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The discrete Hardy type variable exponent inequality with decreasing exponent

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Abstract. The purpose of this note is to obtain the discrete Hardy type variable exponent inequality for the decreasing exponent.

1. Introduction

Let $p : (0,1) \to [1,\infty)$ be a measurable function. We suppose that

$$1 \leq p_- \leq p(x) \leq p_+ < \infty,$$

where $p_- := \text{ess inf}_{x\in(0,1)}\ p(x)$ and $p_+ := \text{ess sup}_{x\in(0,1)}\ p(x)$.

$L^{p(\cdot)}(0,1)$ will denote the Lebesgue space with variable exponent, which is the class of functions defined on $(0,1)$ such that the modular

$$I_p(f) := \int_0^1 |f(x)|^{p(x)}\, dx$$

is finite. The norm in $L^{p(\cdot)}(0,1)$ is defined as follows:

$$\|f\|_{L^{p(\cdot)}(0,1)} = \inf \left\{ \lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$

The classical Hardy inequality states that for $1 < p < \infty$ and for any $f \in L^p(0,\infty)$

$$\int_0^\infty \left| \frac{1}{x}\int_0^x f(y)dy \right|^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty |f(x)|^p dx.$$

This inequality has a long history, and also many generalizations (see [6, 7, 9, 10, 12] and the references therein).

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The Hardy–Littlewood–Pólya inequality is a classical result in analysis and mathematical inequalities, named after mathematicians G. H. Hardy, J. E. Littlewood and G. Pólya. The generality and versatility of Hardy inequalities make them a valuable tool in mathematical analysis, providing a bridge between different branches of mathematics. Their exploration and application continue to be an active area of research, contributing to the development of various mathematical theories and their applications in diverse fields. Many proofs of celebrated Hardy inequality are known, the interested reader is invited to check the survey [7], in which several proofs and historical aspects are given. For more on Hardy inequality, see e.g. [5], where the authors improved the classical Hardy inequality for a sequence of non-negative real numbers, and also [8], where Lefévre obtained a short direct proof of the discrete Hardy inequality provided that the constant p' is optimal.

Hardy's inequality can be stated in the terms of the Hardy operator, which is the linear operator given by

$$Hf(x) = \int_0^x f(y)dy.$$

Then, the inequality becomes

$$\|x^{-1}Hf\|_p \leq \frac{p}{p-1}\|f\|_p.$$

There are also studies in the framework of Hardy type inequalities and their applications to evolution problems. For example, in [1] the authors deal with local and nonlocal weighted improved Hardy inequalities related to the study of Kolmogorov operators perturbed by singular potentials. In [3], the authors extended the reverse Hardy inequalities to a general metric measure space with two negative exponents. For the study of fractional Hardy inequality on the integers and the optimality of the Hardy weight, see [4].

In this paper, we state and prove a discrete version of the Hardy inequality in the setting of variable Lebesgue spaces with decreasing exponent. To this end, we recall the continuous version of this theorem (see Theorem 1 below) in order to establish our main result, which is Theorem 2.

2. Main result

In [11], Mamedova established and proved the following theorem.

Theorem 1. *Let $f \in L^{p(\cdot)}(0,1)$, $f \geq 0$, where the exponent $p : (0,1) \to (1,\infty)$ is a decreasing function on some interval $(0,\varepsilon)$, $\varepsilon > 0$. Then*

$$\left\| \frac{1}{x}\int_0^x f(y)dy \right\|_{L^{p(\cdot)}(0,1)} \leq C\|f\|_{L^{p(\cdot)}(0,1)}. \tag{1}$$

We deal with the discrete version of Theorem 1, which is stated as follows.

Theorem 2. *Let* $\{a_n\}_{n\in\mathbb{N}}$ *be a sequence of positive real numbers. Let* $p:(0,1)\to(1,\infty)$ *be a decreasing function such that* $\sum_{n=1}^{\infty} a_n^{p(t)} < \infty$. *Then*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^{p_-} \leq C\sum_{n=1}^{\infty} a_n^{p(t)}.$$

Proof. Let us define $A_N := \sum_{n=1}^{N} a_n$ where $N = 1, 2, \ldots$. First of all, let us observe that if some function $\phi(u)$ is a positive and increasing function of u, then $\sum_{N=1}^{\infty}\phi(\frac{A_N}{N})$ is maximum when $\{a_n\}_{n\in\mathbb{N}}$ is decreasing. Indeed, if we set $s > t$ and $a_t > a_s$, the substitution of a_t for a_s in $\{a_n\}_{n\in\mathbb{N}}$ does not alter A_N if $N < t$ or $N \geq s$, it also does not increase if $t \leq N < s$. Taking into account these observations, let us suppose that $\{a_n\}_{n\in\mathbb{N}}$ is a decreasing sequence such that $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$, and define

$$f(t) = \sum_{n=1}^{\infty} a_n \chi_{(n-1,n)}(t).$$

Then, if $0 < a_1 \leq 1$ and $\varepsilon > 0$ is any positive real number, we have

$$\int_0^1 (f(t))^{p(t)}\, dt = \int_0^1 \left(\sum_{n=1}^{\infty} a_n \chi_{(n-1,n)}(t)\right)^{p(t)} dt$$
$$= \int_0^1 \left(a_1\chi_{(0,1)}(t)\right)^{p(t)}\, dt = \int_0^1 a_1^{p(t)}\, dt \leq \int_0^1 a_1^{p_-}\, dt = a_1^{p_-}$$
$$\leq a_1^{p(t)+\varepsilon} = Ca_1^{p(t)} \leq C\sum_{n=1}^{\infty} a_n^{p(t)}, \text{where } C = a_1^{\varepsilon}.$$

On the other hand, in case $a_1 > 1$, we have

$$\int_0^1 (f(t))^{p(t)}\, dt = \int_0^1 \left(\sum_{n=1}^{\infty} a_n \chi_{(n-1,n)}(t)\right)^{p(t)} dt$$
$$= \int_0^1 \left(a_1\chi_{(0,1)}(t)\right)^{p(t)}\, dt = \int_0^1 a_1^{p(t)}\, dt$$
$$\leq \int_0^1 a_1^{p_+}\, dt = a_1^{p_+} \leq a_1^{p(t)-\varepsilon} = Ca_1^{p(t)} \leq C\sum_{n=1}^{\infty} a_n^{p(t)}, \text{where } C = a_1^{-\varepsilon}.$$

So, in any case,

$$\int_0^1 (f(t))^{p(t)}\, dt \leq C\sum_{n=1}^{\infty} a_n^{p(t)}. \tag{2}$$

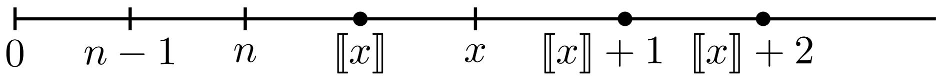

FIGURE 1. Relationship between x and $[\![x]\!]$.

In the sequel $[\![x]\!]$ will denote the greatest integer which is less or equal than the real number x. Having clarified this, let us observe that

$$\begin{aligned}
F(x) =& \frac{1}{x}\int_0^x \sum_{n=1}^{\infty} a_n \chi_{(n-1,n)}(t)\,dt\\
=& \frac{1}{x}\left[\int_0^{[\![x]\!]} \sum_{n=1}^{[\![x]\!]} a_n \chi_{(n-1,n)}(t)dt + \int_{[\![x]\!]}^{x} \sum_{n=[\![x]\!]+1}^{\infty} a_n \chi_{(n-1,n)}(t)\,dt\right]\\
=& \frac{1}{x}\left[\int_0^{1} \sum_{n=1}^{[\![x]\!]} a_n \chi_{(n-1,n)}(t)\chi_{(0,[\![x]\!])}(t)dt + \int_{[\![x]\!]}^{x} a_{[\![x]\!]+1} \chi_{([\![x]\!],[\![x]\!]+1)}(t)\,dt\right.\\
&\left. + \int_{[\![x]\!]}^{x} \sum_{n=[\![x]\!]+2}^{\infty} a_n \chi_{(n-1,n)}(t)\,dt\right].
\end{aligned}$$

Now, consider Figure 1 regarding $[\![x]\!]$. Accordingly, we can observe that

$$\begin{aligned}
([\![x]\!], x) \cap ([\![x]\!]+1, [\![x]\!]+2) &= \emptyset,\\
([\![x]\!], x) \cap ([\![x]\!]+2, [\![x]\!]+3) &= \emptyset,\\
&\vdots\\
([\![x]\!], x) \cap ([\![x]\!]+n, [\![x]\!]+n+1) &= \emptyset.
\end{aligned}$$

Thus

$$\sum_{n=[\![x]\!]+2}^{\infty} a_n \chi_{(n-1,n)}\chi_{([\![x]\!],x)} = \sum_{n=[\![x]\!]+2}^{\infty} a_n \chi_{\emptyset} = 0.$$

Also, since

$$(n-1,n) \subset (0,[\![x]\!]),$$

we have

$$\chi_{(n-1,n)\cap(0,[\![x]\!])} = \chi_{(n-1,n)}.$$

Similarly, given that

$$([\![x]\!], x) \subset ([\![x]\!], [\![x]\!]+1),$$

we have

$$\chi_{([\![x]\!],x)\cap([\![x]\!],[\![x]\!]+1)} = \chi_{([\![x]\!],x)}.$$

Using these results we get

$$\begin{aligned} F(x) =& \frac{1}{x}\left[\int_{n-1}^{n}\sum_{n=1}^{[\![x]\!]} a_n dt + \int_{[\![x]\!]}^{x} a_{[\![x]\!]+1} dt\right] \\ =& \frac{1}{x}\left[\sum_{n=1}^{[\![x]\!]} a_n + (x-[\![x]\!])a_{[\![x]\!]+1}\right] \\ =& \frac{1}{x}A_{[\![x]\!]} + \left(1-\frac{[\![x]\!]}{x}\right)a_{[\![x]\!]+1}. \end{aligned}$$

For $N \leq x < N+1$, the derivative of F is negative (since $\{a_n\}_{n\in\mathbb{N}}$ are decreasing). So that

$$F(x) > \sum_{N=1}^{\infty} \frac{1}{N} A_N \chi_{(N-1,N)}(x). \tag{3}$$

Now, we recall that the modular $I_p(h)$ and the norm $\|h\|_{L^{p(\cdot)}(0,1)}$ are related by the following inequalities (see [2, Corollary 2.23]):

$$\|h\|^{p^+}_{L^{p(\cdot)}(0,1)} \leq I_p(h) \leq \|h\|^{p^-}_{L^{p(\cdot)}(0,1)}, \quad 1 \geq \|h\|_{L^{p(\cdot)}(0,1)}, \tag{4}$$

$$\|h\|^{p^-}_{L^{p(\cdot)}(0,1)} \leq I_p(h) \leq \|h\|^{p^+}_{L^{p(\cdot)}(0,1)}, \quad 1 \leq \|h\|_{L^{p(\cdot)}(0,1)}.$$

Without loss of generality, suppose that $1 \geq \|F\|_{L^{p(\cdot)}(0,1)}$ (the case $1 \leq \|F\|_{L^{p(\cdot)}(0,1)}$ is similar). Then, going back to (3), we obtain

$$\begin{aligned} \left(\sum_{N=1}^{\infty}\frac{A_N}{N}\right)^{p_-} \leq \sum_{N=1}^{\infty}\left(\frac{A_N}{N}\right)^{p_+} &\leq \int_0^1 [F(t)]^{p_+}\,dt \leq \int_0^1 [F(t)]^{p(t)}\,dt = I_p(F) \\ &\leq \|F\|^{p_-}_{L_{p(\cdot)(0,1)}} && \text{(by (4))} \\ &\leq C\,\|f\|^{p_-}_{L_{p(\cdot)(0,1)}} && \text{(by (1))} \\ &\leq C I_p(f) = C\int_0^1 (f(t))^{p(t)}\,dt \\ &\leq C\sum_{n=1}^{\infty} a_n^{p(t)}. && \text{(by (2))} \end{aligned}$$

Hence

$$\sum_{N=1}^{\infty}\left(\frac{A_N}{N}\right)^{p_-} \leq C\sum_{n=1}^{\infty} a_n^{p(\cdot)},$$

and therefore

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^{p_-} \leq C\sum_{n=1}^{\infty}a_n^{p(\cdot)}.$$

The observation made at the beginning of the proof is applicable to the function $\phi(u) = u^{p(\cdot)}$ and thus the general case is established. $\square$

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