) \\ &= \sum_{\sigma \in sh_{(p,q-1)}} h(x_{\sigma^{-1}(1)}) \otimes ... \otimes h(x_{\sigma^{-1}(p+q-1)}) \otimes g(x_{p+q}), \end{aligned}$$

and

$$\begin{aligned} X\diamondsuit Y &= \big(x_1 \otimes ... \otimes x_p\big)\diamondsuit\big(x_{p+1} \otimes ... \otimes x_{p+q}\big) \\ &= \sum_{\sigma \in sh_{(p-1,q-1)}} h(x_{\sigma^{-1}(1)}) \otimes ... \otimes h(x_{\sigma^{-1}(p-1)}) \otimes h(x_{\sigma^{-1}(p+1)}) \otimes ... \end{aligned}$$

$$
\begin{aligned}
&\ldots \otimes h(x_{\sigma^{-1}(p+q-1)}) \otimes \big(x_p \diamond x_{p+q}\big)\\
&+ \sum_{k=1}^{p+q-2} \sum_{\substack{\sigma\in sh_{(p,q-1)}\\ 1\leq\sigma^{-1}(k)\leq p\\ p+1\leq\sigma^{-1}(k+1)<p+q}} h(x_{\sigma^{-1}(1)}) \otimes \ldots\\
&\ldots \otimes h(x_{\sigma^{-1}(k-1)}) \otimes \big[x_{\sigma^{-1}(k)}, x_{\sigma^{-1}(k+1)}\big] \otimes h(x_{\sigma^{-1}(k+2)}) \otimes \ldots\\
&\ldots \otimes h(x_{\sigma^{-1}(p+q-1)}) \otimes g(x_{p+q}),
\end{aligned}
$$

are defined in Propositions 8 and 9.

Proof. We know that $\big(T(A), \curlywedge, f, g\big)$ is a BiHom-Zinbiel algebra and $\big(T(A), \diamondsuit, f, g\big)$ is a BiHom-pre-Lie algebra. It remains to demonstrate the following compatibility conditions between $\diamondsuit$ and $\curlywedge$:

(i) $\big(g(X)\diamondsuit f(Y) - g(Y)\diamondsuit f(X)\big) \curlywedge g(Z) = fg(X)\diamondsuit\big(f(Y)\curlywedge Z\big) - fg(Y)\curlywedge \big(f(X)\diamondsuit Z\big)$,

(ii) $\big(g(X)\curlywedge f(Y) + g(Y)\curlywedge f(X)\big)\diamondsuit g(Z) = fg(X)\curlywedge\big(f(Y)\diamondsuit Z\big) + fg(Y)\curlywedge \big(f(X)\diamondsuit Z\big)$.

As far as the proof of these two conditions is concerned, we use the simplification between multiple terms.
More precisely,
- the term $\ldots\otimes g\big(f(x_i)\diamond x_j\big)\otimes\ldots$ is simplified to $\ldots\otimes g\big(f(x_i)\diamond x_j\big)\otimes\ldots$,
- the term $\ldots\otimes f\big(\big[g(x_i), f(x_j)\big]\big)\otimes\ldots$ is simplified to the two following terms $\ldots\otimes f\big(g(x_i)\diamond f(x_j)\big)\otimes\ldots$ and $\ldots\otimes f\big(g(x_j)\diamond f(x_i)\big)\otimes\ldots$. □

3.2.2. Rota–Baxter operator and BiHom-pre-Poisson algebras.
We examine certain properties of Rota–Baxter operator on both BiHom-commutative algebra and BiHom-Lie algebra.

Definition 16. Let $\big(A, \mu, f, g\big)$ be a BiHom-commutative algebra. A linear map $\beta : A \longrightarrow A$ is called a *Rota–Baxter operator on a BiHom-commutative algebra*, if

$$
\begin{aligned}
\beta f &= f\beta,\\
\beta g &= g\beta,\\
\mu\big(\beta(x), \beta(y)\big) &= \beta\Big(\mu\big(\beta(x), y\big) + \mu\big(x, \beta(y)\big)\Big),
\end{aligned}
$$

for all $x, y \in A$.

Proposition 10. *Let $\big(A, \mu, f, g\big)$ be a BiHom-commutative algebra and let $\beta : A \longrightarrow A$ be a Rota–Baxter operator on A. Define a new operation on A by*

$$x \wedge y = \mu\big(\beta(x), y\big), \quad \forall x, y \in A.$$

Then $\big(A, \wedge, f, g\big)$ *is a BiHom-Zinbiel algebra.*

Proof. For all $x, y, z \in A$, we compute

$$\begin{aligned}
& fg(x) \wedge \big(f(y) \wedge z\big) \\
&= fg(x) \wedge \mu\big(\beta\big(f(y)\big), z\big) \\
&= \mu\Big(\beta\big(fg(x)\big), \mu\big(\beta\big(f(y)\big), z\big)\Big) \\
&= \mu\Big(f\big(\beta\big(g(x)\big)\big), \mu\big(\beta\big(f(y)\big), z\big)\Big) \\
&= \mu\Big(\mu\Big(\beta\big(g(x)\big), \beta\big(f(y)\big)\Big), g(z)\Big) \\
&= \mu\Big(\beta\Big(\mu\big(\beta\big(g(x)\big), f(y)\big) + \mu\big(g(x), \beta\big(f(y)\big)\big)\Big), g(z)\Big) \\
&= \mu\Big(\beta\Big(\mu\big(\beta\big(g(x)\big), f(y)\big)\Big), g(z)\Big) + \mu\Big(\beta\Big(\mu\big(g(x), f\big(\beta(y)\big)\big)\Big), g(z)\Big) \\
&= \mu\Big(\beta\Big(\mu\big(\beta\big(g(x)\big), f(y)\big)\Big), g(z)\Big) + \mu\Big(\beta\Big(\mu\big(g\big(\beta(y)\big), f(x)\big)\Big), g(z)\Big) \\
&= \mu\big(\beta\big(g(x)\big), f(y)\big) \wedge g(z) + \mu\big(g\big(\beta(y)\big), f(x)\big) \wedge g(z) \\
&= \big(g(y) \wedge f(x)\big) \wedge g(z) + \big(g(x) \wedge f(y)\big) \wedge g(z).
\end{aligned}$$

Similarly, we compute the condition (4). This completes the proof. □

Definition 17. A linear map $\beta : A \longrightarrow A$ is called a *Rota–Baxter operator on a BiHom-Lie algebra* $\big(A, [-,-], f, g\big)$, if

$$\begin{aligned}
\beta f &= f\beta, \\
\beta g &= g\beta, \\
\big[\beta(x), \beta(y)\big] &= \beta\Big(\big[\beta(x), y\big] + \big[x, \beta(y)\big]\Big),
\end{aligned}$$

for all $x, y, z \in A$.

Proposition 11. *Let* $\big(A, [-,-], f, g\big)$ *be a BiHom-Lie algebra and* $\beta : A \longrightarrow A$ *be a Rota–Baxter operator on A. Define a new product on A by*

$$x \diamond y = [\beta(x), y], \quad \forall x, y \in A.$$

Then $\big(A, \diamond, f, g\big)$ *is a BiHom-pre-Lie algebra.*

Bringing the previous results together, we have the following theorem.

Theorem 2. *Let* $\big(A, \mu, \{-,-\}, f, g\big)$ *be a BiHom-Poisson algebra and let* $\beta : A \longrightarrow A$ *be a Rota–Baxter operator on both BiHom-algebra* $\big(A, \mu, f, g\big)$ *and* $\big(A, \{-,-\}, f, g\big)$*. Define a new operations on A by*

$$x \wedge y = \mu\big(\beta(x), y\big) \quad \text{and} \quad x \diamond y = \{\beta(x), y\}, \quad \text{for all } x, y \in A.$$

Then $\big(A, \wedge, \diamond, f, g\big)$ *is a BiHom-pre-Poisson algebra.*

Proof. We already demonstrated that $\big(A, \wedge, f, g\big)$ is a BiHom-Zinbiel algebra and $\big(A, \diamond, f, g\big)$ is a BiHom-pre-Lie algebra. It remains to confirm the compatibility conditions.

For any $x, y, z \in A$, we compute

$$\begin{aligned}
&\big(g(x)\diamond f(y)-g(y)\diamond f(x)\big)\wedge g(z)-fg(x)\diamond\big(f(y)\wedge z\big)+fg(y)\wedge\big(f(x)\diamond z\big)\\
&=\Big(\big\{\beta\big(g(x)\big),f(y)\big\}-\big\{\beta\big(g(y)\big),f(x)\big\}\Big)\wedge g(z)-fg(x)\diamond\mu\big(\beta\big(f(y)\big),z\big)\big)\\
&\quad+fg(y)\wedge\big\{\beta\big(f(x)\big),z\big\}\\
&=\big\{\beta\big(g(x)\big),f(y)\big\}\wedge g(z)-\big\{\beta\big(g(y)\big),f(x)\big\}\wedge g(z)-fg(x)\diamond\mu\big(\beta\big(f(y)\big),z\big)\\
&\quad+fg(y)\wedge\big\{\beta\big(f(x)\big),z\big\}\\
&=\mu\Big(\beta\Big(\big\{\beta\big(g(x)\big),f(y)\big\}\Big),g(z)\Big)-\mu\Big(\beta\Big(\big\{\beta\big(g(y)\big),f(x)\big\}\Big),g(z)\Big)\\
&\quad-\big\{\beta\big(fg(x)\big),\mu\big(\beta\big(f(y)\big),z\big)\big\}+\mu\Big(\beta\big(fg(y)\big),\big\{\beta\big(f(x)\big),z\big\}\Big)\\
&=\mu\Big(\beta\Big(\big\{\beta\big(g(x)\big),f(y)\big\}-\big\{g\big(\beta(y)\big),f(x)\big\}\Big),g(z)\Big)\\
&\quad-\big\{fg\big(\beta(x)\big),\mu\big(f\big(\beta(y)\big),z\big)\big\}+\mu\Big(fg\big(\beta(y)\big),\big\{f\big(\beta(x)\big),z\big\}\Big)\\
&=\mu\Big(\beta\Big(\big\{\beta\big(g(x)\big),f(y)\big\}+\big\{g(x),f\big(\beta(y)\big)\big\}\Big),g(z)\Big)\\
&\quad-\Big(\big\{fg\big(\beta(x)\big),\mu\big(f\big(\beta(y)\big),z\big)\big\}-\mu\Big(gf\big(\beta(y)\big),\big\{f\big(\beta(x)\big),z\big\}\Big)\\
&=\mu\Big(\big\{\beta\big(g(x)\big),\beta\big(f(y)\big)\big\},g(z)\Big)-\mu\Big(\big\{g\big(\beta(x)\big),f\big(\beta(y)\big)\big\},g(z)\Big)\\
&=\mu\Big(\big\{\beta\big(g(x),\beta\big(f(y)\big)\big\},g(z)\Big)-\mu\Big(\big\{\beta\big(g(x),\beta\big(f(y)\big)\big\},g(z)\Big)\\
&=0,
\end{aligned}$$

which implies that
$\big(g(x)\diamond f(y)-g(y)\diamond f(x)\big)\wedge g(z)=fg(x)\diamond\big(f(y)\wedge z\big)-fg(y)\wedge\big(f(x)\diamond z\big)$.
Likewise, we prove
$\big(g(x)\wedge f(y)+g(y)\wedge f(x)\big)\diamond g(z)=fg(x)\wedge\big(f(y)\diamond z\big)+fg(y)\wedge\big(f(x)\diamond z\big)$.
The proof is thus complete. □

4. Dual BiHom-pre-Poisson algebras

In this section, we introduce the structures of a BiHom-permutative algebra, a BiHom-Leibniz algebra and a dual BiHom-pre-Poisson algebra. Additionally, we provide an example of a dual BiHom-pre-Poisson algebra using an averaging operator on a BiHom-Poisson algebra.

4.1. Dual BiHom-pre-Poisson algebras.

Definition 18. Let A be a vector space, let $\bullet : A\times A \longrightarrow A$ be a bilinear map and let f, $g : A \longrightarrow A$ be two endomorphisms. The quadruple $\big(A, \bullet, f, g\big)$ is called a *BiHom-permutative algebra* if it satisfies

$$f(x \bullet y) = f(x) \bullet f(y),$$
$$g(x \bullet y) = g(x) \bullet g(y),$$
$$fg(x) \bullet \big(f(y) \bullet z\big) = \big(g(x) \bullet f(y\big) \bullet g(z) = \big(g(y) \bullet f(x)\big) \bullet g(z),$$
for any $x, y, z \in A$.

Definition 19. Let $\big(A, \bullet, f, g\big)$ and let $\big(A', \bullet', \varphi, \psi\big)$ be two BiHom-permutative algebras. A linear map $\alpha : A \longrightarrow A'$ is said to be a *morphism of BiHom-permutative algebras* if

$\alpha(x \bullet y) = \alpha(x) \bullet' \alpha(y)$, for all $x, y \in A$, as well as $\alpha f = \varphi\alpha$ and $\alpha g = \psi\alpha$.

Proposition 12. *Let $(A, \bullet)$ be a permutative algebra and let $f,\ g : A \longrightarrow A$ be two commuting morphisms of a permutative algebra. Then $\big(A, \bullet_{(f,g)}, f, g\big)$ is a BiHom-permutative algebra obtained by composition, where $x \bullet_{(f,g)} y = f(x) \bullet g(y)$ for all $x, y \in A$.*

Proof. We depart from the assumption that f and g are multiplicative. By straightforward computations, we obtain
$fg(x) \bullet_{(f,g)} \big(f(y) \bullet_{(f,g)} z\big) = \big(g(x) \bullet_{f,g} f(y)\big) \bullet_{f,g} g(z)$ and
$\big(g(x) \bullet_{f,g} f(y\big) \bullet_{f,g} g(z) = \big(g(y) \bullet_{f,g} f(x)\big) \bullet_{f,g} g(z)$, $\forall x, y, z \in A$,
which is the desired result. □

Remark 10. Let $(A, \bullet)$ be a permutative algebra and let $f,\ g : A \longrightarrow A$ be two commuting morphisms of a permutative algebra. Assume that $(A', \bullet')$ is another permutative algebra and $\varphi,\ \psi : A' \longrightarrow A'$ are two commuting morphisms of a permutative algebra satisfying $\alpha \circ f = \varphi \circ \alpha$ and $\alpha \circ g = \psi \circ \alpha$. Then $\alpha : \big(A, \bullet_{(f,g)}, f, g\big) \longrightarrow \big(A', \bullet'_{(\varphi,\psi)}, \varphi, \psi\big)$ is a morphism of BiHom-permutative algebras.

Definition 20 ([11])**.** Let A be a vector space, let $\{-, -\} : A \times A \longrightarrow A$ be a bilnear map and let $f,\ g : A \longrightarrow A$ be two endomorphisms. The 4-uple $\big(A, \{-, -\}, f, g\big)$ is called a *BiHom-Leibniz algebra*, if it satisfies

$$f\big(\{x, y\}\big) = \big\{f(x), f(y)\big\},$$
$$g\big(\{x, y\}\big) = \big\{g(x), g(y)\big\},$$
$$\big\{\big\{g(x), y\big\}, g(z)\big\} = \big\{fg(x), \big\{y, z\big\}\big\} - \big\{g(y), \big\{f(x), z\big\}\big\},$$
for all $x, y, z \in A$.

Definition 21. Let $\big(A, \{-, -\}, f, g\big)$ and $\big(A', \{-, -\}', \varphi, \psi\big)$ be two BiHom-Leibniz algebras. A linear map $\alpha : A \longrightarrow A'$ is said to be a *morphism of BiHom-Leibniz algebras* if

$\alpha(\{x, y\}) = \{\alpha(x), \alpha(y)\}$ for all $x, y \in A$, as well as $\alpha f = \varphi\alpha$ and $\alpha g = \psi\alpha$.

Proposition 13. *Let $(A, \{-, -\})$ be a Leibniz algebra and let $f,\ g : A \longrightarrow A$ be two commuting morphisms of A. Then $\big(A, \{-, -\}_{(f,g)}, f, g\big)$ is a BiHom-Leibniz algebra obtained by composition, where $\{x, y\}_{(f,g)} = \big\{f(x), g(y)\big\}$ for all $x, y \in A$.*

Proof. It is clear that f and g are multiplicative and through a direct computation, for all $x, y, z \in A$, it follows that

$$\begin{aligned}
&\big\{\{g(x), y\}_{(f,g)}, g(z)\big\}_{(f,g)}\\
&= \big\{\big\{fg(x), g(y)\big\}, g(z)\big\}_{(f,g)}\\
&= \big\{\big\{f^2g(x), fg(y)\big\}, g^2(z)\big\}\\
&= \big\{f^2g(x), \{g(y), g(z)\}_{(f,g)}\big\} - \big\{fg(y), \{fg(x), g(z)\}_{(f,g)}\big\}\\
&= \big\{fg(x), \{y, z\}_{(f,g)}\big\}_{(f,g)} - \big\{g(y), \{f(x), z\}_{(f,g)}\big\}_{(f,g)}\\
&= \big\{fg(x), \{y, z\}_{(f,g)}\big\}_{(f,g)} - \big\{g(y), \{f(x), z\}_{(f,g)}\big\}_{(f,g)}.
\end{aligned}$$

□

Remark 11. Let $(A, \{-,-\})$ be a Leibniz algebra and let $f, g : A \longrightarrow A$ be two commuting morphisms of a Leibniz algebra. Assume that $(A', \{-,-\}')$ is another Leibniz algebra and let $\varphi,\ \psi : A' \longrightarrow A'$ be two commuting morphisms of a Leibniz algebra satisfying $\alpha \circ f = \varphi \circ \alpha$ and $\alpha \circ g = \psi \circ \alpha$. Then $\alpha : \big(A, \{-,-\}_{(f,g)}, f, g\big) \longrightarrow \big(A', \{-,-\}'_{(\varphi,\psi)}, \varphi, \psi\big)$ is a morphism of BiHom-Leibniz algebras.

Definition 22. Let A be a vector space, let $\bullet$, $\{-,-\} : A \times A \longrightarrow A$ be two bilinear maps and let $f, g : A \longrightarrow A$ be two endomorphisms. The 5-tuple $\big(A, \bullet, \{-,-\}, f, g\big)$ is called a *dual BiHom-pre-Poisson algebra*, if it satisfies

(1) $\big(A, \bullet, f, g\big)$ is a BiHom-permutative algebra,
(2) $\big(A, \{-,-\}, f, g\big)$ is a BiHom-Leibniz algebra,
(3) the compatibility conditions
    i) $\big\{fg(x), y \bullet z\big\} = \big\{g(x), y\big\} \bullet g(z) + g(y) \bullet \big\{f(x), z\big\}$,
    ii) $\big\{g(x) \bullet y, g(z)\big\} = fg(x) \bullet \big\{y, z\big\} + g(y) \bullet \big\{f(x), z\big\}$,
    iii) $\big\{g(x), f(y)\big\} \bullet fg(z) + \big\{g(y), f(x)\big\} \bullet fg(z) = 0$,
    are satisfied for all $x, y, z \in A$.

Definition 23. Let $\big(A, \mu, \{-,-\}, f, g\big)$ and let $\big(A', \mu', \{-,-\}', \varphi, \psi\big)$ be two dual BiHom-pre-Poisson algebras. A linear map $\alpha : A \longrightarrow A'$ is said to be a *morphism of a dual BiHom-pre-Poisson algebra* if α is a morphism of both a BiHom-permutative algebra and a BiHom-Leibniz algebra, as well as $\alpha \circ f = \varphi \circ \alpha$ and $\alpha \circ g = \psi \circ \alpha$.

Proposition 14. *Let $(A, \bullet, \{-,-\})$ be a Poisson algebra and let $f, g :$ $A \longrightarrow A$ be two commuting morphisms of a pre-Poisson algebra. Define a new products on A by*

$$x \bullet_{(f,g)} y = f(x) \bullet g(y) \quad \textit{and} \quad \{x, y\}_{(f,g)} = \{f(x), f(y)\},\ \forall x, y \in A.$$

Then $\big(A, \bullet, \{-,-\}, f, g\big)$ *is called a dual BiHom-pre-Poisson algebra obtained by composition.*

Remark 12. Let $(A, \bullet, \{-,-\})$ be a dual BiHom-pre-Poisson algebra and let $f,\ g:\ A \longrightarrow A$ be two commuting morphisms of A. Assume that $(A', \bullet', \{-,-\}')$ is another dual BiHom-pre-Poisson algebra and let $\varphi,\ \psi:\ A' \longrightarrow A'$ be two commuting morphisms of a dual BiHom-pre-Poisson algebra satisfying $\alpha \circ f = \varphi \circ \alpha$ and $\alpha \circ g = \psi \circ \alpha$. Then

$$\alpha : \big(A, \bullet_{(f,g)}, \{-,-\}_{(f,g)}, f, g\big) \longrightarrow \big(A', \bullet'_{(\varphi,\psi)}\{-,-\}'_{(\varphi,\psi)}, \varphi, \psi\big)$$

is a morphism of dual BiHom-pre-Poisson algebras.

4.2. Averaging operator and dual BiHom-pre-Poisson algebras. In this subsection, we demonstrate that using an averaging operator over BiHom-commutative, BiHom-associative, BiHom-Lie and BiHom-Poisson algebra respectively, we construct BiHom-permutative, BiHom-diassociative, BiHom-Leibniz and dual BiHom-pre-Poisson algebras respectively.

Definition 24. A linear map $\alpha:\ A \longrightarrow A$ is called an *averaging operator on a BiHom-commutative algebra* $\big(A, \mu, f, g\big)$ if

$$\alpha f = f\alpha,$$
$$\beta g = g\beta,$$
$$\mu\big(\alpha(x), \alpha(y)\big) = \alpha\big(\mu\big(\alpha(x), y\big)\big) = \alpha\big(\mu\big(x, \alpha(y)\big)\big),$$

for all $x, y \in A$.

Proposition 15. *Let* $\big(A, \mu, f, g\big)$ *be a BiHom-commutative algebra and let* $\alpha:\ A \longrightarrow A$ *be an averaging operator on* $\big(A, \mu, f, g\big)$*. Define a new operation on* A *by*

$$x \bullet y = \mu\big(\alpha(x), y\big),\ \forall x, y \in A.$$

Then $\big(A, \bullet, f, g\big)$ *is a BiHom-permutative algebra.*

Definition 25. Let $\big(A, [-,-], f, g\big)$ be a BiHom-Lie algebra. A linear map $\alpha:\ A \longrightarrow A$ is called an *averaging operator on* A if

$$\alpha f = f\alpha,$$
$$\alpha g = g\alpha,$$
$$[\alpha(x), \alpha(y)] = \alpha\big([\alpha(x), y]\big),$$

for all $x, y, z \in A$.

Proposition 16. *Let* $\big(A, [-,-], f, g\big)$ *be a BiHom-Lie algebra such that* $f,\ g$ *are commuting and let* $\alpha:\ A \longrightarrow A$ *be an averaging operator on* $\big(A, [-,-], f, g\big)$*. Define a new operation on* A *by*

$$\{x, y\} = [\alpha(x), y],\ \forall x, y \in A.$$

Then $\big(A, \{-,-\}, f, g\big)$ *is a BiHom-Leibniz algebra.*

Proof. We use the assumption that f and g are multiplicative. By straightforward calculation, we obtain

$$\big\{fg(x), \{y, z\}\big\} = \big\{g(y), \{f(x), z\}\big\} + \big\{\{g(x), y\}, g(z)\big\}.$$ □

Now, we build up the structure of a dual BiHom-pre-Poisson algebra using an averaging operator on a BiHom-Poisson algebra. The following theorem holds.

Theorem 3. *Let $\big(A, \mu, [-,-], f, g\big)$ be a BiHom-Poisson algebra and let $\alpha : A \longrightarrow A$ be an averaging operator on both BiHom-commutative algebra $\big(A, \mu, f, g\big)$ and BiHom-Lie algebra $\big(A, [-,-], f, g\big)$. Define new operations on A by*

$$x \bullet y = \mu\big(\alpha(x), y\big) \quad \text{and} \quad \{x, y\} = [\alpha(x), y], \ \forall x, y \in A.$$

Then $\big(A, \bullet, \{-,-\}, f, g\big)$ is a dual BiHom-pre-Poisson algebra.

Proof. Grounded on Propositions 15 and 16 we know that, $\big(A, \bullet, f, g\big)$ is a BiHom-permutative algebra and $\big(A, \{-,-\}, f, g\big)$ is a BiHom-Leibniz algebra. We only prove the first compatibility condition and leave the rest to the reader.

For all $x, y, z \in A$, we compute:

$$\begin{aligned}
&\big\{fg(x), y \bullet z\big\} - \big\{g(x), y\big\} \bullet g(z) - g(y) \bullet \big\{f(x), z\big\} \\
&= \big\{fg(x), \mu\big(\alpha(y), z\big)\big\} - \big[\alpha\big(g(x)\big), y\big] \bullet g(z) - g(y) \bullet \big[\alpha\big(f(x)\big), z\big] \\
&= \big[\alpha\big(fg(x)\big), \mu\big(\alpha(y), z\big)\big] - \mu\Big(\alpha\big(\big[\alpha\big(g(x)\big), y\big]\big), g(z)\Big) \\
&\quad - \mu\Big(\alpha\big(g(y)\big), \big[\alpha\big(f(x)\big), z\big]\Big) \\
&= \big[fg\big(\alpha(x)\big), \mu\big(\alpha(y), z\big)\big] - \mu\Big(\big[g\big(\alpha(x)\big), \alpha(y)\big], g(z)\Big) \\
&\quad - \mu\Big(g\big(\alpha(y)\big), \big[f\big(\alpha(x)\big), z\big]\Big) \\
&= \mu\Big(\big[g\big(\alpha(x)\big), \alpha(y)\big], g(z)\Big) + \mu\Big(g\big(\alpha(y)\big), \big[f\big(\alpha(x)\big), z\big]\Big) \\
&\quad - \mu\Big(\big[g\big(\alpha(x)\big), \alpha(y)\big], g(z)\Big) - \mu\Big(g\big(\alpha(y)\big), \big[f\big(\alpha(x)\big), z\big]\Big) \\
&= 0.
\end{aligned}$$

□

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