ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 28, Number 2, December 2024
Available online at https://ojs.utlib.ee/index.php/ACUTM

A note on modified third-order Jacobsthal quaternions and their properties

Gamaliel Morales

Dedicated to my daughter Julieta

Abstract. Modified third-order Jacobsthal quaternion sequence is defined in this study. Some properties involving this sequence, including the Binet-style formula and the generating function are presented.

1. Introduction

The Jacobsthal numbers have many interesting properties and applications in many fields of science (see, e.g., [1, 3, 4]). The Jacobsthal numbers J_n are defined by the recurrence relation

$$J_0 = 0,\ J_1 = 1,\ J_{n+2} = J_{n+1} + 2J_n,\ n \geq 0. \tag{1}$$

Another important sequence is the Jacobsthal–Lucas sequence. This sequence is defined by the recurrence relation $j_{n+2} = j_{n+1} + 2j_n$, where $j_0 = 2$ and $j_1 = 1$ (see [4]).

In [2] the Jacobsthal recurrence relation is extended to higher order recurrence relations and the basic list of identities provided by Horadam [4] is expanded and extended to several identities for some of the higher order cases. For example, the third-order Jacobsthal numbers, $\{J_n^{(3)}\}_{n\geq 0}$, and the third-order Jacobsthal–Lucas numbers, $\{j_n^{(3)}\}_{n\geq 0}$, are defined by

$$J_{n+3}^{(3)} = J_{n+2}^{(3)} + J_{n+1}^{(3)} + 2J_n^{(3)},\ J_0^{(3)} = 0,\ J_1^{(3)} = J_2^{(3)} = 1, \tag{2}$$

and

$$j_{n+3}^{(3)} = j_{n+2}^{(3)} + j_{n+1}^{(3)} + 2j_n^{(3)},\ j_0^{(3)} = 2,\ j_1^{(3)} = 1,\ j_2^{(3)} = 5, \tag{3}$$

respectively.

Received April 2, 2024.
2020 *Mathematics Subject Classification.* 11B37, 11B39, 11B83.
Key words and phrases. Modified third-order Jacobsthal number, recurrence relation, third-order Jacobsthal number, quaternion.
https://doi.org/10.12697/ACUTM.2024.28.13

Using standard techniques for solving recurrence relations, the auxiliary equation, and its roots are given by

$$x^3 - x^2 - x - 2 = 0;\ x = 2,\ \text{and}\ x = \frac{-1 \pm i\sqrt{3}}{2}.$$

Note that the latter two are the complex conjugate cube roots of unity. Call them ω_1 and ω_2, respectively. Thus the Binet formulas can be written as

$$J_n^{(3)} = \frac{1}{7}\left[2^{n+1} - \left(\frac{3+2i\sqrt{3}}{3}\right)\omega_1^n + \left(\frac{3-2i\sqrt{3}}{3}\right)\omega_2^n\right] \tag{4}$$

and

$$j_n^{(3)} = \frac{1}{7}\left[2^{n+3} + \left(3+2i\sqrt{3}\right)\omega_1^n + \left(3-2i\sqrt{3}\right)\omega_2^n\right], \tag{5}$$

respectively. For more details on these sequences see [5, 6, 7, 8, 2].

On the other hand, the real quaternions are a number system which extends the complex numbers. In [5], Cerda-Morales defined a new type of quaternions with the third-order Jacobsthal and third-order Jacobsthal–Lucas number components as

$$JQ_n^{(3)} = J_n^{(3)} + J_{n+1}^{(3)}\mathbf{i} + J_{n+2}^{(3)}\mathbf{j} + J_{n+3}^{(3)}\mathbf{k}$$

and

$$jQ_n^{(3)} = j_n^{(3)} + j_{n+1}^{(3)}\mathbf{i} + j_{n+2}^{(3)}\mathbf{j} + j_{n+3}^{(3)}\mathbf{k},$$

respectively, where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$, and studied the properties of these quaternions. In particular, these sequences satisfy the recurrence relations

$$JQ_{n+3}^{(3)} = JQ_{n+2}^{(3)} + JQ_{n+1}^{(3)} + 2JQ_n^{(3)},\ n \geq 0,$$

and

$$jQ_{n+3}^{(3)} = jQ_{n+2}^{(3)} + jQ_{n+1}^{(3)} + 2jQ_n^{(3)},\ n \geq 0,$$

respectively. Furthermore, the generating functions and many other identities for the third-order Jacobsthal and the third-order Jacobsthal–Lucas quaternions were derived.

Motivated essentially by the recent works [5], [7] and [2], in this paper we introduce the modified third-order Jacobsthal quaternion sequences and we give some properties, including the Binet-style formula and the generating functions for these sequences. Some identities involving these sequences of quaternions are also provided.

2. The modified third-order Jacobsthal quaternion sequence, Binet's formula and the generating function

The principal goal of this section will be to define the modified third-order Jacobsthal quaternion sequence and to present some elementary results involving it.

First of all, in [7], the author defined the modified third-order Jacobsthal numbers

$$K_{n+3}^{(3)} = K_{n+2}^{(3)} + K_{n+1}^{(3)} + 2K_{n}^{(3)},$$

with initial conditions $K_0^{(3)} = 3$, $K_1^{(3)} = 1$ and $K_2^{(3)} = 3$. Furthermore, this sequence appears when we study the third-order Jacobsthal numbers with indices from an arithmetic progression, for example,

$$J_{a(n+3)+r}^{(3)} = (2^a + \omega_1^a + \omega_2^a)\, J_{a(n+2)+r}^{(3)} - (2^a(\omega_1^a + \omega_2^a) + 1)\, J_{a(n+1)+r}^{(3)} + 2^a J_{an+r}^{(3)},$$

for fixed integers a, r with $0 \leq r < a$. Note that

$$2^n + \omega_1^n + \omega_2^n = J_n^{(3)} + 2J_{n-1}^{(3)} + 6J_{n-2}^{(3)}, \quad (n \geq 2),$$

where $J_n^{(3)}$ is the n-th third-order Jacobsthal number.

Here, we define the modified third-order Jacobsthal sequence, denoted by $\{KQ_n^{(3)}\}_{n\geq 0}$, whose first terms are $\{3+\mathbf{i}+3\mathbf{j}+10\mathbf{k},\ 1+3\mathbf{i}+10\mathbf{j}+15\mathbf{k},\ 3+10\mathbf{i}+15\mathbf{j}+31\mathbf{k},\ 10+15\mathbf{i}+31\mathbf{j}+66\mathbf{k},\ ...\}$. This sequence is defined recursively by

$$KQ_{n+3}^{(3)} = KQ_{n+2}^{(3)} + KQ_{n+1}^{(3)} + 2KQ_{n}^{(3)}, \tag{6}$$

with initial conditions $KQ_0^{(3)} = 3+\mathbf{i}+3\mathbf{j}+10\mathbf{k}$, $KQ_1^{(3)} = 1+3\mathbf{i}+10\mathbf{j}+15\mathbf{k}$ and $KQ_2^{(3)} = 3+10\mathbf{i}+15\mathbf{j}+31\mathbf{k}$.

In order to find the generating function for the modified third-order Jacobsthal quaternion sequence, we shall write the sequence as a power series, where each term of the sequence corresponds to coefficients of the series. As a consequence of the definition, the generating function associated to $\{KQ_n^{(3)}\}_{n\geq 0}$, denoted by $\{gKQ(t)\}$, is defined by

$$gKQ(t) = \sum_{n\geq 0} KQ_n^{(3)} t^n.$$

Consequently, we obtain the following result.

Theorem 1. *The generating function for the modified third-order Jacobsthal quaternions* $\{KQ_n^{(3)}\}_{n\geq 0}$ *is*

$$gKQ(t) = \frac{3+\boldsymbol{i}+3\boldsymbol{j}+10\boldsymbol{k}+(-2+2\boldsymbol{i}+7\boldsymbol{j}+5\boldsymbol{k})t+(-1+6\boldsymbol{i}+2\boldsymbol{j}+6\boldsymbol{k})t^2}{1-t-t^2-2t^3}.$$

Proof. Using definition of generating function associated to $\{KQ_n^{(3)}\}_{n\geq 0}$, we have $gKQ(t) = KQ_0^{(3)} + KQ_1^{(3)}t + KQ_2^{(3)}t^2 + \cdots + KQ_n^{(3)}t^n + \cdots$. Multiplying both sides of this identity by $-t$, $-t^2$ and by $-2t^3$, and then from equation (6) we have $(1-t-t^2-2t^3)gKQ(t) = KQ_0^{(3)} + (KQ_1^{(3)} - KQ_0^{(3)})t + (KQ_2^{(3)} - KQ_1^{(3)} - KQ_0^{(3)})t^2$, and the result follows. □

The following result gives the Binet-style formula for $KQ_n^{(3)}$.

Theorem 2. *For $n \geq 0$, we have*

$$KQ_n^{(3)} = 2^n\Phi + \omega_1^n\Phi_1 + \omega_2^n\Phi_2 = 2^n\Phi + M_n^{(2)},$$

where

$$M_n^{(2)} = \begin{cases} 2 - \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k} & \text{if } n \equiv 0 \pmod 3, \\ -1 - \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} & \text{if } n \equiv 1 \pmod 3, \\ -1 + 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k} & \text{if } n \equiv 2 \pmod 3, \end{cases} \tag{7}$$

and ω_1, ω_2 are the roots of the characteristic equation $x^2 + x + 1 = 0$, $\Phi = 1 + 2\boldsymbol{i} + 4\boldsymbol{j} + 8\boldsymbol{k}$, $\Phi_1 = 1 + \omega_1\boldsymbol{i} + \omega_2\boldsymbol{j} + \boldsymbol{k}$, and $\Phi_2 = 1 + \omega_2\boldsymbol{i} + \omega_1\boldsymbol{j} + \boldsymbol{k}$.

Proof. Since the characteristic equation has three distinct roots, the sequence $KQ_n^{(3)} = 2^nP + \omega_1^nQ + \omega_2^nR$ is the solution of (6). Considering $n = 0, 1, 2$ in this identity and solving this system of linear equations, we obtain a unique value for P, Q and R, which are, in this case,

$$P = 1 + 2\mathbf{i} + 4\mathbf{j} + 8\mathbf{k},$$
$$Q = 1 + \omega_1\mathbf{i} + \omega_1^2\mathbf{j} + \omega_1^3\mathbf{k},$$
$$R = 1 + \omega_2\mathbf{i} + \omega_2^2\mathbf{j} + \omega_2^3\mathbf{k}.$$

So, using these values in the expression of $KQ_n^{(3)}$ stated before, we get the required result. □

Using the fact that $\omega_1 + \omega_2 = -\omega_1\omega_2 = -1$, we have

$$\omega_1^n + \omega_2^n = -\frac{1}{7}\left(4Z_{n+1}^{(2)} - Z_n^{(2)}\right) \tag{8}$$

and

$$Z_n^{(2)} = \left(\frac{3 + 2i\sqrt{3}}{3}\right)\omega_1^n - \left(\frac{3 - 2i\sqrt{3}}{3}\right)\omega_2^n.$$

Furthermore, we have $M_{n+2}^{(2)} = -M_{n+1}^{(2)} - M_n^{(2)}$, $M_0^{(2)} = 2 - \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $M_1^{(2)} = -1 - \mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Then, we easily obtain the identities stated in the following proposition.

Proposition 1. *For a natural number n and m, if $JQ_n^{(3)}$, $jQ_n^{(3)}$, and $KQ_n^{(3)}$ are, respectively, the n-th third-order Jacobsthal, third-order Jacobsthal–Lucas, and modified third-order Jacobsthal quaternions, then the following identities are true:*

$$JQ_{n+2}^{(3)} = \frac{1}{147}\left(13KQ_{n+2}^{(3)} + 48KQ_{n+1}^{(3)} + 20KQ_n^{(3)}\right), \tag{9}$$

$$KQ_{n+2}^{(3)} = \frac{1}{6}\left(5jQ_{n+2}^{(3)} + 3jQ_{n+1}^{(3)} - 5jQ_n^{(3)}\right), \tag{10}$$

$$jQ_{n+2}^{(3)} = \frac{1}{49}\left(43KQ_{n+2}^{(3)} + 8KQ_{n+1}^{(3)} + 36KQ_n^{(3)}\right), \tag{11}$$

$$KQ_n^{(3)}KQ_m^{(3)} + KQ_{n+1}^{(3)}KQ_{m+1}^{(3)} + KQ_{n+2}^{(3)}KQ_{m+2}^{(3)} = \left\{ \begin{array}{c} 2^n\Phi\left(M_{m+1}^{(2)} + 3M_{m+2}^{(2)}\right) + 2^m\left(M_{n+1}^{(2)} + 3M_{n+2}^{(2)}\right)\Phi \\ +21 \cdot 2^{n+m}\Phi^2 + M_n^{(2)}M_m^{(2)} + M_{n+1}^{(2)}M_{m+1}^{(2)} + M_{n+2}^{(2)}M_{m+2}^{(2)} \end{array} \right\}, \quad (12)$$

$$\left(KQ_n^{(3)}\right)^2 + \left(KQ_{n+1}^{(3)}\right)^2 + \left(KQ_{n+2}^{(3)}\right)^2 = \left\{ \begin{array}{c} 2^n\Phi\left(M_{n+1}^{(2)} + 3M_{n+2}^{(2)}\right) + 2^n\left(M_{n+1}^{(2)} + 3M_{n+2}^{(2)}\right)\Phi \\ +21 \cdot 2^{2n}\Phi^2 + \left(M_n^{(2)}\right)^2 + \left(M_{n+1}^{(2)}\right)^2 + \left(M_{n+2}^{(2)}\right)^2 \end{array} \right\}, \quad (13)$$

and $M_n^{(2)}$ *as in* (7).

Proof. First, we will just prove equalities (9) and (12) since (10) and (11) can be dealt with in the same manner, and equality (13) is obtained from (12) if $m = n$.

(9): To prove (9), we use induction on n. Let $n = 0$, we get

$$\begin{aligned} JQ_2^{(3)} &= 1 + 2\mathbf{i} + 5\mathbf{j} + 9\mathbf{k} \\ &= \frac{1}{147}\left\{ \begin{array}{c} 13(3 + 10\mathbf{i} + 15\mathbf{j} + 31\mathbf{k}) \\ +48(1 + 3\mathbf{i} + 10\mathbf{j} + 15\mathbf{k}) \\ +20(3 + \mathbf{i} + 3\mathbf{j} + 10\mathbf{k}) \end{array} \right\} \\ &= \frac{1}{147}\left[13KQ_2^{(3)} + 48KQ_1^{(3)} + 20KQ_0^{(3)}\right]. \end{aligned}$$

In the same way for the case $n = 1, 2$. Let us assume that

$$147JQ_{m+2}^{(3)} = 13KQ_{m+2}^{(3)} + 48KQ_{m+1}^{(3)} + 20KQ_m^{(3)}$$

is true for all values $m \geq 0$ less than or equal $n \geq 2$. Then

$$\begin{aligned} 147JQ_{(n+1)+2}^{(3)} &= 147JQ_{n+3}^{(3)} \\ &= 147\left(JQ_{n+2}^{(3)} + JQ_{n+1}^{(3)} + 2JQ_n^{(3)}\right) \\ &= 13KQ_{n+2}^{(3)} + 48KQ_{n+1}^{(3)} + 20KQ_n^{(3)} \\ &\quad + 13KQ_{n+1}^{(3)} + 48KQ_n^{(3)} + 20KQ_{n-1}^{(3)} \\ &\quad + 26KQ_n^{(3)} + 96KQ_{n-1}^{(3)} + 40KQ_{n-2}^{(3)} \\ &= 13KQ_{n+3}^{(3)} + 48KQ_{n+2}^{(3)} + 20KQ_{n+1}^{(3)}. \end{aligned}$$

(12): Using the Binet formula of $KQ_n^{(3)}$ in Theorem 2 and $M_{n+2}^{(2)} + M_{n+1}^{(2)} + M_n^{(2)} = 0$, we have

$$KQ_n^{(3)}KQ_m^{(3)} + KQ_{n+1}^{(3)}KQ_{m+1}^{(3)} + KQ_{n+2}^{(3)}KQ_{m+2}^{(3)}$$

$$= \left\{ \begin{array}{c} \left(2^n\Phi + M_n^{(2)}\right)\left(2^m\Phi + M_m^{(2)}\right) \\ + \left(2^{n+1}\Phi + M_{n+1}^{(2)}\right)\left(2^{m+1}\Phi + M_{m+1}^{(2)}\right) \\ + \left(2^{n+2}\Phi + M_{n+2}^{(2)}\right)\left(2^{m+2}\Phi + M_{m+2}^{(2)}\right) \end{array} \right\},$$

where $\Phi = 1 + 2\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}$. Then we obtain

$$KQ_n^{(3)}KQ_m^{(3)} + KQ_{n+1}^{(3)}KQ_{m+1}^{(3)} + KQ_{n+2}^{(3)}KQ_{m+2}^{(3)}$$

$$= \left\{ \begin{array}{c} 21 \cdot 2^{n+m}\Phi^2 \\ +2^n\Phi\left(M_m^{(2)} + 2M_{m+1}^{(2)} + 4M_{m+2}^{(2)}\right) \\ +2^m\left(M_n^{(2)} + 2M_{n+1}^{(2)} + 4M_{n+2}^{(2)}\right)\Phi \\ +M_n^{(2)}M_m^{(2)} + M_{n+1}^{(2)}M_{m+1}^{(2)} + M_{n+2}^{(2)}M_{m+2}^{(2)} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} 2^n\Phi\left(M_{m+1}^{(2)} + 3M_{m+2}^{(2)}\right) + 2^m\left(M_{n+1}^{(2)} + 3M_{n+2}^{(2)}\right)\Phi \\ +21 \cdot 2^{n+m}\Phi^2 + M_n^{(2)}M_m^{(2)} + M_{n+1}^{(2)}M_{m+1}^{(2)} + M_{n+2}^{(2)}M_{m+2}^{(2)} \end{array} \right\}.$$

Finally, we obtain (13) if $m = n$ in (12). □

3. Some identities involving the modified third-order Jacobsthal quaternion sequence

In this section, we state some identities related to the considered third-order sequence. As a consequence of the Binet formula of Theorem 2, we get for this sequence the following interesting identities.

Proposition 2 (Catalan-like identity)**.** *For natural numbers n, s with $n \geq s$, if $KQ_n^{(3)}$ is the n-th modified third-order Jacobsthal quaternion, then the following identity is true:*

$$KQ_{n+s}^{(3)}KQ_{n-s}^{(3)} - \left(KQ_n^{(3)}\right)^2 = \left\{ \begin{array}{c} 2^{n+s}\Phi\left(\left(U_{s+1}^{(2)} - 2^{-s}\right)M_n^{(2)} - U_s^{(2)}M_{n+1}^{(2)}\right) \\ +2^{n-s}\left(U_s^{(2)}M_{n+1}^{(2)} - \left(U_{s-1}^{(2)} + 2^s\right)M_n^{(2)}\right)\Phi \\ +M_{n+s}^{(2)}M_{n-s}^{(2)} - \left(M_n^{(2)}\right)^2 \end{array} \right\},$$

where $M_n^{(2)}$ is in (7), $U_n^{(2)} = \frac{\omega_1^n - \omega_2^n}{\omega_1 - \omega_2}$ and ω_1, ω_2 are the roots of the characteristic equation associated with the recurrence relation $x^2 + x + 1 = 0$.

Proof. Using the expression (7) and the Binet formula of $KQ_n^{(3)}$ in Theorem 2, we have

$$KQ_{n+s}^{(3)}KQ_{n-s}^{(3)} - \left(KQ_n^{(3)}\right)^2 = \left(2^{n+s}\Phi + M_{n+s}^{(2)}\right)\left(2^{n-s}\Phi + M_{n-s}^{(2)}\right) - \left(2^n\Phi + M_n^{(2)}\right)^2$$

$$= \left\{ \begin{array}{c} 2^{n+s}\Phi M_{n-s}^{(2)} + 2^{n-s}M_{n+s}^{(2)}\Phi \\ -2^{n}\left(\Phi M_{n}^{(2)} + M_{n}^{(2)}\Phi\right) \\ +M_{n+s}^{(2)}M_{n-s}^{(2)} - \left(M_{n}^{(2)}\right)^{2} \end{array} \right\}.$$

Using the following identity for the sequence $M_n^{(2)}$,

$$\begin{aligned} M_{n+s}^{(2)} &= \frac{1}{\omega_1-\omega_2}\left(\omega_1^{n+s+1}\Phi_1 - \omega_1^{n+s-1}\Phi_1 + \omega_2^{n+s-1}\Phi_2 - \omega_2^{n+s+1}\Phi_2\right) \\ &= \left(\frac{\omega_1^{s}-\omega_2^{s}}{\omega_1-\omega_2}\right)\left(\omega_1^{n+1}\Phi_1 + \omega_2^{n+1}\Phi_2\right) - \left(\frac{\omega_1^{s-1}-\omega_2^{s-1}}{\omega_1-\omega_2}\right)\left(\omega_1^{n}\Phi_1 + \omega_2^{n}\Phi_2\right) \\ &= U_s^{(2)}M_{n+1}^{(2)} - U_{s-1}^{(2)}M_n^{(2)}, \end{aligned}$$

where $U_s^{(2)} = \frac{\omega_1^s-\omega_2^s}{\omega_1-\omega_2}$ and $U_{-s}^{(2)} = -U_s^{(2)}$, we obtain the statement of the theorem. □

Note that for $s = 1$ in the obtained Catalan-like identity, we get the Cassini-like identity for the modified third-order Jacobsthal quaternion sequence. Furthermore, for $s = 1$, the identity stated in Proposition 2 yields

$$KQ_{n+1}^{(3)}KQ_{n-1}^{(3)} - \left(KQ_n^{(3)}\right)^2 = \left\{ \begin{array}{c} 2^{n+1}\Phi\left(\left(U_2^{(2)} - 2^{-1}\right)M_n^{(2)} - U_1^{(2)}M_{n+1}^{(2)}\right) \\ +2^{n-1}\left(U_1^{(2)}M_{n+1}^{(2)} - \left(U_0^{(2)} + 2\right)M_n^{(2)}\right)\Phi \\ +M_{n+1}^{(2)}M_{n-1}^{(2)} - \left(M_n^{(2)}\right)^2 \end{array} \right\},$$

and using $U_0^{(2)} = 0$, $U_1^{(2)} = 1$ and $U_2^{(2)} = -1$ in Proposition 2, we obtain the next result.

Proposition 3 (Cassini-like identity)**.** *For a natural number n, if $KQ_n^{(3)}$ is the n-th modified third-order Jacobsthal quaternion, then the following identity is true:*

$$KQ_{n+1}^{(3)}KQ_{n-1}^{(3)} - \left(KQ_n^{(3)}\right)^2 = \left\{ \begin{array}{c} 2^{n}\Phi\left(-3M_n^{(2)} - 2M_{n+1}^{(2)}\right) \\ +2^{n-1}\left(M_{n+1}^{(2)} - 2M_n^{(2)}\right)\Phi \\ +M_{n+1}^{(2)}M_{n-1}^{(2)} - \left(M_n^{(2)}\right)^2 \end{array} \right\}.$$

The d'Ocagne-like identity can also be obtained using the Binet formula.

Proposition 4 (d'Ocagne-like identity)**.** *For natural numbers m, n with $m \geq n$, if $KQ_n^{(3)}$ is the n-th modified third-order Jacobsthal quaternion, then the following identity is true:*

$$KQ_{m+1}^{(3)}KQ_n^{(3)} - KQ_m^{(3)}KQ_{n+1}^{(3)}$$

$$= \left\{ \begin{array}{c} 2^m\Phi\left(2M_n^{(2)} - M_{n+1}^{(2)}\right) + 2^n\left(M_{m+1}^{(2)} - 2M_m^{(2)}\right)\Phi \\ +M_{m+1}^{(2)}M_n^{(2)} - M_m^{(2)}M_{n+1}^{(2)} \end{array} \right\}.$$

Proof. The result follows by using (7) of Theorem 2. □

In addition, some formulae involving sums of terms of the modified third-order Jacobsthal quaternion sequence will be provided in the following proposition.

Proposition 5. *For natural numbers m, n with $n \geq m$, if $KQ_n^{(3)}$ is the n-th modified third-order Jacobsthal quaternion, then the following identities are true:*

$$\sum_{s=m}^{n} KQ_s^{(3)} = \frac{1}{3}\left(KQ_{n+2}^{(3)} + 2KQ_n^{(3)} + KQ_m^{(3)} - KQ_{m+2}^{(3)}\right), \tag{14}$$

$$\sum_{s=0}^{n} KQ_s^{(3)} = \begin{cases} KQ_{n+1}^{(3)} + 2 - 2\boldsymbol{i} - 7\boldsymbol{j} - 5\boldsymbol{k} & \text{if} \quad n \equiv 0 \pmod 3, \\ KQ_{n+1}^{(3)} + 1 - 6\boldsymbol{i} - 2\boldsymbol{j} - 6\boldsymbol{k} & \text{if} \quad n \equiv 1 \pmod 3, \\ KQ_{n+1}^{(3)} - 3 - \boldsymbol{i} - 3\boldsymbol{j} - 10\boldsymbol{k} & \text{if} \quad n \equiv 2 \pmod 3. \end{cases} \tag{15}$$

Proof. (14): Using (6), we obtain

$$\begin{aligned} \sum_{s=m}^{n} KQ_s^{(3)} &= KQ_m^{(3)} + KQ_{m+1}^{(3)} + KQ_{m+2}^{(3)} + \sum_{s=m+3}^{n} KQ_s^{(3)} \\ &= KQ_m^{(3)} + KQ_{m+1}^{(3)} + KQ_{m+2}^{(3)} \\ &\quad + \sum_{s=m+2}^{n-1} KQ_s^{(3)} + \sum_{s=m+1}^{n-2} KQ_s^{(3)} + 2\sum_{s=m}^{n-3} KQ_{s-3}^{(3)} \\ &= 4\sum_{s=m}^{n} KQ_s^{(3)} + KQ_{m+2}^{(3)} - KQ_m^{(3)} - 4KQ_n^{(3)} - 3KQ_{n-1}^{(3)} - 2KQ_{n-2}^{(3)} \\ &= 4\sum_{s=m}^{n} KQ_s^{(3)} + KQ_{m+2}^{(3)} - KQ_m^{(3)} - 2KQ_n^{(3)} - KQ_{n+2}. \end{aligned}$$

Thus the equality (14) is proved.

(15): As a consequence of (7) of Theorem 2 and

$$\sum_{s=0}^{n}(\omega_1^s\Phi_1 + \omega_2^s\Phi_2) = \frac{\omega_1^{n+1}-1}{\omega_1-1}\Phi_1 + \frac{\omega_2^{n+1}-1}{\omega_2-1}\Phi_2 = \frac{1}{3}\left(M_n^{(2)} - M_{n+1}^{(2)}\right) + (1 - \mathbf{i} + \mathbf{k}),$$

we have

$$\sum_{s=0}^{n} KQ_s^{(3)} = \sum_{s=0}^{n} 2^s\Phi + \sum_{s=0}^{n} M_s^{(2)}$$

$$
\begin{aligned}
&= 2^{n+1}\Phi - \Phi + \frac{1}{3}\left(M_n^{(2)} - M_{n+1}^{(2)}\right) + 1 - \mathbf{i} + \mathbf{k} \\
&= KQ_{n+1}^{(3)} + \frac{1}{3}\left(M_n^{(2)} - 4M_{n+1}^{(2)}\right) - 3\mathbf{i} - 4\mathbf{j} - 7\mathbf{k} \\
&= \begin{cases} KQ_{n+1}^{(3)} + 2 - 2\mathbf{i} - 7\mathbf{j} - 5\mathbf{k} & \text{if } n \equiv 0 \pmod 3, \\ KQ_{n+1}^{(3)} + 1 - 6\mathbf{i} - 2\mathbf{j} - 6\mathbf{k} & \text{if } n \equiv 1 \pmod 3, \\ KQ_{n+1}^{(3)} - 3 - \mathbf{i} - 3\mathbf{j} - 10\mathbf{k} & \text{if } n \equiv 2 \pmod 3). \end{cases}
\end{aligned}
$$

Hence we obtained the result. □

Next, for negative subscripts terms of the sequence of modified third-order Jacobsthal quaternions we can establish the following result.

Proposition 6. *For a natural number n, the following identity is true:*

$$KQ_{-n}^{(3)} = KQ_n^{(3)} + (2^{-n} - 2^n)\Phi + U_n^{(2)}\left(3\boldsymbol{i} - 3\boldsymbol{j}\right), \tag{16}$$

where $U_n^{(2)} = \frac{\omega_1^n - \omega_2^n}{\omega_1 - \omega_2}$.

Proof. (16): Since $M_{-n}^{(2)} = M_{0-n}^{(2)} = U_{-n}^{(2)} M_1^{(2)} - U_{-n-1}^{(2)} M_0^{(2)}$, $U_n^{(2)} + U_{n+1}^{(2)} + U_{n+2}^{(2)} = 0$ and $U_{-n}^{(2)} = -U_n^{(2)}$, using the Binet formula stated in Theorem 2 and the fact that $\omega_1\omega_2 = 1$, the claim of the proposition follows. In fact,

$$
\begin{aligned}
KQ_{-n}^{(3)} &= 2^{-n}\Phi + M_{-n}^{(2)} \\
&= 2^{-n}\Phi - U_n^{(2)} M_1^{(2)} + U_{n+1}^{(2)} M_0^{(2)} \\
&= 2^{-n}\Phi + M_n^{(2)} - M_n^{(2)} - U_n^{(2)} M_1^{(2)} + U_{n+1}^{(2)} M_0^{(2)} \\
&= 2^{-n}\Phi + M_n^{(2)} - U_n^{(2)} M_1^{(2)} + U_{n-1}^{(2)} M_0^{(2)} - U_n^{(2)} M_1^{(2)} + U_{n+1}^{(2)} M_0^{(2)} \\
&= 2^{-n}\Phi + M_n^{(2)} - U_n^{(2)}\left(2M_1^{(2)} + M_0^{(2)}\right) \\
&= 2^{-n}\Phi + M_n^{(2)} + U_n^{(2)}\left(3\mathbf{i} - 3\mathbf{j}\right) \\
&= 2^{n}\Phi + M_n^{(2)} - 2^n\Phi + 2^{-n}\Phi + U_n^{(2)}\left(3\mathbf{i} - 3\mathbf{j}\right) \\
&= KQ_n^{(3)} + (2^{-n} - 2^n)\Phi + U_n^{(2)}\left(3\mathbf{i} - 3\mathbf{j}\right).
\end{aligned}
$$

So, the proof is completed. □

4. Conclusion

Sequences of quaternions have been studied over several years, including the well known third-order Jacobsthal quaternion sequence and, consequently, the third-order Jacobsthal–Lucas quaternion sequence. In this paper we have contributed for the study of modified third-order Jacobsthal quaternion sequence, deducing some formulae for the sums of such numbers, presenting the generating functions and their Binet-style formula. It is our intention to continue the study of this type of sequences, exploring some of their applications in the science domain. For example, a new type of sequences in the generalized quaternion algebra with the use of these numbers and their combinatorial properties.

Acknowledgement

The author is thankful to the referee for valuable suggestions leading to improving the quality of the paper.

References

[1] P. Barry, *Triangle geometry and Jacobsthal numbers*, Irish Math. Soc. Bull. **51** (2003), 45–57.
[2] C. K. Cook and M. R. Bacon, *Some identities for Jacobsthal and Jacobsthal–Lucas numbers satisfying higher order recurrence relations*, Ann. Math. Inform. **41** (2013), 27–39.
[3] A. F. Horadam, *Jacobsthal and Pell curves*, Fibonacci Quart. **26** (1988), 79–83.
[4] A. F. Horadam, *Jacobsthal representation numbers*, Fibonacci Quart. **43** (1996), 40–54.
[5] G. Cerda-Morales, *Identities for third order Jacobsthal quaternions*, Adv. Appl. Clifford Algebr. **27** (2017), 1043–1053. DOI
[6] G. Cerda-Morales, *On a generalization of Tribonacci quaternions*, Mediterr. J. Math. **14** (2017), 12 pp. DOI
[7] G. Cerda-Morales, *A note on modified third-order Jacobsthal numbers*, Proyecciones (Antofagasta) **39** (2020), 409–420. DOI
[8] G. Cerda-Morales, On third-order Jacobsthal polynomials and their properties, Miskolc Math. Notes **22** (2021), 123–132. DOI

Instituto de Matemáticas, Pontificia Universidad Católica de Valparaíso, Blanco Viel 596, Valparaíso, Chile
E-mail address: gamaliel.cerda.m@mail.pucv.cl