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I-limit points and I-cluster points of multiset sequences

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Abstract. $\mathcal{I}$-convergence is a type of convergence that generalizes many known types of convergences. In this study, $\mathcal{I}$-convergence and $\mathcal{I}$-boundedness of multiset sequences are defined and some examples are given. $\mathcal{I}$-limit points, $\mathcal{I}$-cluster points, B_{mx} and A_{mx} sets and the concepts of $\mathcal{I}$-limit infimum and $\mathcal{I}$-limit supremum are defined for a multiset sequence. These definitions are supported by examples. It is shown that the set of $\mathcal{I}$-cluster points of a multiset sequence covers the set of $\mathcal{I}$-limit points and $\mathcal{I}$-$\liminf mx \leq \mathcal{I}$-$\limsup mx$. Additionally, necessary and sufficient conditions for $\mathcal{I}$-$\liminf mx = \mathcal{I}$-$\limsup mx$ are proved.

1. Introduction

Although studies on multisets have been carried out for many years; studies on multiset sequences that accept these sets as elements, are quite new. Looking at the year 2021, the convergence of multiset sequences and statistical convergence of multiset sequences were studied by Pachilangode and John [26]. The definition of statistical convergence of set sequences by Nuray and Rhoades [25] in 2011 played a major role in the emergence of these studies. Later on, Gümüş et al. [15] studied lacunary statistical convergence, Demir and Gümüş [8, 9] studied ideal convergence, briefly $\mathcal{I}$-convergence, and lacunary $\mathcal{I}$-statistical convergence of these sequences, respectively. Considering that $\mathcal{I}$-convergence generalizes many types of convergence, including statistical convergence, these studies contain more general results. In addition, limit points and cluster points play an important role in convergence; statistical limit points and statistical cluster points play an important role in

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statistical convergence; $\mathcal{I}$-limit points and $\mathcal{I}$-cluster points play an important role in $\mathcal{I}$-convergence. Fridy's article [13] from 1993 on statistical limit points and statistical cluster points can be considered as an important study in this sense. After this study, Fridy and Orhan [14] examined the results about statistical limit superior and statistical limit inferior. In the article, where Kostyrko et al. [19] defined ideal convergence, they also defined $\mathcal{I}$-limit points and $\mathcal{I}$-cluster points and gave some basic results about them. $\mathcal{I}$-limit superior and $\mathcal{I}$-limit inferior were studied by Demirci [10] in 2001.

Now, we can consider these concepts in a little bit more detail. In classical set theory, the elements of a set are written only once. Unlike this situation, multiset (shortly mset) is a collection of objects in which elements are allowed to repeat. In fact, it is possible to see multisets in many areas of our lives. Some examples of this situation are phone numbers, computer codes, molecules and coincident roots of equations in mathematics. In each example, there are same numbers and same molecules that play different roles. This has made multisets very interesting in many branches of science such as mathematics, logic, physics, computer science, etc. Since the 1970s, Bender [1], Lake [20], Hickman [16], Meyer [22] and Monro [23] investigated some important properties of multisets. In 1981, Knuth [18] studied computer programing and multisets. On the other hand, Blizard [2, 3, 4] studied multisets in his doctoral thesis.

In multisets, it is very important how many times the elements are repeated in the set but the order of the elements is not important. So, the multisets $\{1, 3, 5, 3, 4, 1, 1\}$ and $\{1, 1, 1, 3, 3, 4, 5\}$ are the same. The multiset $\{1, 3, 5, 3, 4, 1, 1\}$ is denoted by $\{1, 3, 4, 5\}_{3,2,1,1}$ or $\{1|3,\ 3|2,\ 4|1,\ 5|1\}$ and it means 1 appearing 3 times, 3 appearing 2 times, 4 appearing 1 times and 5 appearing 1 times. The cardinality of a multiset is the sum of the multiplicities of its elements.

Studies on multisets continued in the 2000s. Among the important contributions we mention the study on mathematics of multisets by Syropoulos [30] in 2001, an overview of the application of multisets by Singh [28] in 2007, the studies of soft multisets by Majumdar [21] in 2012, on multisets and multigroups by Nazmul [24] in 2013 and multigroup actions on multisets by İbrahim [17] in 2017.

After these studies on multisets, the multiset sequences and their properties have started to be a subject of research.

Definition 1 ([26])**.** Let X be a set. A sequence in which all the terms are multisets is known as a *multiset sequence*. For any sequence $x = (x_i) \in X$, the set of multiset sequences is defined by

$$M = \{x_i|c_i : x_i \in X,\ c_i \in \mathbb{N}_0\},$$

where $\mathbb{N}_0$ is the set of non-negative integers.

Example 1 ([26])**.** Let $N_n = \{1|1, 2|2, \cdots, n|n\}$. Then $\{N_n\}$ is a multiset sequence whose n^{th} term has $\frac{n(n+1)}{2}$ elements.

Example 2 ([26])**.** The prime factorises n completely, and let F_n be the mset of these factors, including 1. Then, $F_1 = \{1\}$, $F_2 = \{1, 2\}$, $F_3 = \{1, 3\}$, $F_4 = \{1, 2, 2\}$ and $F_{36} = \{1, 2, 2, 3, 3\}$. In this case $\{F_n\}$ is an mset sequence.

According to Definition 1, the set of multiset sequences in $\mathbb{R}$ can be defined as follows.

Definition 2 ([7])**.** Let $\mathbb{N}_0$ be the set of non-negative integers. Then

$$m\mathbb{R} = \{mx = x_i|c_i : x_i \in \mathbb{R} \text{ and } c_i \in \mathbb{N}_0\}$$

is called the *set of multiset sequences of real numbers.*

Usual convergence of multiset sequences was studied by Pachilangode and John [26] in 2021 and statistical convergence on $\mathbb{R}$ was studied by Debnath and Debnath [7] in 2021.

Before defining $\mathcal{I}$-convergence, which is our other main concept, let us briefly talk about statistical convergence. Statistical convergence was first mentioned in 1935 in Warsaw and it was formally introduced by Fast [11] and Steinhaus [29], independently. After the 1950s, studies on the concept of statistical convergence made rapid progress and many studies were conducted on this subject such as [5],[12].

Definition 3 ([11])**.** A number sequence (x_i) is *statistically convergent* to L provided that for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{i \leq n : |x_i - L| \geq \varepsilon\}| = 0.$$

In this case we write $st\text{-}\lim x_i = L$ and usually the set of statistically convergent sequences is denoted by S.

$\mathcal{I}$-convergence has emerged as a generalized form of many types of convergences. This means that, if we choose different ideals we have different convergences. Kostyrko et al. [19] introduced this concept in a metric space. Later on, $\mathcal{I}$-statistical convergence is defined by Das and Savaş [6], and Savaş and Gümüş [27] examined this issue from a different perspective. We explain this situation with an interesting example later. Before defining $\mathcal{I}$-convergence, the definitions of ideal and filter will be needed.

Definition 4. A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is an *ideal* if the following properties are provided:

i) $\emptyset \in \mathcal{I}$,
ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
iii) for each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

We say that $\mathcal{I}$ is *non-trivial* if $\mathbb{N} \notin \mathcal{I}$ and $\mathcal{I}$ is *admissible* if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Definition 5. A family of sets $F \subseteq 2^{\mathbb{N}}$ is a *filter* if the following properties are provided:

i) $\emptyset \notin F$,
ii) if $A, B \in F$, then we have $A \cap B \in F$,
iii) for each $A \in F$ and each $A \subseteq B$ we have $B \in F$.

Proposition 1. *If $\mathcal{I}$ is an ideal in $\mathbb{N}$ then the collection*

$$F(\mathcal{I}) = \{A \subset \mathbb{N} : \mathbb{N} \backslash A \in \mathcal{I}\}$$

forms in a filter in $\mathbb{N}$ which is called the filter associated with $\mathcal{I}$.

Definition 6 ([19])**.** A sequence of reals $x = (x_i)$ is *$\mathcal{I}$-convergent* to $L \in \mathbb{R}$ if the set

$$A_\varepsilon = \{i \in \mathbb{N} : |x_i - L| \geq \varepsilon\} \in \mathcal{I}$$

for each $\varepsilon > 0$. In this case, we say that L is the $\mathcal{I}$-limit of the sequence x.

Example 3. Denote the set of all finite subsets of $\mathbb{N}$ by $\mathcal{I}_f$. Then, $\mathcal{I}_f$ is an ideal and $\mathcal{I}_f$-convergence coincides with the usual convergence.

Example 4. Define the set $\mathcal{I}_d$ by $\mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$ where $d(A) = \lim_n \frac{|A_n|}{n}$. Then $\mathcal{I}_d$ is an ideal and $\mathcal{I}_d$-convergence gives the statistical convergence.

Definition 7 ([19])**.** A sequence $x = (x_i)$ is *$\mathcal{I}$-bounded* if there exist a positive real number M such that $\{i \in \mathbb{N} : |x_i| \geq M\} \in \mathcal{I}$.

We can now give definitions of $\mathcal{I}$-limit points and $\mathcal{I}$-cluster points.

Definition 8 ([19])**.** Let $x = (x_i)$ be a sequence and $\lambda \in \mathbb{R}$. If there exists a set $M = \{m_1, m_2, ..., m_i, ...\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and

$$\lim_{i \to \infty} x_{m_i} = \lambda,$$

then the number λ is the *$\mathcal{I}$-limit point* of the sequence x.

Definition 9 ([19])**.** An element $\xi \in \mathbb{R}$ is said to be an *$\mathcal{I}$-cluster point* of x if, for each $\varepsilon > 0$,

$$\{i \in \mathbb{N} : |x_i - \xi| < \varepsilon\} \notin \mathcal{I}.$$

Definition 10. For a number sequence $x = (x_i)$, let

$$B_x := \{b \in \mathbb{R} : \{i \in \mathbb{N} : x_i > b\} \notin \mathcal{I}\}$$

and

$$A_x := \{a \in \mathbb{R} : \{i \in \mathbb{N} : x_i < a\} \notin \mathcal{I}\}.$$

Definition 11. Let $\mathcal{I}$ be an admissible ideal and $x = (x_i)$ be a number sequence. Then *$\mathcal{I}$-limit superior* of x is given by

$$\mathcal{I}\text{-}\limsup x = \begin{cases} \sup B_x, & \text{if } B_x \neq \emptyset, \\ -\infty, & \text{if } B_x = \emptyset. \end{cases}$$

Definition 12. Let $\mathcal{I}$ be an admissible ideal and $x = (x_i)$ be a number sequence. Then $\mathcal{I}$*-limit inferior* of x is given by

$$\mathcal{I}\text{-}\liminf x = \begin{cases} \inf A_x, & \text{if } A_x \neq \emptyset, \\ +\infty, & \text{if } A_x = \emptyset. \end{cases}$$

2. Main results

After specifying our purpose, let us start by giving the definition of ideal convergence for multiset sequences on $\mathbb{R}$.

As mentioned, the aim of the study is to generalize the concepts of convergence and statistical convergence which were previously defined for multiset sequences with the help of ideals. As it is known, multisets are sets whose elements can repeat a finite number of times. Due to repetitive elements of the multisets, it is necessery to define a new metric in order to work on multisets. Let (X, d) be a metric space and M a multiset in a metric space (X, d). The metric d is not very functional on M because of the repetitive elements of M. Hence, if a new d_M metric is defined on M, then (M, d_M) is a metric space. In this study, it is defined as

$$d_M(mx, my) = d_M(x_i|c_i, y_i|t_i) = \sqrt{(x_i - y_i)^2 + (c_i - t_i)^2},$$

where $d_M : M \times M \to \mathbb{R}$. It is easily seen that d_M satisfies the metric conditions with Minkowsky Inequality. With the help of all this information, the ideal convergence of multiset sequences on $\mathbb{R}$ is defined.

Definition 13. A multiset sequence $mx = (x_i|c_i)$ of $m\mathbb{R}$ is *ideal convergent* to $l|c$ if

$$\{i \in \mathbb{N} : d_M(x_i|c_i, l|c) \geq \varepsilon\} = \left\{i \in \mathbb{N} : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\} \in \mathcal{I}$$

for each $\varepsilon > 0$. In this case, we write $\mathcal{I}$-lim $mx = l|c$.

Definition 14. A multiset sequence $mx = (x_i|c_i)$ is said to be $\mathcal{I}$*-bounded* if there exists a non-negative real number M such that

$$\left\{i \in \mathbb{N} : \sqrt{x_i^2 + (c_i - 1)^2} > M\right\} \in \mathcal{I}.$$

Example 5. Consider a multisequence $mx = (x_i|c_i)$, given by

$$x_i = \begin{cases} i, & i = n^2; n = 1, 2, 3, ..., \\ 1, & otherwise. \end{cases} \quad \text{and} \quad c_i = \begin{cases} i, & i = n^3; n = 1, 2, 3, ..., \\ 5, & otherwise. \end{cases}$$

Then

$$\sqrt{x_i^2+(c_i-1)^2} \leq \sqrt{x_i^2}+\sqrt{(c_i-1)^2} = |x_i| + |c_i-1|,$$

$$\left\{i \in \mathbb{N} : \sqrt{x_i^2}+\sqrt{(c_i-1)^2} > M\right\} \subseteq \{i \in \mathbb{N} : |x_i| > M\} \cup \{i \in \mathbb{N} : |c_i-1| > M\}.$$

The right hand side belongs to the ideal, so $mx = (x_i|c_i)$ is bounded.

Definition 15. Let $(m\mathbb{R}, d_M)$ be a metric space and $mx = (x_i|c_i) \in m\mathbb{R}$. The number $\xi|e$ of $m\mathbb{R}$ is said to be an *$\mathcal{I}$-limit point* of the multisequence $mx = (x_i|c_i)$ provided that there is a set $M = \{i_1, i_2, ...\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\lim_{k\to\infty}(x_{i_k}|c_{i_k}) = \xi|e$. The set of all $\mathcal{I}$-limit points of mx is denoted by $\mathcal{I}(\Lambda_{mx})$.

Example 6. Consider the multiset sequence $mx = (x_i|c_i)$ where

$$x_i = \begin{cases} i, & \text{when } i \text{ is } n^2, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad c_i = \begin{cases} i, & \text{when } i \text{ is } n^2, \\ 3, & \text{otherwise.} \end{cases}$$

Then

$$(x_i) = (1, 0, 0, 4, 0, 0, 0, 0, 9, 0, 0, 0, ...)$$

and

$$(c_i) = (1, 3, 3, 4, 3, 3, 3, 3, 9, 3, 3, 3, ...).$$

Therefore

$$(x_i|c_i) = (1|1,\ 0|3,\ 0|3,\ 4|4,\ 0|3,\ 0|3,\ 0|3,\ 0|3,\ 9|9,\ 0|3, ...).$$

Considering the ideal convergence definition for this multiset sequence we can find the set $M = \{2, 3, 5, 6, 7, 8, 10, ...\} \notin \mathcal{I}$ and $\lim_{k\to\infty}(x_{i_k}|c_{i_k}) = 0|3$. So, $0|3$ is the $\mathcal{I}$-limit point of mx.

Definition 16. Let $(m\mathbb{R}, d_M)$ be a metric space and $mx = (x_i|c_i) \in m\mathbb{R}$. The number $\gamma|c$ of $m\mathbb{R}$ is said to be an *$\mathcal{I}$-cluster point* of the multisequence $mx = (x_i|c_i)$ provided that, for every $\varepsilon > 0$, the set

$$\left\{i \in \mathbb{N} : \sqrt{(x_i-\gamma)^2+(c_i-c)^2} < \varepsilon\right\} \notin \mathcal{I}.$$

The set of all $\mathcal{I}$-cluster points of mx is denoted by $\mathcal{I}(\Gamma_{mx})$.

Theorem 1. *Let $\mathcal{I}$ be an admissible ideal. Then, for each sequence $mx = (x_i|c_i)$ of $m\mathbb{R}$, we have $\mathcal{I}(\Lambda_{mx}) \subset \mathcal{I}(\Gamma_{mx})$.*

Proof. Let $\xi|e \in \mathcal{I}(\Lambda_{mx})$. In this case the definition of $\mathcal{I}$-limit point is provided i.e. there is a set $M = \{i_1, i_2, ...\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\lim_{k\to\infty}(x_{i_k}|c_{i_k}) = \xi|e$. From the definition of usual convergence of the submultiset sequence, for each $\delta > 0$ there is an $i_{k_0} \in \mathbb{N}$ such that

$\sqrt{(x_{i_k}-\xi)^2+(c_{i_k}-e)^2}<\delta$ for $i_k>i_{k_0}$. Considering the state of the multiset sequence and submultiset sequence,

$$M\backslash\{i_1,i_2,...,i_{k_0}\}\subset\left\{i\in\mathbb{N}:\sqrt{(x_i-\xi)^2+(c_i-e)^2}<\delta\right\}$$

can be written. So,

$$\left\{i\in\mathbb{N}:\sqrt{(x_i-\xi)^2+(c_i-e)^2}<\delta\right\}\notin\mathcal{I}$$

and we have $\xi|e\in\mathcal{I}(\Gamma_{mx})$. □

Now, $\mathcal{I}$- lim inf mx and $\mathcal{I}$- lim sup mx will be introduced, and some of their basic properties will be given.

Definition 17. For any multiset sequence $mx=(x_i|c_i)$, let B_{mx} denote the set

$$B_{mx}=\left\{b|c\in m\mathbb{R}:\left\{i:\sqrt{x_i^2+(c_i-1)^2}>\sqrt{b^2+(c-1)^2}\right\}\notin\mathcal{I}\right\}$$

and A_{mx} denote the set

$$A_{mx}=\left\{a|d\in m\mathbb{R}:\left\{i:\sqrt{x_i^2+(c_i-1)^2}<\sqrt{a^2+(d-1)^2}\right\}\notin\mathcal{I}\right\}.$$

Definition 18. If $mx=(x_i|c_i)$ is a multiset sequence of real numbers then, $\mathcal{I}$-limit superior of mx is given by

$$\mathcal{I}\text{- lim sup }\ mx=\begin{cases}\sup B_{mx}, & B_{mx}\neq\emptyset,\\ -\infty, & B_{mx}=\emptyset,\end{cases}$$

and $\mathcal{I}$-limit inferior of $mx=(x_i|c_i)$ is given by

$$\mathcal{I}\text{- lim inf }\ mx=\begin{cases}\inf A_{mx}, & A_{mx}\neq\emptyset,\\ +\infty, & A_{mx}=\emptyset.\end{cases}$$

From the definition of statistical cluster point we can say that $\mathcal{I}$- lim sup mx and $\mathcal{I}$- lim inf mx are the *greatest and least cluster points* of mx.

Proposition 2. *Let $mx=(x_i|c_i)$ be a multiset sequence of $m\mathbb{R}$. If $b|c=\mathcal{I}$- lim sup mx is finite, then, for every positive number ε,*

$$\left\{i:\sqrt{x_i^2+(c_i-1)^2}>\sqrt{(b-\varepsilon)^2+(c-1)^2}\right\}\notin\mathcal{I}$$

and

$$\left\{i:\sqrt{x_i^2+(c_i-1)^2}>\sqrt{(b+\varepsilon)^2+(c-1)^2}\right\}\in\mathcal{I}.$$

Proposition 3. *Let $mx = (x_i|c_i)$ be a multiset sequence of $m\mathbb{R}$. If $a|c = \mathcal{I}\text{-}\liminf mx$ is finite, then, for every positive number ε,*

$$\left\{i : \sqrt{x_i^2 + (c_i - 1)^2} < \sqrt{(a+\varepsilon)^2 + (d-1)^2}\right\} \notin \mathcal{I}$$

and

$$\left\{i : \sqrt{x_i^2 + (c_i - 1)^2} < \sqrt{(a-\varepsilon)^2 + (d-1)^2}\right\} \in \mathcal{I}.$$

The proofs of Propositions 2 and 3 can be easily understood when we consider that $\mathcal{I}\text{-}\limsup mx$ and $\mathcal{I}\text{-}\liminf mx$ are the largest and least accumulation points of the sequence mx.

Theorem 2. *For any multiset sequence $mx = (x_i|c_i)$ in $m\mathbb{R}$,*

$$\mathcal{I}\text{-}\liminf mx \leq \mathcal{I}\text{-}\limsup mx.$$

Proof. Let us examine this theorem for three cases:
Case 1: $\mathcal{I}\text{-}\limsup mx = -\infty$,
Case 2: $\mathcal{I}\text{-}\limsup mx = \infty$,
Case 3: $\mathcal{I}\text{-}\limsup mx = b|c$ is finite and $\mathcal{I}\text{-}\liminf mx = a|d$.
Suppose that $\mathcal{I}\text{-}\limsup mx = -\infty$. In this case $B_{mx} = \phi$ and considering the definition, for each $b|c \in m\mathbb{R}$,

$$\left\{i : \sqrt{x_i^2 + (c_i - 1)^2} > \sqrt{b^2 + (c-1)^2}\right\} \in \mathcal{I}$$

and therefore

$$\left\{i : \sqrt{x_i^2 + (c_i - 1)^2} \leq \sqrt{b^2 + (c-1)^2}\right\} \in F(\mathcal{I}).$$

Since the same situation will be provided for the $a|d$ element of $m\mathbb{R}$,

$$\left\{i : \sqrt{x_i^2 + (c_i - 1)^2} < \sqrt{a^2 + (d-1)^2}\right\} \notin \mathcal{I}$$

so $\mathcal{I}\text{-}\liminf mx = -\infty$.

Now, let us consider the case where $\mathcal{I}\text{-}\limsup mx = \infty$. In this case, the inequality will be achieved.

Finally, let us assume that $\mathcal{I}\text{-}\limsup mx = b|c$ is finite and $\mathcal{I}\text{-}\liminf mx = a|d$, $\varepsilon > 0$, is given. If it is shown that $(b+\varepsilon)|c \in A_{mx}$ i.e.

$$\left\{i : \sqrt{x_i^2 + (c_i - 1)^2} < \sqrt{(b+\varepsilon)^2 + (c-1)^2}\right\} \notin \mathcal{I},$$

then the proof will be completed because $a|d$ is $\inf A_{mx}$, and if $(b+\varepsilon)|c \in A_{mx}$, then we will have $a \leq b$ (for any arbitrary $\varepsilon > 0$).

Since $\mathcal{I}\text{-}\limsup mx = b|c$ is finite, then by Proposition 2 we have

$$\left\{i : \sqrt{x_i^2 + (c_i - 1)^2} > \sqrt{(b+\varepsilon)^2 + (c-1)^2}\right\} \in \mathcal{I}.$$

Therefore

$$\left\{i : \sqrt{x_i^2 + (c_i - 1)^2} \leq \sqrt{(b+\varepsilon)^2 + (c-1)^2}\right\} \in F(\mathcal{I}),$$

i.e.,

$$\left\{i : \sqrt{x_i^2 + (c_i - 1)^2} < \sqrt{(b+\varepsilon)^2 + (c-1)^2}\right\} \notin \mathcal{I}.$$

□

Theorem 3. *The $\mathcal{I}$-bounded multiset sequence $mx = (x_i|c_i)$ of $m\mathbb{R}$ is $\mathcal{I}$-convergent if and only if*

$$\mathcal{I}\text{-}\liminf mx = \mathcal{I}\text{-}\limsup mx.$$

Proof. Suppose that mx is $\mathcal{I}$-bounded. It gives us that $\mathcal{I}\text{-}\liminf mx$ and $\mathcal{I}\text{-}\limsup mx$ are finite.

($\Rightarrow$) Let us assume that $mx = (x_i|c_i)$ is $\mathcal{I}$-convergent to $l|c$ and $\varepsilon > 0$. To show that $\mathcal{I}\text{-}\liminf mx = \mathcal{I}\text{-}\limsup mx$ we need to prove that $\mathcal{I}\text{-}\liminf mx \leq \mathcal{I}\text{-}\limsup mx$ and $\mathcal{I}\text{-}\liminf mx \geq \mathcal{I}\text{-}\limsup mx$. The case $\mathcal{I}\text{-}\liminf mx \leq \mathcal{I}\text{-}\limsup mx$ is known from the previous theorem. The case $\mathcal{I}\text{-}\limsup mx \leq \mathcal{I}\text{-}\liminf mx$ will be shown using the boundedness and convergence properties.

Let $\mathcal{I}\text{-}\liminf mx = a|c_1$ and $\mathcal{I}\text{-}\limsup mx = b|c_2$. Based on the assumption $\mathcal{I}\text{-}\lim mx = l|c$, for for every $\varepsilon > 0$,

$$\left\{i \in \mathbb{N} : \sqrt{(x_i - l)^2 + (c_i - c)^2} \geq \varepsilon\right\} \in \mathcal{I}.$$

Then

$$\begin{aligned}
&\left\{i \in \mathbb{N} : \sqrt{(x_i - l)^2 + (c_i - c)^2} < \varepsilon\right\} \in F(\mathcal{I}) \\
&\Longrightarrow \left\{i \in \mathbb{N} : |x_i - l| < \tfrac{\varepsilon}{\sqrt{2}} = \varepsilon_1,\ |c_i - c| < \tfrac{\varepsilon}{\sqrt{2}} = \varepsilon_1\right\} \in F(\mathcal{I}) \\
&\Longrightarrow \{i \in \mathbb{N} : |x_i - l| \geq \varepsilon_1,\ |c_i - c| \geq \varepsilon_1\} \in \mathcal{I} \\
&\Longrightarrow \{i \in \mathbb{N} : |x_i - l| > \varepsilon_1,\ |c_i - c| > \varepsilon_1\} \in \mathcal{I} \\
&\Longrightarrow \{i \in \mathbb{N} : x_i > l + \varepsilon_1,\ c_i - 1 > c + \varepsilon_1 - 1\} \in \mathcal{I}
\end{aligned}$$

and continuing we obtain

$$\left\{i \in \mathbb{N} : \sqrt{x_i^2 + (c_i - 1)^2} > \sqrt{(l+\varepsilon_1)^2 + ((c+\varepsilon_1) - 1)^2}\right\} \in \mathcal{I}.$$

Then we have

$$\begin{aligned}
&\sqrt{b^2 + (c_2 - 1)^2} < \sqrt{(l+\varepsilon_1)^2 + (c + \varepsilon_1 - 1)^2} \\
&\Longrightarrow \quad \sqrt{b^2 + (c_2 - 1)^2} \leq \sqrt{l^2 + (c-1)^2}.
\end{aligned}$$

This means

$$\mathcal{I}\text{-}\limsup mx \leq \mathcal{I}\text{-}\lim mx. \tag{1}$$

Similarly,

$$\sqrt{(l-\varepsilon_1)^2+(c-\varepsilon_1-1)^2}<\sqrt{a^2+(c-1)^2}$$

and

$$\sqrt{l^2+(c-1)^2}\le\sqrt{a^2+(c-1)^2}.$$

Thus

$$\mathcal{I}\text{-}\lim mx\le\mathcal{I}\text{-}\liminf mx. \tag{2}$$

Then from (1) and (2) $\mathcal{I}$- $\liminf mx=\mathcal{I}$- $\lim\sup mx$ is obtained.

($\Leftarrow$) Now let $\mathcal{I}$- $\liminf mx=\mathcal{I}$- $\limsup mx=l|c$. Since $\mathcal{I}$- $\limsup mx=l|c$ for every $\varepsilon_1>0$, we have

$$\left\{i\in\mathbb{N}:\sqrt{x_i^2+(c_i-1)^2}>\sqrt{(l+\varepsilon_1)^2+(c-1)^2}\right\}\in\mathcal{I}$$

from Proposition 2. On the other hand, it is possible to find a number ε' greater than zero such that

$$\sqrt{(l+\varepsilon_1)^2+(c-1)^2}<\sqrt{l^2+(c-1)^2}+\varepsilon',$$

and so

$$\left\{i\in\mathbb{N}:\sqrt{x_i^2+(c_i-1)^2}>\sqrt{l^2+(c-1)^2}+\varepsilon'\right\}\in\mathcal{I}.$$

At the same time, since $\mathcal{I}$- $\liminf mx=l|c$ for every $\varepsilon_2>0$, we have

$$\left\{i\in\mathbb{N}:\sqrt{x_i^2+(c_i-1)^2}>\sqrt{(l-\varepsilon_2)^2+(c-1)^2}\right\}\in\mathcal{I}$$

from Proposition 3. Similarly, it is possible to find an $\varepsilon''>0$ such that

$$\sqrt{(l-\varepsilon_2)^2+(c-1)^2}=\sqrt{l^2+(c-1)^2}+\varepsilon''$$

so

$$\left\{i\in\mathbb{N}:\sqrt{x_i^2+(c_i-1)^2}<\sqrt{l^2+(c-1)^2}-\varepsilon''\right\}\in\mathcal{I}.$$

For $\varepsilon=\max(\varepsilon',\varepsilon'')$,

$$\left\{i\in\mathbb{N}:\sqrt{x_i^2+(c_i-1)^2}>\sqrt{l^2+(c-1)^2}+\varepsilon\right\}\in\mathcal{I} \tag{3}$$

and

$$\left\{i\in\mathbb{N}:\sqrt{x_i^2+(c_i-1)^2}<\sqrt{l^2+(c-1)^2}-\varepsilon\right\}\in\mathcal{I}. \tag{4}$$

When we think of (3) and (4) together we have $\mathcal{I}$- $\lim mx=l|c$. □

3. Conclusions

Multisets have a very important place in many areas of science and even in our daily lives. Therefore, the properties of multiset sequences consisting of these sets are very interesting. On the other hand, how the concept of $\mathcal{I}$-convergence, which generalizes many convergence types, can be defined for multiset sequences is the main purpose of this study. We think that, this study will hold an important place for other studies in this subject.

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