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Rate of convergence of Fourier–Legendre series of functions of the class $(n^\alpha)BV^p[-1,1]$

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Abstract. In this paper, the rate of convergence of the Fourier–Legendre series of functions of the class $(n^\alpha)BV^p[-1,1]$ and in particular, the class $BV^p[-1,1]$, are estimated. The result obtained is similar to a result of Bojanić and Vuilleumier for the Fourier–Legendre series of functions of bounded variation, and is applicable to a wider class.

1. Introduction

In this section, we recall certain results about pointwise convergence and rate of convergence of a Fourier–Legendre series. We need the following definitions.

Definition 1. Given a function $f : [a, b] \to \mathbb{R}$, a non-decreasing sequence $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}}$ of positive numbers such that $\sum \frac{1}{\lambda_k}$ diverges, and a real number p, $1 \le p < \infty$, we say that $f \in \Lambda\mathrm{BV}^p[a, b]$ (that is, f is of *p-Λ-bounded variation over* $[a, b]$) if

$$V_{p\Lambda}(f, [a, b]) = \sup \left\{ \sum_{k=1}^{n} \frac{|f(a_k) - f(b_k)|^p}{\lambda_k} \right\}^{1/p} < \infty,$$

where the supremum is extended over all sequences $\{I_k\}$ of non-overlapping intervals with $I_k = [a_k, b_k] \subset [a, b],\ k = 1, \ldots, n$.

When $\Lambda = \{1\}$ and $p = 1$, the class is referred to as the class of functions of bounded variation (BV) and we denote the corresponding variation of f over an interval $[a, b]$ by $V(f, a, b)$. When $\Lambda = \{n^\alpha\}$, $0 < \alpha < 1$ and $p = 1$, we denote the class by $(n^\alpha)\mathrm{BV}$ and the corresponding variation of f over

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an interval $[a, b]$ by $V_{n^\alpha}(f, a, b)$. When $\Lambda = \{n^\alpha\}$, $0 \leq \alpha < 1$, we denote the class by $(n^\alpha)\mathrm{BV}^p$ and the corresponding variation of f over an interval $[a, b]$ by $V_{pn^\alpha}(f, a, b)$. When $\Lambda = \{1\}$, the class is referred to as the class of functions of p-bounded variation (BV^p) and we denote the corresponding variation of f over an interval $[a, b]$ by $V_p(f, a, b)$.

We note that if f is of p-Λ-bounded variation, then $f(x+0)$ and $f(x-0)$ exist at every point x of $[a, b]$ (see [6, Theorem 2]). We define, for $x \in [a, b]$,

$$s(f, x) = \frac{1}{2}(f(x+0) + f(x-0))$$

and

$$\phi_x(t) = \begin{cases} f(t) - f(x-0), & a \leq t < x, \\ 0, & t = x, \\ f(t) - f(x+0), & x < t \leq b. \end{cases}$$

Definition 2. Let $P_n(x)$ be the Legendre polynomial of degree n normalized so that $P_n(1) = 1$. If f is an integrable function on $[-1, 1]$, then the *Fourier–Legendre series* (see, e.g., [5, p. 237, Section 8.3]) of f is the series

$$\sum_{k=0}^{\infty} a_k(f) P_k(x)$$

where

$$a_k(f) = \left(k + \frac{1}{2}\right) \int_{-1}^{1} f(t) P_k(t) dt, \quad k = 0, 1, 2, \ldots.$$

The n^{th} symmetric partial sum of the Fourier–Legendre series of f, denoted by $S_n(f, x)$, is defined as

$$S_n(f, x) = \sum_{k=0}^{n} a_k(f) P_k(x), \quad n = 0, 1, 2, \ldots,$$

which can be written as

$$S_n(f, x) = \int_{-1}^{1} f(t) K_n(x, t) dt,$$

where

$$K_n(x, t) = \sum_{k=0}^{n} \left(k + \frac{1}{2}\right) P_k(x) P_k(t),$$

or equivalently (see [2]),

$$K_n(x, t) = \frac{n+1}{2} \left(\frac{P_{n+1}(x) P_n(t) - P_n(x) P_{n+1}(t)}{x - t} \right).$$

Definition 3. The (*ordinary*) *oscillation* of a function $h : [a, b] \to \mathbb{C}$ over a subinterval J of $[a, b]$ is defined as

$$\operatorname{osc}(h, J) = \sup\{|h(t) - h(t')| : t, t' \in J\}.$$

We also define nodes $s_{j,n}$ and $t_{j,n}$ as

$$s_{j,n} = x + \frac{j(1-x)}{n}, \quad t_{j,n} = x - \frac{j(1+x)}{n}, \quad j = 0, 1, \dots, n,$$

for $x \in (-1, 1)$.

Hobson [3] proved the following theorem concerning the pointwise convergence of Fourier–Legendre series of functions of bounded variation.

Theorem 1. *If f is of bounded variation on $[-1,1]$, then its Fourier–Legendre series converges to $s(f,x)$ at each point $x \in (-1,1)$, i.e.,*

$$\lim_{n\to\infty} (S_n(f,x) - s(f,x)) = 0.$$

Bojanić and Vuilleumier [2, Theorem 1] has quantified Theorem 1 by estimating the rate of convergence of the Fourier–Legendre series at that point by proving the following theorem.

Theorem 2. *Let f be a function of bounded variation on $[-1,1]$. Then, for $x \in (-1,1)$ and $n \geq 2$, we have*

$$\begin{aligned} |S_n(f,x) - s(f,x)| \leq & \frac{28}{n(1-x^2)^{3/2}} \sum_{j=1}^{n} V(\phi_x, t_{1,j}, s_{1,j}) \\ & + \frac{1}{\pi n(1-x^2)} |f(x+0) - f(x-0)|. \end{aligned} \tag{1}$$

The right-hand side of (1) converges to zero as $n \to \infty$, since continuity of $\phi_x(t)$ at $t = x$ implies that

$$V(\phi_x, x-\delta, x+\delta) \to 0 \text{ as } \delta \to 0_+.$$

In [1], we have extended Theorem 2 for functions of the class (n^α)BV as follows.

Theorem 3. *Let $f \in (n^\alpha)BV[-1,1]$, $0 < \alpha < 1$. Then, for $x \in (-1,1)$ and $n \geq 2$, we have*

$$\begin{aligned} & |S_n(f,x) - s(f,x)| \\ & \leq \frac{c_\alpha}{n^{1-\alpha}} \sum_{j=1}^{n-1} \frac{1}{j^\alpha} [c_{1,x} V_{n^\alpha}(\phi_x, t_{1,j}, x) + c_{2,x} V_{n^\alpha}(\phi_x, x, s_{1,j})] \\ & \quad + \sum_{j=[n/2]+1}^{n-1} \frac{(c_{1,x} V_{n^\alpha}(\phi_x, t_{j+1,n}, t_{j,n}) + c_{2,x} V_{n^\alpha}(\phi_x, s_{j,n}, s_{j+1,n}))}{(j(n-j))^{1/2}} \\ & \quad + \frac{(1-x^2)^{-1}}{\pi n} |f(x+0) - f(x-0)|, \end{aligned} \tag{2}$$

where $c_\alpha = 1 - \alpha + 2^{1-\alpha}$ and $c_{i,x} = \frac{2}{(1-x^2)^{3/2}} \left(\frac{8(1-x^2)^{1/2}}{\pi} + 3(1 + (-1)^i x) \right)$, for $i = 1, 2$.

The right-hand side of (2) converges to zero as $n \to \infty$, since continuity of $\phi_x(t)$ at $t = x$ implies (in view of [8, Theorem 3]) that

$$V_\alpha(\phi_x, x-\delta, x+\delta) \to 0 \text{ as } \delta \to 0_+$$

and also $V_\alpha(\phi_x, x+\delta', x+\delta)$ and $V_\alpha(\phi_x, x-\delta, x-\delta')$ tends to zero as $0 < \delta' < \delta \to 0_+$.

In the present paper, our main goal is to estimate the rate of convergence of Fourier–Legendre series of functions of the class $(n^\alpha)BV^p$ and in particular, of the p-bounded variation class.

2. Main result

Our main result is the following.

Theorem 4. *Let $f \in (n^\alpha)BV^p[-1,1]$ for $p > 1$ and $0 < \alpha < 1/p$. Then, for $x \in (-1,1)$ and $n \geq 2$, we have*

$$\begin{aligned}
&|S_n(f,x) - s(f,x)| \\
&\leq \left[\frac{C_{\alpha,p}}{n^{1/p-\alpha}} \sum_{j=1}^{n-1} \frac{1}{j^{1+\alpha-1/p}} \left[C_1^p V_{pn^\alpha}^p(\phi_x, t_{1,j}, x) + C_2^p V_{pn^\alpha}^p(\phi_x, x, s_{1,j})\right]\right]^{1/p} \\
&\quad + \frac{4\sqrt{2}}{1-x^2} \sum_{j=[n/2]+1}^{n-1} \frac{(V_{pn^\alpha}(\phi_x, t_{j+1,n}, t_{j,n}) + V_{pn^\alpha}(\phi_x, s_{j,n}, s_{j+1,n}))}{(j(n-j))^{1/2}} \\
&\quad + \frac{1}{\pi n(1-x^2)} |f(x+0) - f(x-0)|, \qquad (3)
\end{aligned}$$

where

$$C_1 = \frac{1}{1-x^2}\left(4\sqrt{2} + 6\sqrt{\left(\frac{1+x}{1-x}\right)}\right),$$

$$C_2 = \frac{1}{1-x^2}\left(4\sqrt{2} + 6\sqrt{\left(\frac{1-x}{1+x}\right)}\right),$$

and $$C_{\alpha,p} = 2^p(1-\alpha+1/p)\left(\sum_{j=1}^{n} j^{-1-1/p}\right)^{p-1}.$$

The right-hand side of (3) converges to zero as $n \to \infty$, since continuity of $\phi_x(t)$ at $t = x$ implies (in view of ([4, Lemma 2.2])) that

$$V_{pn^\alpha}(\phi_x, x-\delta, x+\delta) \to 0 \text{ as } \delta \to 0_+$$

and also $V_{pn^\alpha}(\phi_x, x+\delta', x+\delta)$ and $V_{pn^\alpha}(\phi_x, x-\delta, x-\delta')$ tend to zero as $0 < \delta' < \delta \to 0_+$.

In particular, for $\alpha = 0$, our Theorem 4 may be viewed as a quantitative result for convergence of Fourier–Legendre series of functions of p-bounded variation class. It sounds as follows.

Corollary 1. *Let $f \in BV^p[-1,1]$ for $p > 1$. Then, for $x \in (-1,1)$ and $n \geq 2$, we have*

$$
\begin{aligned}
&|S_n(f,x) - s(f,x)| \\
&\quad \leq \left[\frac{C_{0,p}}{n^{1/p}} \sum_{j=1}^{n-1} \frac{1}{j^{1-1/p}} \left[C_1^p V_p^p\left(\phi_x, t_{1,j}, x\right) + C_2^p V_p^p\left(\phi_x, x, s_{1,j}\right) \right] \right]^{1/p} \\
&\quad\quad + \frac{4\sqrt{2}}{1-x^2} \sum_{j=[n/2]+1}^{n-1} \frac{\left(V_p(\phi_x, t_{j+1,n}, t_{j,n}) + V_p(\phi_x, s_{j,n}, s_{j+1,n})\right)}{(j(n-j))^{1/2}} \\
&\quad\quad + \frac{1}{\pi n(1-x^2)} |f(x+0) - f(x-0)|.
\end{aligned}
$$

Remark 1. Our Theorem 4 may be viewed as a generalization of Theorem 3 for functions of $(n^{\alpha})BV^p$ and, in particular, Corollary 1 is a quantitative analogue of Theorem 2 for functions of p-bounded variation, except for exact constant.

3. Proof

The proof of Theorem 4 is based on a number of properties of Legendre polynomials. The proofs of these properties can be found in [2, Section 2] and [1]. For $x \in (-1,1)$ and $n \geq 2$, we have

$$
|P_n(x)| \leq \left(\frac{2}{\pi n(1-x^2)} \right)^{1/2}, \tag{4}
$$

$$
\int_x^1 K_n(x,t)dt = \frac{1}{2} - \frac{1}{2} P_n(x)P_{n+1}(x), \tag{5}
$$

$$
\int_{-1}^x K_n(x,t)dt = \frac{1}{2} + \frac{1}{2} P_n(x)P_{n+1}(x), \tag{6}
$$

$$
\left| \int_{-1}^t K_n(x,\tau)d\tau \right| \leq \frac{6}{n(x-t)}(1-x^2)^{-1/2}, \ \ t \in [-1,x), \tag{7}
$$

$$
\int_{t_{1,n}}^{s_{1,n}} |K_n(x,t)|dt \leq \frac{4}{1-x^2}, \tag{8}
$$

$$
\int_{t_{j+1,n}}^{t_{j,n}} |K_n(x,u)|du \leq \frac{4\sqrt{2n}}{\pi j(1-x^2)(n-j)^{1/2}}, \text{ for } j = 1, 2, \ldots, n-1. \tag{9}
$$

Proof of Theorem 4. For any fixed $x \in (-1,1)$, using equalities (5) and (6), we have

$$S_n(f,x) - s(f,x) = \int_{-1}^{1} \phi_x(t) K_n(x,t)dt - \frac{1}{2}(f(x+0) - f(x-0))P_n(x)P_{n+1}(x). \tag{10}$$

We decompose the integral on the right-hand side of (10) in two parts, as follows:

$$\int_{-1}^{1} \phi_x(u)K_n(x,u)du = \left(\int_{-1}^{x} + \int_{x}^{1}\right) \phi_x(u)K_n(x,u)du = A_1 + A_2, \text{ say.} \tag{11}$$

Now, we have

$$\begin{aligned} A_1 &= \int_{-1}^{x} \phi_x(u)K_n(x,u)du = \sum_{j=0}^{n-1} \int_{t_{j+1,n}}^{t_{j,n}} \phi_x(u)K_n(x,u)du \\ &= \sum_{j=0}^{n-1} \int_{t_{j+1,n}}^{t_{j,n}} \left(\phi_x(u) - \phi_x(t_{j,n})\right) K_n(x,u)du \\ &\quad + \sum_{j=1}^{n-1} \int_{t_{j+1,n}}^{t_{j,n}} \phi_x(t_{j,n})K_n(x,u)du \\ &= A_{11} + A_{12}, \text{ say.} \end{aligned} \tag{12}$$

Using inequalities (8) and (9), first we estimate A_{11}, as follows:

$$\begin{aligned} |A_{11}| &\leq \sum_{j=0}^{n-1} \operatorname{osc}(\phi_x, [t_{j+1,n}, t_{j,n}]) \int_{t_{j+1,n}}^{t_{j,n}} |K_n(x,u)|du \\ &= \sum_{j=1}^{n-1} \operatorname{osc}(\phi_x, [t_{j+1,n}, t_{j,n}]) \int_{t_{j+1,n}}^{t_{j,n}} |K_n(x,u)|du \\ &\quad + \operatorname{osc}(\phi_x, [t_{1,n}, t_{0,n}]) \int_{t_{1,n}}^{t_{0,n}} |K_n(x,u)|du \\ &\leq \frac{4\sqrt{2n}}{\pi(1-x^2)} \sum_{j=1}^{n-1} \frac{1}{j\sqrt{n-j}} \operatorname{osc}(\phi_x, [t_{j+1,n}, t_{j,n}]) + \operatorname{osc}(\phi_x, [t_{1,n}, t_{0,n}]) \frac{4}{1-x^2} \\ &\leq \frac{4}{(1-x^2)} \left\{ \frac{\sqrt{2}}{\pi} \sum_{j=1}^{n-1} \frac{\sqrt{n}}{j\sqrt{n-j}} \operatorname{osc}(\phi_x, [t_{j+1,n}, t_{j,n}]) + \operatorname{osc}(\phi_x, [t_{1,n}, t_{0,n}]) \right\} \\ &\leq \frac{4}{(1-x^2)} \left\{ \sum_{j=1}^{n-1} \frac{\sqrt{n}}{(j+1)\sqrt{n-j}} \operatorname{osc}(\phi_x, [t_{j+1,n}, t_{j,n}]) + \operatorname{osc}(\phi_x, [t_{1,n}, t_{0,n}]) \right\} \end{aligned}$$

$$
\begin{aligned}
&=\frac{4}{(1-x^2)}\sum_{j=0}^{n-1}\frac{\sqrt{n}}{(j+1)\sqrt{n-j}}\operatorname{osc}(\phi_x,[t_{j+1,n},t_{j,n}])\\
&\le\frac{4\sqrt{2}}{(1-x^2)}\left\{\sum_{j=0}^{[n/2]}\frac{1}{j+1}+\sum_{j=[n/2]+1}^{n-1}\frac{1}{\sqrt{j(n-j)}}\right\}\operatorname{osc}(\phi_x,[t_{j+1,n},t_{j,n}]).
\end{aligned}
\tag{13}
$$

Now, we estimate A_{12}. We have

$$
\begin{aligned}
A_{12}&=\sum_{j=1}^{n-1}\phi_x(t_{j,n})\left(\int_{-1}^{t_{j,n}}K_n(x,u)du-\int_{-1}^{t_{j+1,n}}K_n(x,u)du\right)\\
&=\sum_{j=1}^{n-1}\int_{-1}^{t_{j,n}}\phi_x(t_{j,n})K_n(x,u)du-\sum_{j=2}^{n}\int_{-1}^{t_{j,n}}\phi_x(t_{j-1,n})K_n(x,u)du\\
&=\sum_{j=1}^{n-1}\int_{-1}^{t_{j,n}}(\phi_x(t_{j,n})-\phi_x(t_{j-1,n}))K_n(x,u)du\\
&\quad+\int_{-1}^{t_{1,n}}\phi_x(t_{0,n})K_n(x,u)du-\int_{-1}^{t_{n,n}}\phi_x(t_{n-1,n})K_n(x,u)du.
\end{aligned}
$$

Since $t_{0,n}=x$, $\phi_x(t_{0,n})=\phi_x(x)=0$ and $t_{n,n}=-1$, the last two terms on the right-hand side of the above equation vanish. Also, in view of (7), we have

$$
\begin{aligned}
|A_{12}|&\le\sum_{j=1}^{n-1}|\phi_x(t_{j,n})-\phi_x(t_{j-1,n})|\left|\int_{-1}^{x-\frac{j(1+x)}{n}}K_n(x,u)du\right|\\
&\le\frac{6}{(1+x)(1-x^2)^{1/2}}\sum_{j=1}^{n-1}\frac{1}{j}\operatorname{osc}(\phi_x,[t_{j,n},t_{j-1,n}])\\
&=\frac{6(1-x)}{(1-x^2)^{3/2}}\sum_{j=0}^{n-2}\frac{1}{j+1}\operatorname{osc}(\phi_x,[t_{j+1,n},t_{j,n}]).
\end{aligned}
\tag{14}
$$

Using (13) and (14) in (12), we get

$$
\begin{aligned}
|A_1|&\le|A_{11}|+|A_{12}|\\
&\le C_1\sum_{j=0}^{n-1}\frac{\operatorname{osc}(\phi_x,[t_{j+1,n},t_{j,n}])}{j+1}+\frac{4\sqrt{2}}{(1-x^2)}\sum_{j=[n/2]+1}^{n-1}\frac{\operatorname{osc}(\phi_x,[t_{j+1,n},t_{j,n}])}{(j(n-j))^{1/2}},
\end{aligned}
\tag{15}
$$

where $C_1=\frac{1}{(1-x^2)}\left(4\sqrt{2}+6\sqrt{\frac{1-x}{1+x}}\right)$.

Similarly, one can prove

$$|A_2| \le C_2 \sum_{j=0}^{n-1} \frac{\operatorname{osc}(\phi_x, [s_{j,n}, s_{j+1,n}])}{j+1} + \frac{4\sqrt{2}}{(1-x^2)} \sum_{j=[n/2]+1}^{n-1} \frac{\operatorname{osc}(\phi_x, [s_{j,n}, s_{j+1,n}])}{(j(n-j))^{1/2}}, \tag{16}$$

where $C_2 = \frac{1}{(1-x^2)}\left(4\sqrt{2} + 6\sqrt{\frac{1+x}{1-x}}\right)$.

Therefore, from (11), (15), and (16), we have

$$\begin{aligned}
\left| \int_{-1}^{1} \phi_x(u) K_n(x,u)\,du \right| &\le |A_1| + |A_2| \\
&\le \sum_{j=0}^{n-1} \frac{C_1 \operatorname{osc}(\phi_x, [t_{j+1,n}, t_{j,n}]) + C_2 \operatorname{osc}(\phi_x, [s_{j,n}, s_{j+1,n}])}{j+1} \\
&\quad + \frac{4\sqrt{2}}{(1-x^2)} \sum_{j=[n/2]+1}^{n-1} \frac{\operatorname{osc}(\phi_x, [t_{j+1,n}, t_{j,n}]) + \operatorname{osc}(\phi_x, [s_{j,n}, s_{j+1,n}])}{(j(n-j))^{1/2}} \\
&\le \sum_{j=0}^{n-1} \frac{C_1 \operatorname{osc}(\phi_x, [t_{j+1,n}, t_{j,n}]) + C_2 \operatorname{osc}(\phi_x, [s_{j,n}, s_{j+1,n}])}{j+1} \\
&\quad + \frac{4\sqrt{2}}{(1-x^2)} \sum_{j=[n/2]+1}^{n-1} \frac{(V_{pn^\alpha}(\phi_x, t_{j+1,n}, t_{j,n}) + V_{pn^\alpha}(\phi_x, s_{j,n}, s_{j+1,n}))}{(j(n-j))^{1/2}},
\end{aligned} \tag{17}$$

because by Definition 1, we have $\operatorname{osc}(\phi_x, [t_{j+1,n}, t_{j,n}]) \le V_{pn^\alpha}(\phi_x, t_{j+1,n}, t_{j,n})$ and $\operatorname{osc}(\phi_x, [s_{j,n}, s_{j+1,n}]) \le V_{pn^\alpha}(\phi_x, s_{j,n}, s_{j+1,n})$.

Now for $p \ge 1$, applying Holder's inequality in the first sum of the right hand side of the inequality (17), we have

$$\begin{aligned}
&\sum_{j=0}^{n-1} \frac{C_1 \operatorname{osc}(\phi_x, [t_{j+1,n}, t_{j,n}]) + C_2 \operatorname{osc}(\phi_x, [s_{j,n}, s_{j+1,n}])}{(j+1)^{1-1/p^2+1/p^2}} \\
&\quad \le \left(\sum_{j=0}^{n-1} \frac{(C_1 \operatorname{osc}(\phi_x, [t_{j+1,n}, t_{j,n}]) + C_2 \operatorname{osc}(\phi_x, [s_{j,n}, s_{j+1,n}]))^p}{(j+1)^{1/p}} \right)^{1/p} \\
&\qquad \times \left(\sum_{j=0}^{n-1} \frac{1}{(j+1)^{1+1/p}} \right)^{1-1/p}
\end{aligned}$$

$$\leq 2\left(\sum_{j=0}^{n-1}\frac{(C_1\mathrm{osc}(\phi_x,[t_{j+1,n},t_{j,n}]))^p+(C_2\mathrm{osc}(\phi_x,[s_{j,n},s_{j+1,n}]))^p}{(j+1)^{1/p}}\right)^{1/p}$$
$$\times\left(\sum_{j=1}^{n}\frac{1}{j^{1+1/p}}\right)^{1-1/p}. \tag{18}$$

Now, for fixed n, let

$$\mathrm{M}_j=\sum_{i=0}^{j}\frac{1}{(i+1)^\alpha}(\mathrm{osc}(\phi_x,[t_{i+1,n},t_{i,n}]))^p,\quad j=0,1,\ldots,n-1.$$

Then it follows from Definition 1 that

$$\mathrm{M}_j\leq V^p_{pn^\alpha}(\phi_x,t_{j+1,n},x). \tag{19}$$

Also, define a function on the interval $(-1,t_{1,n}]$ by

$$\mathrm{M}(u)=\mathrm{M}_{\left[\frac{n(x-u)}{(1+x)}\right]-1},\quad u\in(-1,t_{1,n}].$$

Now, for $j=0,1,\ldots,n-2$, we have

$$\begin{aligned}u\in(t_{j+2,n},t_{j+1,n}] &\implies x-\frac{(j+2)(1+x)}{n}<u\leq x-\frac{(j+1)(1+x)}{n}\\ &\implies \frac{(j+1)(1+x)}{n}\leq x-u<\frac{(j+2)(1+x)}{n}\\ &\implies j+1\leq\frac{n(x-u)}{1+x}<j+2\implies \mathrm{M}(u)=\mathrm{M}_j.\end{aligned} \tag{20}$$

For $0<\alpha<1/p$, using the partial summation formula (see [7, Theorem 3.41]) with $a_j=\frac{1}{(j+1)^\alpha}(\mathrm{osc}(\phi_x,[t_{j+1,n},t_{j,n}]))^p$ and $b_j=(j+1)^{\alpha-1/p}$, we can write the given summation as follows:

$$\begin{aligned}\sum_{j=0}^{n-1}\frac{(\mathrm{osc}(\phi_x,[t_{j+1,n},t_{j,n}]))^p}{(j+1)^{1/p}} &=\sum_{j=0}^{n-1}\frac{(j+1)^{\alpha-1/p}}{(j+1)^\alpha}(\mathrm{osc}(\phi_x,[t_{j+1,n},t_{j,n}]))^p\\ &=\sum_{j=0}^{n-2}\mathrm{M}_j\left((j+1)^{\alpha-1/p}-(j+2)^{\alpha-1/p}\right)\\ &\qquad\qquad+n^{\alpha-1/p}\mathrm{M}_{n-1}\\ &=B_1+B_2,\quad\text{say.}\end{aligned} \tag{21}$$

We will use the properties of the Riemann–Stieltjes integral to estimate B_1. Since $0 < \alpha < 1/p$, the function $\left(-u^{\alpha-1/p}\right)$ is continuous and nondecreasing for $u > 0$. Therefore, we have

$$B_1 = \sum_{j=0}^{n-2} \mathrm{M}_j \left((j+1)^{\alpha-1/p} - (j+2)^{\alpha-1/p}\right) = \sum_{j=0}^{n-2} \mathrm{M}_j \int_{j+1}^{j+2} d(-u^{\alpha-1/p})$$
$$= \sum_{j=0}^{n-2} \mathrm{M}_j \int_{j+1}^{j+2} (1/p-\alpha)(u^{-1+\alpha-1/p})du. \tag{22}$$

Put $u = \frac{n(x-s)}{1+x}$. Then $\frac{du}{ds} = \frac{-n}{1+x}$, $u \to j+1 \iff s \to x - \frac{(j+1)(1+x)}{n} = t_{j+1,n}$, and $u \to j+2 \iff s \to x - \frac{(j+2)(1+x)}{n} = t_{j+2,n}$. Therefore

$$\int_{j+1}^{j+2} u^{-1+\alpha-1/p}du = \int_{t_{j+1,n}}^{t_{j+2,n}} \left(\frac{n(x-s)}{1+x}\right)^{-1+\alpha-1/p} \left(\frac{-n}{1+x}\right) ds$$
$$= \left(\frac{1+x}{n}\right)^{1/p-\alpha} \int_{t_{j+2,n}}^{t_{j+1,n}} (x-s)^{-1+\alpha-1/p} ds. \tag{23}$$

Using (23) in (22), and in view of (20), we have

$$B_1 = (1/p-\alpha)\left(\frac{1+x}{n}\right)^{1/p-\alpha} \sum_{j=0}^{n-2} \mathrm{M}_j \int_{t_{j+2,n}}^{t_{j+1,n}} (x-s)^{-1+\alpha-1/p} ds$$
$$= (1/p-\alpha)\left(\frac{1+x}{n}\right)^{1/p-\alpha} \int_{-1}^{x-\frac{(1+x)}{n}} \mathrm{M}(s)(x-s)^{-1+\alpha-1/p} ds. \tag{24}$$

Now, put $s = x - \frac{(1+x)}{u}$. Then, we have $s \to -1 \iff u \to 1$, $s \to x - \frac{(1+x)}{n} \iff u \to n$, and $\frac{ds}{du} = (1+x)u^{-2}$. Therefore, from (24), we have

$$B_1 \le (1/p-\alpha)\left(\frac{1+x}{n}\right)^{1/p-\alpha}$$
$$\times \int_1^n \mathrm{M}\left(x - \frac{1+x}{u}\right)\left(x - x + \frac{1+x}{u}\right)^{-1+\alpha-1/p} (1+x)u^{-2}du$$
$$= \frac{(1/p-\alpha)}{n^{1/p-\alpha}} \sum_{j=1}^{n-1} \int_j^{j+1} \mathrm{M}\left(x - \frac{1+x}{u}\right) \frac{1}{u^{1+\alpha-1/p}} du. \tag{25}$$

From the definition of $\mathrm{M}(u)$, (19) and (20), for $j \le u \le j+1$, we have

$$\mathrm{M}\left(x - \frac{1+x}{u}\right) = \mathrm{M}_{[\frac{n}{u}]-1} \le \mathrm{M}_{[\frac{n}{j}]-1} \le V^p_{pn^\alpha}\left(\phi_x, t_{[\frac{n}{j}]}, x\right)$$
$$= V^p_{pn^\alpha}\left(\phi_x, x - \frac{[\frac{n}{j}](1+x)}{n}, x\right)$$

$$\leq V^p_{pn^\alpha}\left(\phi_x, x-\frac{(1+x)}{j}, x\right) \tag{26}$$

and also

$$\frac{1}{u^{1+\alpha-1/p}} \leq \frac{1}{j^{1+\alpha-1/p}}. \tag{27}$$

Using (26) and (27) in (25), we get

$$\begin{aligned} B_1 &\leq \frac{(1/p-\alpha)}{n^{1/p-\alpha}}\sum_{j=1}^{n-1}\int_j^{j+1} V^p_{pn^\alpha}\left(\phi_x, x-\frac{(1+x)}{j}, x\right)\frac{1}{j^{1+\alpha-1/p}}du \\ &= \frac{(1/p-\alpha)}{n^{1/p-\alpha}}\sum_{j=1}^{n-1}\frac{1}{j^{1+\alpha-1/p}}V^p_{pn^\alpha}\left(\phi_x, x-\frac{(1+x)}{j}, x\right). \end{aligned} \tag{28}$$

Also, from (19), we get

$$\begin{aligned} B_2 = n^{\alpha-1/p}\mathrm{M}_{n-1} &\leq n^{\alpha-1/p}V^p_{pn^\alpha}(\phi_x, -1, x) \\ &\leq n^{\alpha-1/p}\sum_{j=1}^{n-1}\frac{1}{j^{1+\alpha-1/p}}V^p_{pn^\alpha}\left(\phi_x, x-\frac{(1+x)}{j}, x\right). \end{aligned} \tag{29}$$

Using (28) and (29) in (21), we have

$$\sum_{j=0}^{n-1}\frac{(\mathrm{osc}(\phi_x, [t_{j+1,n}, t_{j,n}]))^p}{(j+1)^{1/p}} \leq \frac{1-\alpha+1/p}{n^{1/p-\alpha}}\sum_{j=1}^{n-1}\frac{1}{j^{1+\alpha-1/p}}V^p_{pn^\alpha}\left(\phi_x, t_{1,j}, x\right). \tag{30}$$

Similarly, one can prove

$$\sum_{j=0}^{n-1}\frac{(\mathrm{osc}(\phi_x, [s_{j,n}, s_{j+1,n}]))^p}{(j+1)^{1/p}} \leq \frac{1-\alpha+1/p}{n^{1/p-\alpha}}\sum_{j=1}^{n-1}\frac{1}{j^{1+\alpha-1/p}}V^p_{pn^\alpha}\left(\phi_x, x, s_{1,j}\right). \tag{31}$$

Using (30) and (31) in (18), and then (18) in (17) we get

$$\begin{aligned} &\left|\int_{-1}^1 \phi_x(u)K_n(x,u)du\right| \\ &\leq \left[\frac{C_{\alpha,p}}{n^{1/p-\alpha}}\sum_{j=1}^{n-1}\frac{1}{j^{1+\alpha-1/p}}\left[C_1^pV^p_{pn^\alpha}\left(\phi_x, t_{1,j}, x\right)+C_2^pV^p_{pn^\alpha}\left(\phi_x, x, s_{1,j}\right)\right]\right]^{1/p} \\ &\quad + \frac{4\sqrt{2}}{1-x^2}\sum_{j=[n/2]+1}^{n-1}\frac{(V_{pn^\alpha}(\phi_x, t_{j+1,n}, t_{j,n})+V_{pn^\alpha}(\phi_x, s_{j,n}, s_{j+1,n}))}{(j(n-j))^{1/2}}, \end{aligned} \tag{32}$$

where C_1 and C_2 are as in (15) and (16), respectively. Also, using (4) in the second term on the right-hand side of (10), we get

$$\frac{1}{2}|f(x+0)-f(x-0)||P_n(x)||P_{n+1}(x)| \leq \frac{|f(x+0)-f(x-0)|}{n\pi(1-x^2)}. \tag{33}$$

This completes the proof of Theorem 4 in view of (10), (32) and (33). □

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