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# Cohomology of modified $\lambda$-differential Jacobi–Jordan algebras and its applications

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Abstract. The purpose of the present paper is to investigate cohomology of modified $\lambda$-differential Jacobi–Jordan algebras. First, we introduce the concept and representations of modified $\lambda$-differential Jacobi–Jordan algebras. Moreover, we define a lower order cohomology theory for modified $\lambda$-differential Jacobi–Jordan algebras. As applications of the proposed cohomology theory, formal deformations of modified $\lambda$-differential Jacobi–Jordan algebras are obtained and the rigidity of a modified $\lambda$-differential Jacobi–Jordan algebra is characterized by the vanishing of the second cohomology group. Also, abelian extensions of modified $\lambda$-differential Jacobi–Jordan algebras are classified by second-order cohomology. Furthermore, we study $T^*$-extensions of modified $\lambda$-differential Jacobi–Jordan algebras.

## 1. Introduction

A Jacobi–Jordan algebra is a commutative algebra satisfying the Jacobi identity, introduced first in [37], where an important example of infinite-dimensional solvable but non-nilpotent Jacobi–Jordan algebra was given. They are rather special objects in the jungle of non-associative algebras. Different names are used to study these algebras, indeed they are called mock-Lie algebras, Jordan algebras of nil index 3, Lie-Jordan, pathological algebras or Jacobi–Jordan algebras in the literature [2, 3, 5, 12, 21, 30, 36]. In contrast to associative and non-associative algebras, results on cohomology theories of Jacobi–Jordan algebras have been relatively scarce for a long time. Recently, analogously to the existing theories for associative and Lie algebras, cohomology and deformation theories for Jacobi–Jordan algebras have been developed [4] where it is observed that they have several properties not

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enjoyed by the Hochschild theory. Actually, this cohomology is called a zigzag cohomology since its complex is defined by two sequences of operators.

Derivations play an important role in studying algebraic structure. For example, derivations can be used to construct homotopy of Lie algebras, deformation formulas, and differential Galois theories [6, 18, 29]. Loday studied algebras with derivations from the operadic point of view [17]. Recently many scholars began to study algebras with derivations assiociated to cohomology theories. The paper [11] considered deformations and extensions of associative algebras with derivations. The theory of Lie algebras with derivations about cohomology, deformations and extensions was studied in [25]. Das studied Leibniz algebras with derivations in [10]. Other algebras like 3-Lie algebras, n-Lie algebras, pre Lie algebras, Lie triple systems, Leibniz triple systems, all with derivations, can be found in [13, 16, 23, 24, 32, 33, 34].

In recent years, more and more scholars started to pay attention to the structure with arbitrary weights. Rota–Baxter Lie algebras of any weight were studied in [8, 9, 31]. After that, for $\lambda \in \mathbb{K}$, the cohomology, extension and deformation theory of Lie algebras with differential operators of weight $\lambda$ were introduced by Li and Wang [15]. In addition, the cohomology and deformation theory of modified Rota–Baxter associative algebras and modified Rota–Baxter Leibniz algebras of weight $\lambda$ are given in [7, 14, 20].

The concept of modified $\lambda$-differential Lie algebras is introduced in [22], modified $\lambda$-differential 3-Lie algebras in [26], modified $\lambda$-differential Lie triple systems in [27] and deformations and extension of modified $\lambda$-differential Lie–Yamaguti algebras in [28]. In [4, 35], the authors developed the lower order cohomology of Jacobi–Jordan algebras, and the matrix form of derivation on Jacobi–Jordan algebras in low dimension was given in [19]. This inspired us to discover the cohomology of modified $\lambda$-differential Jacobi–Jordan algebras.

The paper is organized as follows. In Section 2, we introduce the concept of a modified $\lambda$-differential and give its representation. In Section 3, we define a cohomology theory for modified $\lambda$-differential Jacobi–Jordan algebras. In Section 4, we study 1-parameter formal deformations of a modified $\lambda$-differential Jacobi–Jordan algebra and show that a modified $\lambda$-differential Jacobi–Jordan algebra is rigid if its second cohomology group is trivial. In Section 5, we study abelian extensions of a modified $\lambda$-differential Jacobi–Jordan algebra and show that equivalent classes of abelian extensions are classified by the second cohomology group of a modified $\lambda$-differential Jacobi–Jordan algebra. In Section 6, we study $T^*$-extensions of modified $\lambda$-differential Jacobi–Jordan algebras.

Throughout this paper, $\mathbb{K}$ denotes a field of characteristic zero. All the algebras, vector spaces, linear maps and tensor products are taken over $\mathbb{K}$.

## 2. Representations of modified $\lambda$-differential Jacobi–Jordan algebras

In this section, first, we recall some basic concepts of Jacobi–Jordan algebras from [5, 4, 35]. Then, we introduce the concept of a modified $\lambda$-differential Jacobi–Jordan algebra and its representations.

**Definition 2.1.** An algebra $\mathfrak{J}$ over a field $\mathbb{K}$ is called a *Jacobi–Jordan algebra* if there is a bilinear map $\cdot : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}$ satisfying the following identities:

$$x \cdot y = y \cdot x, \tag{1}$$

$$x \cdot (y \cdot z) + y \cdot (z \cdot x) + z \cdot (x \cdot y) = 0, \tag{2}$$

for all $x, y, z \in \mathfrak{J}$.

**Definition 2.2.** A *homomorphism between two Jacobi–Jordan algebras* $(\mathfrak{J}_1, \cdot_1)$ and $(\mathfrak{J}_2, \cdot_2)$ is a linear map $\zeta : \mathfrak{J}_1 \to \mathfrak{J}_2$ satisfying $\zeta(x \cdot_1 y) = \zeta(x) \cdot_2 \zeta(y)$ for all $x, y \in \mathfrak{J}_1$.

**Definition 2.3.** A *representation of a Jacobi–Jordan algebra* $(\mathfrak{J}, \cdot)$ on a vector space V is a linear map $\rho : \mathfrak{J} \to \mathrm{End}(V)$, such that

$$\rho(x \cdot y)u = -\rho(x)\rho(y)u - \rho(y)\rho(x)u, \tag{3}$$

for all $x, y \in \mathfrak{J}$ and $u \in V$. Then $(V; \rho)$ is called a *representation* of $\mathfrak{J}$. In this case, we also call V a $\mathfrak{J}$*-module*.

**Example 2.4.** Any Jacobi–Jordan algebra $(\mathfrak{J}, \cdot)$ is a representation over itself with

$$\mathrm{ad} : \mathfrak{J} \to \mathrm{End}(\mathfrak{J}), y \mapsto (x \mapsto x \cdot y).$$

It is called the *adjoint representation* over the Jacobi–Jordan algebra.

**Definition 2.5.** Let $\lambda \in \mathbb{K}$ and $(\mathfrak{J}, \cdot)$ be a Jacobi–Jordan algebra. A *modified $\lambda$-differential operator* (also called a *modified differential operator of weight $\lambda$*) on $\mathfrak{J}$ is a linear operator $d : \mathfrak{J} \to \mathfrak{J}$, such that

$$d(x \cdot y) = d(x) \cdot y + x \cdot d(y) + \lambda x \cdot y, \tag{4}$$

for all $x, y \in \mathfrak{J}$.

**Definition 2.6.** A *modified $\lambda$-differential Jacobi–Jordan algebra* (also called a *modified differential Jacobi–Jordan algebra of weight $\lambda$*) is a triple $(\mathfrak{J}, \cdot, d)$ consisting of a Jacobi–Jordan algebra $(\mathfrak{J}, \cdot)$ and a modified $\lambda$-differential operator d.

**Definition 2.7.** A *homomorphism between two modified $\lambda$-differential Jacobi–Jordan algebras* $(\mathfrak{J}_1, \cdot_1, d_1)$ and $(\mathfrak{J}_2, \cdot_2, d_2)$ is a Jacobi–Jordan algebra homomorphism $\zeta : (\mathfrak{J}_1, \cdot_1) \to (\mathfrak{J}_2, \cdot_2)$ such that $\zeta \circ d_1 = d_2 \circ \zeta$. Furthermore, if $\zeta$ is nondegenerate, then $\zeta$ is called an *isomorphism* from $\mathfrak{J}_1$ to $\mathfrak{J}_2$.

**Remark 2.8.** Let d be a modified $\lambda$-differential operator on $(\mathfrak{J}, \cdot)$. If $\lambda = 0$, then d is a derivation on $\mathfrak{J}$. We denote the set of all derivations on $\mathfrak{J}$ by $\mathrm{Der}(\mathfrak{J})$. One can refer to [19] for more information about Jacobi–Jordan algebras with derivations.

Moreover, there is a close relationship between derivations and modified $\lambda$-differential operators.

**Proposition 2.9.** *Let* $(\mathfrak{J}, \cdot)$ *be a Jacobi–Jordan algebra. Then, a linear operator* $\mathrm{d} : \mathfrak{J} \to \mathfrak{J}$ *is a modified* $\lambda$*-differential operator if and only if* $\mathrm{d} + \lambda \mathrm{id}_{\mathfrak{J}}$ *is a derivation on* $\mathfrak{J}$.

*Proof.* Eq. (4) is equivalent to

$$(\mathrm{d} + \lambda \mathrm{id}_{\mathfrak{J}})(\mathrm{x} \cdot \mathrm{y}) = (\mathrm{d} + \lambda \mathrm{id}_{\mathfrak{J}})(\mathrm{x}) \cdot \mathrm{y} + \mathrm{x} \cdot (\mathrm{d} + \lambda \mathrm{id}_{\mathfrak{J}})(\mathrm{y}).$$

The proposition follows. □

**Example 2.10.** Let $(\mathfrak{J}, \cdot, \mathrm{d})$ be a modified $\lambda$-differential Jacobi–Jordan algebra. Then, for $\mathrm{k} \in \mathbb{K}$, $(\mathfrak{J}, \cdot, \mathrm{kd})$ is a modified $(\mathrm{k}\lambda)$-differential Jacobi–Jordan algebra.

**Example 2.11.** Let $(\mathfrak{J}, \cdot)$ be a 4-dimensional Jacobi–Jordan algebra with a basis $\{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \mathfrak{e}_4\}$ defined by $\mathfrak{e}_1 \cdot \mathfrak{e}_1 = \mathfrak{e}_2, \mathfrak{e}_1 \cdot \mathfrak{e}_3 = \mathfrak{e}_4$. Then the operator

$$\mathrm{d} = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & 2a_{11} + \lambda & a_{23} & 0 \\ a_{31} & 0 & a_{33} & 0 \\ a_{41} & 2a_{31} & a_{43} & a_{11} + a_{33} + \lambda \end{pmatrix}$$

is a modified $\lambda$-differential operator on $\mathfrak{J}$, for $\lambda \in \mathbb{K}$.

**Definition 2.12.** A *representation of the modified* $\lambda$*-differential Jacobi–Jordan algebra* $(\mathfrak{J}, \cdot, \mathrm{d})$ is a triple $(\mathrm{V}; \rho, \mathrm{d}_\mathrm{V})$, where $(\mathrm{V}; \rho)$ is a representation of the Jacobi–Jordan algebra $(\mathfrak{J}, \cdot)$ and $\mathrm{d}_\mathrm{V}$ is a linear operator on V, satisfying the equation

$$\mathrm{d}_\mathrm{V}(\rho(\mathrm{x})\mathrm{u}) = \rho(\mathrm{d}(\mathrm{x}))\mathrm{u} + \rho(\mathrm{x})\mathrm{d}_\mathrm{V}(\mathrm{u}) + \lambda\rho(\mathrm{x})\mathrm{u}, \tag{5}$$

for any $\mathrm{x} \in \mathfrak{J}$ and $\mathrm{u} \in \mathrm{V}$.

Obviously, $(\mathfrak{J}; \mathrm{ad}, \mathrm{d})$ is a representation of the modified $\lambda$-differential Jacobi–Jordan algebra $(\mathfrak{J}, \cdot, \mathrm{d})$.

**Remark 2.13.** Let $(\mathrm{V}; \rho, \mathrm{d}_\mathrm{V})$ be a representation of the modified $\lambda$-differential Jacobi–Jordan algebra $(\mathfrak{J}, \cdot, \mathrm{d})$. If $\lambda = 0$, then $(\mathrm{V}; \rho, \mathrm{d}_\mathrm{V})$ is a representation of the Jacobi–Jordan algebra with a derivation $(\mathfrak{J}, \cdot, \mathrm{d})$.

Moreover, the following result finds the relation between representations over modified $\lambda$-differential Jacobi–Jordan algebras and over Jacobi–Jordan algebras with derivations.

**Proposition 2.14.** *Let* $(\mathrm{V}; \rho)$ *be a representation of the Jacobi–Jordan algebra* $(\mathfrak{J}, \cdot)$. *Then* $(\mathrm{V}; \rho, \mathrm{d}_\mathrm{V})$ *is a representation of the modified* $\lambda$*-differential Jacobi–Jordan algebra* $(\mathfrak{J}, \cdot, \mathrm{d})$ *if and only if* $(\mathrm{V}; \rho, \mathrm{d}_\mathrm{V} + \lambda \mathrm{id}_\mathrm{V})$ *is a representation of the Jacobi–Jordan algebra with a derivation* $(\mathfrak{J}, \cdot, \mathrm{d} + \lambda \mathrm{id}_{\mathfrak{J}})$.

*Proof.* Equality (5) is equivalent to

$$(d_V + \lambda id_V)(\rho(x)u) = \rho((d + \lambda id_{\mathfrak{J}})(x))u + \rho(x)(d_V + \lambda id_V)(u).$$

The proposition follows. □

**Example 2.15.** Let $(V; \rho)$ be a representation of the Jacobi–Jordan algebra $(\mathfrak{J}, \cdot)$. Then, for $k \in \mathbb{K}$, $(V; \rho, id_V)$ is a representation of the modified $(-\lambda)$-differential Jacobi–Jordan algebra $(\mathfrak{J}, \cdot, \lambda id_{\mathfrak{J}})$.

**Example 2.16.** Let $(V; \rho, d_V)$ be a representation of the modified $\lambda$-differential Jacobi–Jordan algebra $(\mathfrak{J}, \cdot, d)$. Then, for $k \in \mathbb{K}$, $(V; \rho, kd_V)$ is a representation of the modified $(k\lambda)$-differential Jacobi–Jordan algebra $(\mathfrak{J}, \cdot, kd)$.

Next we construct the semi-direct product in the context of modified $\lambda$-differential Jacobi–Jordan algebras.

**Proposition 2.17.** *Let* $(\mathfrak{J}, \cdot, d)$ *be a modified* $\lambda$*-differential Jacobi–Jordan algebra and* $(V; \rho, d_V)$ *be a representation of it. Then* $\mathfrak{J} \oplus V$ *is a modified* $\lambda$*-differential Jacobi–Jordan algebra under the following maps:*

$$(x + u) \bullet (y + v) := x \cdot y + \rho(x)v + \rho(y)u,$$
$$d \oplus d_V(x + u) := d(x) + d_V(u),$$

*for all* $x, y \in \mathfrak{J}$ *and* $u, v \in V$.

*Proof.* First, as we all know, $(\mathfrak{J} \oplus V, \bullet)$ is a Jacobi–Jordan algebra. Next, for any $x, y \in \mathfrak{J}, u, v \in V$, by equalities (4) and (5), we have

$$\begin{aligned}
&d \oplus d_V((x + u) \bullet (y + v)) \\
&= d(x \cdot y) + d_V(\rho(x)v) + d_V(\rho(y)u) \\
&= d(x) \cdot y + x \cdot d(y) + \lambda x \cdot y + \rho(d(x))v + \rho(x)d_V(v) + \lambda\rho(x)v \\
&\quad + \rho(d(y))u + \rho(y)d_V(u) + \lambda\rho(y)u \\
&= d \oplus d_V(x + u) \bullet (y + v) + (x + u) \bullet d \oplus d_V(y + v) + \lambda(x + u) \bullet (y + v).
\end{aligned}$$

Therefore, $(\mathfrak{J} \oplus V, \bullet, d \oplus d_V)$ is a modified $\lambda$-differential Jacobi–Jordan algebra. □

Let $(V; \rho, d_V)$ be a representation of a modified $\lambda$-differential Jacobi–Jordan algebra $(\mathfrak{J}, \cdot, d)$, and $V^*$ be a dual space of $V$. We define a linear map $\rho^* : \mathfrak{J} \to End(V^*)$ and a linear map $d_V^* : V^* \to V^*$, respectively, by

$$\langle \rho^*(x)u^*, v \rangle = \langle u^*, \rho(x)v \rangle \text{ and } \langle d_V^* u^*, v \rangle = \langle u^*, d_V(v) \rangle, \tag{6}$$

for any $x \in \mathfrak{J}, v \in V$ and $u^* \in V^*$.

**Proposition 2.18.** *With the above notations,* $(V^*; \rho^*, -d_V^*)$ *is a representation of modified* $\lambda$*-differential Jacobi–Jordan algebra* $(\mathfrak{J}, \cdot, d)$*. We call it the dual representation of* $(V; \rho, d_V)$

*Proof.* Following [1], we can easily see that $(V^*;\rho^*)$ is a representation of the Jacobi–Jordan algebra $(\mathfrak{J},\cdot)$. Moreover, for any $x\in\mathfrak{J}$, $v\in V$ and $u^*\in V^*$, by equalities (5) and (6), we have

$$\begin{aligned}
&\langle\rho^*(d(x))u^*,v\rangle+\langle\rho^*(x)(-d_V^*)u^*,v\rangle+\langle\lambda\rho^*(x)u^*,v\rangle-\langle(-d_V^*)\rho^*(x)u^*,v\rangle\\
&=\langle u^*,\rho(d(x))v\rangle+\langle(-d_V^*)u^*,\rho(x)v\rangle+\langle u^*,\lambda\rho(x)v\rangle+\langle\rho^*(x)u^*,d_V(v)\rangle\\
&=\langle u^*,\rho(d(x))v\rangle-\langle u^*,d_V(\rho(x)v)\rangle+\langle u^*,\lambda\rho(x)v\rangle+\langle u^*,\rho(x)d_V(v)\rangle\\
&=\langle u^*,\rho(d(x))v-d_V(\rho(x)v)+\lambda\rho(x)v+\rho(x)d_V(v)\rangle\\
&=0,
\end{aligned}$$

which implies that $\rho^*(d(x))u^*+\rho^*(x)(-d_V^*)u^*+\lambda\rho^*(x)u^*-(-d_V^*)\rho^*(x)u^*=0$. So, we have the result. ◻

**Example 2.19.** Let $(\mathfrak{J};\mathrm{ad},d)$ be an adjoint representation of the modified $\lambda$-differential Jacobi–Jordan algebra $(\mathfrak{J},\cdot,d)$. Then $(\mathfrak{J}^*;\mathrm{ad}^*,-d^*)$ is a dual adjoint representation of $(\mathfrak{J},\cdot,d)$.

## 3. Cohomology of modified $\lambda$-differential Jacobi–Jordan algebras

Let $\mathfrak{J}$ and $V$ be two vector spaces. A k-linear map $f:\mathfrak{J}\times\cdots\times\mathfrak{J}\to V$ is called *symmetric* if it satisfies

$$f(x_{\sigma(1)},\cdots,x_{\sigma(k)})=f(x_1,\cdots,x_k),\ \text{for all } \sigma\in\mathfrak{S}_k,$$

where $\mathfrak{S}_k$ is the group of permutations of $\{1,\cdots,k\}$. The set of symmetric k-linear maps is denoted by $\mathfrak{C}^k(\mathfrak{J},V)$ for $k\in\mathbb{N}^*$. Until now, it is known that the cohomology theories of Jordan algebras are incomplete. As for a special Jordan algebra, in [4, 35] low order cohomologies of Jacobi–Jordan algebras are constructed. Let $V$ be a vector space and $(\mathfrak{J},\cdot)$ a Jacobi–Jordan algebra, $(\rho;V)$ be a representation of $\mathfrak{J}$, and let $\mathfrak{C}^n(\mathfrak{J},V)$ represent the space of symmetric n-linear maps. Consider the complex

$$\mathfrak{C}^1(\mathfrak{J},V)\xrightarrow{\partial_1}\mathfrak{C}^2(\mathfrak{J},V)\xrightarrow{\partial_2}\mathfrak{C}^3(\mathfrak{J},V).$$

The 1-coboundary operator of $\mathfrak{J}$ is the map

$$\partial_1:\mathfrak{C}^1(\mathfrak{J},V)\to\mathfrak{C}^2(\mathfrak{J},V),\quad f\mapsto\partial_1 f,$$

given by

$$\partial_1 f(x,y)=f(x\cdot y)-\rho(x)f(y)-\rho(y)f(x),$$

for all $x,y\in\mathfrak{J}$ and $f\in\mathfrak{C}^1(\mathfrak{J},V)$. The 2-coboundary operator of $\mathfrak{J}$ is the map

$$\partial_2:\mathfrak{C}^2(\mathfrak{J},V)\to\mathfrak{C}^3(\mathfrak{J},V),\quad g\mapsto\partial_2 g,$$

given by

$$\begin{aligned}
&\partial_2 g(x,y,z)\\
&=g(x,y\cdot z)+g(y,z\cdot x)+g(z,x\cdot y)+\rho(x)g(y,z)+\rho(y)g(z,x)+\rho(z)g(x,y),
\end{aligned}$$

for all $x,y,z\in\mathfrak{J}$ and $g\in\mathfrak{C}^2(\mathfrak{J},V)$. The first cohomology group

$$\mathcal{H}^1(\mathfrak{J},V)=Z^1(\mathfrak{J},V),$$

where $Z^1(\mathfrak{J}, V) = \{f \in \mathfrak{C}^1(\mathfrak{J}, V) \mid \partial_1 f = 0\}$. The second cohomology group

$$\mathcal{H}^2(\mathfrak{J}, V) = \frac{Z^2(\mathfrak{J}, V)}{B^2(\mathfrak{J}, V)},$$

where $Z^2(\mathfrak{J}, V) = \{g \in \mathfrak{C}^2(\mathfrak{J}, V) \mid \partial_2 g = 0\}$ is the space of 2-cocycles, and $B^2(\mathfrak{J}, V) = \{\partial_1 f \mid f \in \mathfrak{C}^1(\mathfrak{J}, V)\}$ is the space of 2-coboundaries.

Based on the lower order cohomology of Jacobi–Jordan algebras, we want to consider lower order cohomology of modified $\lambda$-differential Jacobi–Jordan algebras. Let $(\rho; V, d_v)$ be a representation of a modified $\lambda$-differential Jacobi–Jordan algebra $(\mathfrak{J}, \cdot, d)$. We define the set of modified $\lambda$-differential Jacobi–Jordan algebra 1-cochains $\mathfrak{C}^1_{mDJJ^\lambda}(\mathfrak{J}, V) = \mathfrak{C}^1(\mathfrak{J}, V)$. For $1 < n \leq 3$, the n-cochains are defined to be

$$\mathfrak{C}^n_{mDJJ^\lambda}(\mathfrak{J}, V) = \mathfrak{C}^n(\mathfrak{J}, V) \times \mathfrak{C}^{n-1}(\mathfrak{J}, V).$$

We define an operator $\delta : \mathfrak{C}^n(\mathfrak{J}, V) \to \mathfrak{C}^n(\mathfrak{J}, V)$ $(n = 1, 2)$ by

$$\delta f(x) = f \circ d(x) - d_V \circ f(x), \ \forall f \in \mathfrak{C}^1(\mathfrak{J}, V),$$

$$\delta g(x, y) = g(d(x), y) + g(x, d(y)) + \lambda g(x, y) - d_V \circ g(x, y), \ \forall g \in \mathfrak{C}^2(\mathfrak{J}, V).$$

Here we consider the complex

$$\mathfrak{C}^1_{mDJJ^\lambda}(\mathfrak{J}, V) \xrightarrow{\phi_1} \mathfrak{C}^2_{mDJJ^\lambda}(\mathfrak{J}, V) \xrightarrow{\phi_2} \mathfrak{C}^3_{mDJJ^\lambda}(\mathfrak{J}, V).$$

Define $\phi_1 : \mathfrak{C}^1_{mDJJ^\lambda}(\mathfrak{J}, V) \longrightarrow \mathfrak{C}^2_{mDJJ^\lambda}(\mathfrak{J}, V)$ by

$$\phi_1 f = (-\partial_1 f, -\delta f), \ f \in \mathfrak{C}^1(\mathfrak{J}, V),$$

and define $\phi_2 : \mathfrak{C}^2_{mDJJ^\lambda}(\mathfrak{J}, V) \to \mathfrak{C}^3_{mDJJ^\lambda}(\mathfrak{J}, V)$ by

$$\phi_2(g, h) = (-\partial_2 g, \partial_1 h - \delta g), g \in \mathfrak{C}^2(\mathfrak{J}, V), \ h \in \mathfrak{C}^1(\mathfrak{J}, V),$$

where $\partial_1, \partial_2$ correspond to the coboundary operators of $\mathfrak{J}$ above.

**Proposition 3.1.** *The operators $\partial_i$ and $\delta$ are commutative, i.e., $\partial_i \circ \delta = \delta \circ \partial_i$, $i = 1, 2$.*

*Proof.* For $i = 1$, we have

$$\begin{aligned}
&\partial_1 \circ \delta f(x, y) \\
&= \delta f(x \cdot y) - \rho(x)\delta f(y) - \rho(y)\delta f(x) \\
&= fd(x \cdot y) - d_V f(x \cdot y) - \rho(x) fd(y) \\
&\quad + \rho(x) d_V f(y) - \rho(y) fd(x) + \rho(y) d_V f(x),
\end{aligned}$$

and

$$\begin{aligned}
&\delta \circ \partial_1 f(x, y) \\
&= \partial_1 f(d(x), y) + \partial_1 f(x, d(y)) - d_V \partial_1 f(x, y) \\
&= f(d(x) \cdot y) - \rho(d(x)) f(y) - \rho(y) f(d(x)) \\
&\quad + f(x \cdot d(y)) - \rho(x) f(d(y)) - \rho(d(y)) f(x) \\
&\quad + d_V \rho(x) f(y) + d_V \rho(y) f(x) - d_V f(x \cdot y).
\end{aligned}$$

Since d is a modified $\lambda$-differential operator of $\mathfrak{J}$, and by equality (5), we have $\partial_1 \circ \delta = \delta \circ \partial_1$.

Similarly, for i = 2, we have

$$\begin{aligned}
&\partial_2 \circ \delta g(x, y, z)\\
&= \delta g(x, y \cdot z) + \delta g(y, z \cdot x) + \delta g(z, x \cdot y)\\
&\quad + \rho(x)\delta g(y, z) + \rho(y)\delta g(z, x) + \rho(z)\delta g(x, y)\\
&= g(d(x), y \cdot z) + g(x, d(y \cdot z)) + \lambda g(x, y \cdot z) - d_V g(x, y \cdot z) + g(d(y), z \cdot x)\\
&\quad + g(y, d(z \cdot x)) + \lambda g(y, z \cdot x) - d_V g(y, z \cdot x) + g(d(z), x \cdot y) + \lambda g(z, x \cdot y)\\
&\quad - d_V g(z, x \cdot y) - d_V g(z, x \cdot y) + g(z, d(x \cdot y)) + \rho(x)(g(d(y), z) + g(y, d(z))\\
&\quad + \lambda g(y, z) - d_V g(y, z)) + \rho(y)(g(d(z), x) + g(z, d(x)) + \lambda g(z, x) - d_V g(z, x))\\
&\quad + \rho(z)(g(d(x), y) + g(x, d(y)) + \lambda g(x, y) - d_V g(x, y)),
\end{aligned}$$

and

$$\begin{aligned}
&\delta \circ \partial_2 g(x, y, z)\\
&= \partial_2 g(d(x), y, z) + \partial_2 g(x, d(y), z) + \partial_2 g(x, y, d(z)) + \lambda \partial_2 g(x, y, z) - d_V \partial_2 g(x, y, z)\\
&= g(d(x), y \cdot z) + g(y, z \cdot d(x)) + g(z, d(x) \cdot y) + \rho(d(x))g(y, z)\\
&\quad + \rho(y)g(z, d(x)) + \rho(z)g(d(x), y) + g(x, d(y) \cdot z) + g(d(y), z \cdot x)\\
&\quad + g(z, x \cdot d(y)) + \rho(x)g(d(y), z) + \rho(d(y))g(z, x) + \rho(z)g(x \cdot d(y))\\
&\quad + g(x, y \cdot d(z)) + g(y, d(z) \cdot x) + g(d(z), x \cdot y) + \rho(x)g(y, d(z))\\
&\quad + \rho(y)g(d(z), x) + \rho(d(z))g(x, y) + \lambda g(x, y \cdot z) + \lambda g(y, z \cdot x)\\
&\quad + \lambda g(z, x \cdot y) + \lambda\rho(x)g(y, z) + \lambda\rho(y)g(z, x) + \lambda\rho(z)g(x, y) - d_V g(x, y \cdot z)\\
&\quad - d_V g(y, z \cdot x) - d_V g(z, x \cdot y) - d_V \rho(x)g(y, z) - d_V \rho(y)g(z, x) - d_V \rho(z)g(x, y).
\end{aligned}$$

Comparing the above two formulas, we have $\partial_2 \circ \delta = \delta \circ \partial_2$. □

**Proposition 3.2.** *With the above notation, we have $\phi_2 \circ \phi_1 = 0$.*

*Proof.* Since $\partial_1 \circ \delta = \delta \circ \partial_1$, for any $f \in \mathfrak{C}^1(\mathfrak{J}, V)$, we have

$$\begin{aligned}
&\phi_2 \circ \phi_1(f)\\
&= \phi_2(-\partial_1 f, -\delta f)\\
&= (-\partial_2(-\partial_1 f), \partial_1(-\delta f) - \delta(-\partial_1 f))\\
&= (0, \partial_1(-\delta f) + \delta(\partial_1 f))\\
&= 0
\end{aligned}$$

and the proof is complete. □

We denote the space of n-cocycles by $Z^n_{mDJJ^\lambda}(\mathfrak{J}, V)$ and the set of n-coboundaries by $B^n_{mDJJ^\lambda}(\mathfrak{J}, V)$, n = 1, 2. The first cohomology group is defined by

$$\mathcal{H}^1_{mDJJ^\lambda}(\mathfrak{J}, V) = Z^1_{mDJJ^\lambda}(\mathfrak{J}, V),$$

where $Z^1_{mDJJ^\lambda}(\mathfrak{J}, V) = \{f \in \mathfrak{C}^1_{mDJJ^\lambda}(\mathfrak{J}, V) \mid \phi_1 f = 0\}$. The second cohomology group of a modified $\lambda$-differential algebra $(\mathfrak{J}, \cdot, d)$ is the quotient

$$\mathcal{H}^2_{mDJJ^\lambda}(\mathfrak{J}, V) = \frac{Z^2_{mDJJ^\lambda}(\mathfrak{J},V)}{B^2_{mDJJ^\lambda}(\mathfrak{J},V)},$$

where

$$Z^2_{mDJJ^\lambda}(\mathfrak{J}, V) = \{g \in \mathfrak{C}^2_{mDJJ^\lambda}(\mathfrak{J}, V) \mid \phi_2 g = 0\} \text{ and}$$
$$B^2_{mDJJ^\lambda}(\mathfrak{J}, V) = \{\phi_1 f \mid f \in \mathfrak{C}^1_{mDJJ^\lambda}(\mathfrak{J}, V)\}.$$

**Theorem 3.3.** *Let* $(\rho; V, d_V)$ *be a representation of* $(\mathfrak{J}, \cdot, d)$*. Then we have*

$$\mathcal{H}^1_{mDJJ^\lambda}(\mathfrak{J}, V) = \{f \in Z^1(\mathfrak{J}, V) \mid f \circ g = d_V \circ f\}.$$

*Proof.* For any $f \in \mathfrak{C}^1_{mDJJ^\lambda}(\mathfrak{J}, V)$, we have

$$\phi_1 f = (-\partial_1 f, -\delta f),$$

so f is closed if and only if $f \in Z^1(\mathfrak{J}, V)$, and satisfies $f \circ g = d_V \circ f$. Thus the conclusion holds. □

## 4. Deformations of modified $\lambda$-differential Jacobi–Jordan algebras

In this section, we introduce formal deformations of the modified $\lambda$-differential Jacobi–Jordan algebra. Furthermore, we show that if the second cohomology group $\mathcal{H}^2_{mDJJ^\lambda}(\mathfrak{J}, \mathfrak{J}) = 0$, then the modified $\lambda$-differential Jacobi–Jordan algebra $(\mathfrak{J}, \cdot, d)$ is rigid.

Let $(\mathfrak{J}, \cdot, d)$ be a modified $\lambda$-differential Jacobi–Jordan algebra. Denote by $\nu$ the multiplication of $\mathfrak{J}$, i.e., $\nu = \cdot$. Consider the 1-parameterized family

$$\nu_t = \sum_{i=0}^{\infty} \nu_i t^i, \ \ \nu_i \in \mathfrak{C}^2(\mathfrak{J}, \mathfrak{J}), \qquad d_t = \sum_{i=0}^{\infty} d_i t^i, \ \ d_i \in \mathfrak{C}^1(\mathfrak{J}, \mathfrak{J}).$$

**Definition 4.1.** A *1-parameter formal deformation* of the modified $\lambda$-differential Jacobi–Jordan algebra $(\mathfrak{J}, \nu, d)$ is a pair $(\nu_t, d_t)$ which endows the $\mathbb{K}[[t]]$-module $(\mathfrak{J}[[t]], \nu_t, d_t)$ with the modified $\lambda$-differential Jacobi–Jordan algebra over $\mathbb{K}[[t]]$ such that $(\nu_0, d_0) = (\nu, d)$.

Obviously, $(\mathfrak{J}[[t]], \nu_t = \nu, d_t = d)$ is a 1-parameter formal deformation of $(\mathfrak{J}, \nu, d)$. The pair $(\nu_t, d_t)$ generates a 1-parameter formal deformation of the modified $\lambda$-differential Jacobi–Jordan algebra $(\mathfrak{J}, \cdot, d)$ if and only if for all $x, y, z \in \mathfrak{J}$, the following equations hold:

$$\nu_t(x, \nu_t(y, z)) + \nu_t(y, \nu_t(z, x)) + \nu_t(z, \nu_t(x, y)) = 0, \tag{7}$$
$$d_t(\nu_t(x, y)) = \nu_t(d_t(x), y) + \nu_t(x, d_t(y)) + \lambda\nu_t(x, y). \tag{8}$$

Comparing the coefficients of $t^n$ on both sides, equations (7)–(8) are equivalent to the following equations:

$$\sum_{i+j=n} \nu_i(x, \nu_j(y, z)) + \nu_i(y, (\nu_j(z, x)) + \nu_i(z, \nu_j(x, y)) = 0, \tag{9}$$

$$\sum_{i+j=n} d_i(\nu_j(x, y)) = \sum_{i+j=n} (\nu_i(d_j(x), y) + \nu_i(x, d_j(y)) + \lambda\nu_n(x, y). \tag{10}$$

**Proposition 4.2.** *Let* $(\mathfrak{J}[[t]], \nu_t, d_t)$ *be a* 1*-parameter formal deformation of the modified* $\lambda$*-differential Jacobi–Jordan algebra* $(\mathfrak{J}, \nu, d)$*. Then* $(\nu_1, d_1)$ *is a* 2*-cocycle of* $(\mathfrak{J}, \nu, d)$ *with the coefficient in the adjoint representation.*

*Proof.* For $n = 1$, equation (9) can be written as

$$\nu_1(x, y \cdot z) + \nu_1(y, z \cdot x) + \nu_1(z, x \cdot y) + x \cdot \nu_1(y, z) + y \cdot \nu_1(z, x) + z \cdot \nu_1(x, y) = 0,$$

i.e., $\partial_2\nu_1 = 0$. In addition, for $n = 1$, equation (10) is equivalent to

$$\begin{aligned} & d_1(x \cdot y) + d(\nu_1(x, y)) \\ = & \, d_1(x) \cdot y + x \cdot d_1(y) + \nu_1(d(x), y) + \nu_1(x, d(y)) + \lambda\nu_1(x, y), \end{aligned}$$

that is, $\partial_1 d_1 - \delta\nu_1 = 0$. In other words, equations (9) and (10) are equivalent to $\phi_2(\nu_1, d_1) = (-\partial_2\nu_1, \partial_1 d_1 - \delta\nu_1) = 0$. Therefore, $(\nu_1, d_1)$ is a 2-cocycle. □

**Definition 4.3.** The 2-cocycle $(\nu_1, d_1)$ is called the *infinitesimal* of the 1-parameter formal deformation $(\mathfrak{J}[[t]], \nu_t, d_t)$ of $(\mathfrak{J}, \nu, d)$.

**Definition 4.4.** Let $(\mathfrak{J}[[t]], \nu_t, d_t)$ and $(\mathfrak{J}[[t]], \nu'_t, d'_t)$ be two 1-parameter formal deformations of $(\mathfrak{J}, \nu, d)$. A *formal isomorphism* from $(\mathfrak{J}[[t]], \nu'_t, d'_t)$ to $(\mathfrak{J}[[t]], \nu_t, d_t)$ is a power series $\varphi_t = \mathrm{id}_{\mathfrak{J}} + \sum_{i=1}^{\infty} \varphi_i t^i : (\mathfrak{J}[[t]], \nu'_t, d'_t) \to (\mathfrak{J}[[t]], \nu_t, d_t)$, where $\varphi_i \in \mathrm{End}(\mathfrak{J})$, such that

$$\varphi_t \circ \nu'_t = \nu_t \circ (\varphi_t \times \varphi_t), \tag{11}$$

$$\varphi_t \circ d'_t = d_t \circ \varphi_t. \tag{12}$$

Two 1-parameter formal deformations $(\mathfrak{J}[[t]], \nu_t, d_t)$ and $(\mathfrak{J}[[t]], \nu'_t, d'_t)$ are said to be *equivalent* if there exists a formal isomorphism $\varphi_t : (\mathfrak{J}[[t]], \nu'_t, d'_t) \to (\mathfrak{J}[[t]], \nu_t, d_t)$.

**Theorem 4.5.** *The infinitesimals of two equivalent* 1*-parameter formal deformations of* $(\mathfrak{J}, \nu, d)$ *are in the same cohomology class of* $\mathcal{H}^2_{mDJJ^\lambda}(\mathfrak{J}, \mathfrak{J})$.

*Proof.* Let $\varphi_t : (\mathfrak{J}[[t]], \nu'_t, d'_t) \to (\mathfrak{J}[[t]], \nu_t, d_t)$ be a formal isomorphism. For all $x, y \in \mathfrak{J}$, we have

$$\begin{aligned} \varphi_t \circ \nu'_t(x, y) &= \nu_t(\varphi_t(x), \varphi_t(y)), \\ \varphi_t \circ d'_t(x) &= d_t \circ \varphi_t(x). \end{aligned}$$

Comparing the coefficients of t on both sides of the above equations, we get

$$\begin{aligned} \nu'_1(x, y) - \nu_1(x, y) &= \varphi_1(x) \cdot y + x \cdot \varphi_1(y) - \varphi_1(x \cdot y), \\ d'_1(x) - d_1(x) &= d(\varphi_1(x)) - \varphi_1(d(x)). \end{aligned}$$

Therefore, we have

$$(\nu'_1, d'_1) = (\nu_1 - \partial_1\varphi_1, \varphi_1 - \delta\varphi_1) = (\nu_1, \varphi_1) + (\partial_1\varphi_1 - \delta\varphi_1) = (\nu_1, \varphi_1) + \alpha_1(\varphi_1),$$

which implies that $[(\nu'_1, d'_1)] = [(\nu_1, d_1)] \in \mathcal{H}^2_{\mathrm{mDJJ}^\lambda}(\mathfrak{J}, \mathfrak{J})$. □

**Definition 4.6.** A 1-parameter formal deformation $(\mathfrak{J}[[t]], \nu_t, d_t)$ of $(\mathfrak{J}, \nu, d)$ is said to be *trivial* if it is equivalent to the deformation $(\mathfrak{J}[[t]], \nu, d)$.

**Definition 4.7.** A modified $\lambda$-differential Jacobi–Jordan algebra $(\mathfrak{J}, \nu, d)$ is called *rigid* if every 1-parameter formal deformation is trivial.

**Theorem 4.8.** *If* $\mathcal{H}^2_{\mathrm{mDJJ}^\lambda}(\mathfrak{J}, \mathfrak{J}) = 0$*, then the modified $\lambda$-differential Jacobi–Jordan algebra* $(\mathfrak{J}, \nu, d)$ *is rigid.*

*Proof.* Let $(\mathfrak{J}[[t]], \nu_t, d_t)$ be a 1-parameter formal deformation of $(\mathfrak{J}, \nu, d)$. By Proposition 4.2, $(\nu_1, d_1)$ is a 2-cocycle. Since $\mathcal{H}^2_{\mathrm{mDJJ}^\lambda}(\mathfrak{J}, \mathfrak{J}) = 0$, there exists a 1-cochain $\varphi_1 \in \mathfrak{C}^1_{\mathrm{mDJJ}^\lambda}(\mathfrak{J}, \mathfrak{J})$ such that

$$(\nu_1, d_1) = -\phi_1(\varphi_1). \tag{13}$$

Then setting $\varphi_t = \mathrm{id}_{\mathfrak{J}} + \varphi_1 t$, we have a deformation $(\mathfrak{J}[[t]], \nu'_t, d'_t)$, where

$$\begin{aligned} \nu'_t &= \varphi_t^{-1} \circ \nu_t \circ (\varphi_t \otimes \varphi_t), \\ d'_t &= \varphi_t^{-1} \circ d_t \circ \varphi_t. \end{aligned}$$

Thus, $(\mathfrak{J}[[t]], \nu'_t, d'_t)$ is equivalent to $(\mathfrak{J}[[t]], \nu_t, d_t)$. Moreover, we have

$$\begin{aligned} \nu'_t &= (\mathrm{id}_{\mathfrak{J}} - \varphi_1 t + \varphi_1^2 t^2 + \cdots + (-1)^i \varphi_1^i t^i + \cdots) \circ \nu_t \circ ((\mathrm{id}_{\mathfrak{J}} + \varphi_1 t) \otimes (\mathrm{id}_{\mathfrak{J}} + \varphi_1 t)), \\ d'_t &= (\mathrm{id}_{\mathfrak{J}} - \varphi_1 t + \varphi_1^2 t^2 + \cdots + (-1)^i \varphi_1^i t^i + \cdots) \circ d_t \circ (\mathrm{id}_{\mathfrak{J}} + \varphi_1 t). \end{aligned}$$

By equality (13), we have

$$\begin{aligned} \nu'_t &= \nu + \nu'_2 t^2 + \cdots, \\ d'_t &= d + d'_2 t^2 + \cdots. \end{aligned}$$

Then, by repeating the argument, we can show that $(\mathfrak{J}[[t]], \nu_t, d_t)$ is equivalent to $(\mathfrak{J}[[t]], \nu, d)$. Therefore $(\mathfrak{J}, \nu, d)$ is rigid. □

## 5. Abelian extensions of modified $\lambda$-differential Jacobi–Jordan algebras

Extensions of Jacobi–Jordan algebra were studied in [2, 3]. In this section, we study abelian extensions of modified $\lambda$-differential Jacobi–Jordan algebras and show that they are classified by the second cohomology.

**Definition 5.1.** Let $(\mathfrak{J}, \cdot, \mathrm{d})$ and $(\mathrm{V}, \cdot_{\mathrm{V}}, \mathrm{d_V})$ be two modified $\lambda$-differential Jacobi–Jordan algebras. An *abelian extension* of $(\mathfrak{J}, \cdot, \mathrm{d})$ by $(\mathrm{V}, \cdot_{\mathrm{V}}, \mathrm{d_V})$ is a short exact sequence of homomorphisms of modified $\lambda$-differential Jacobi–Jordan algebras

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{\mathrm{i}} & \hat{\mathfrak{J}} & \xrightarrow{\mathrm{p}} & \mathfrak{J} & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{d}_V}\big\downarrow & & {\scriptstyle \hat{\mathrm{d}}}\big\downarrow & & {\scriptstyle \mathrm{d}}\big\downarrow & & \\ 0 & \longrightarrow & V & \xrightarrow{\mathrm{i}} & \hat{\mathfrak{J}} & \xrightarrow{\mathrm{p}} & \mathfrak{J} & \longrightarrow & 0 \end{array}$$

such that $\mathrm{V} \cdot_{\hat{\mathfrak{J}}} \mathrm{V} = 0$, i.e., V is an abelian ideal of $\hat{\mathfrak{J}}$.

**Definition 5.2.** Let $(\hat{\mathfrak{J}}_1, \cdot_{\hat{\mathfrak{J}}_1}, \hat{\mathrm{d}}_1)$ and $(\hat{\mathfrak{J}}_2, \cdot_{\hat{\mathfrak{J}}_2}, \hat{\mathrm{d}}_2)$ be two abelian extensions of $(\mathfrak{J}, \cdot, \mathrm{d})$ by $(\mathrm{V}, \cdot_{\mathrm{V}}, \mathrm{d_V})$. They are said to be *equivalent* if there is an isomorphism of modified $\lambda$-differential Jacobi–Jordan algebras $\zeta : (\hat{\mathfrak{J}}_1, \cdot_{\hat{\mathfrak{J}}_1}, \hat{\mathrm{d}}_1) \to (\hat{\mathfrak{J}}_2, \cdot_{\hat{\mathfrak{J}}_2}, \hat{\mathrm{d}}_2)$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (V, \mathrm{d}_V) & \xrightarrow{\mathrm{i}_1} & (\hat{\mathfrak{J}}_1, \hat{\mathrm{d}}_1) & \xrightarrow{\mathrm{p}_1} & (\mathfrak{J}, \mathrm{d}) & \longrightarrow & 0 \\ & & \Big\| & & {\scriptstyle \zeta}\big\downarrow & & \Big\| & & \\ 0 & \longrightarrow & (V, \mathrm{d}_V) & \xrightarrow{\mathrm{i}_2} & (\hat{\mathfrak{J}}_2, \hat{\mathrm{d}}_2) & \xrightarrow{\mathrm{p}_2} & (\mathfrak{J}, \mathrm{d}) & \longrightarrow & 0. \end{array}$$

A section of an abelian extension $(\hat{\mathfrak{J}}, \cdot_{\hat{\mathfrak{J}}}, \hat{\mathrm{d}})$ of $(\mathfrak{J}, \cdot, \mathrm{d})$ by $(\mathrm{V}, \cdot_{\mathrm{V}}, \mathrm{d_V})$ is a linear map $\sigma : \mathfrak{J} \to \hat{\mathfrak{J}}$ such that $\mathrm{p} \circ \sigma = \mathrm{id}_{\mathfrak{J}}$.

Now, for an abelian extension $(\hat{\mathfrak{J}}, \cdot_{\hat{\mathfrak{J}}}, \hat{\mathrm{d}})$ of $(\mathfrak{J}, \cdot, \mathrm{d})$ by $(\mathrm{V}, \cdot_{\mathrm{V}}, \mathrm{d_V})$ with a section $\sigma : \mathfrak{J} \to \hat{\mathfrak{J}}$, we define a linear map $\vartheta : \mathfrak{J} \to \mathrm{End(V)}$ by

$$\vartheta(\mathrm{x})\mathrm{u} := \sigma(\mathrm{x}) \cdot_{\hat{\mathfrak{J}}} \mathrm{u}, \quad \forall \mathrm{x} \in \mathfrak{J}, \mathrm{u} \in \mathrm{V}.$$

**Proposition 5.3.** *With the above notations,* $(\mathrm{V}, \vartheta, \mathrm{d_V})$ *is a representation over the modified $\lambda$-differential Jacobi–Jordan algebra* $(\mathfrak{J}, \cdot, \mathrm{d})$ *and it does not depend on the choice of the section $\sigma$.*

*Proof.* First, for any other section $\sigma' : \mathfrak{J} \to \hat{\mathfrak{J}}$, we have

$$\mathrm{p}(\sigma(\mathrm{x}) - \sigma'(\mathrm{x})) = \mathrm{p}(\sigma(\mathrm{x})) - \mathrm{p}(\sigma'(\mathrm{x})) = \mathrm{x} - \mathrm{x} = 0.$$

Thus, there is $\mathrm{u} \in \mathrm{V}$, such that $\sigma'(\mathrm{x}) = \sigma(\mathrm{x}) + \mathrm{u}$. Note that V is an abelian ideal of $\hat{\mathfrak{J}}$. This yields that

$$\sigma'(\mathrm{x}) \cdot_{\hat{\mathfrak{J}}} \mathrm{v} = (\sigma(\mathrm{x}) + \mathrm{u}) \cdot_{\hat{\mathfrak{J}}} \mathrm{v} = \sigma(\mathrm{x}) \cdot_{\hat{\mathfrak{J}}} \mathrm{v}.$$

This means that $\vartheta$ does not depend on the choice of the section $\sigma$.
Secondly, for any $\mathrm{x}, \mathrm{y} \in \mathfrak{J}$ and $\mathrm{u} \in \mathrm{V}$, from $(\sigma(\mathrm{x}) \cdot_{\hat{\mathfrak{J}}} \sigma(\mathrm{y})) - \sigma(\mathrm{x} \cdot \mathrm{y}) \in \mathrm{V} \cong \ker(\mathrm{p})$,

furthermore, by equations (1) and (2), we obtain

$$\begin{aligned}
& -\vartheta(x)\vartheta(y)u - \vartheta(y)\vartheta(x)u \\
= & -\sigma(x) \cdot_{\hat{\mathfrak{J}}} (\sigma(y) \cdot_{\hat{\mathfrak{J}}} u) - \sigma(y) \cdot_{\hat{\mathfrak{J}}} (\sigma(x) \cdot_{\hat{\mathfrak{J}}} u) \\
= & (\sigma(x) \cdot_{\hat{\mathfrak{J}}} \sigma(y)) \cdot_{\hat{\mathfrak{J}}} u \\
= & (\sigma(x) \cdot_{\hat{\mathfrak{J}}} \sigma(y) + \sigma(x \cdot y) - \sigma(x) \cdot_{\hat{\mathfrak{J}}} \sigma(y)) \cdot_{\hat{\mathfrak{J}}} u \\
= & \sigma(x \cdot y) \cdot_{\hat{\mathfrak{J}}} u \\
= & \vartheta(x \cdot y)u.
\end{aligned}$$

In addition, $\hat{d}(\sigma(x)) - \sigma(d(x)) \in V$, we have

$$\begin{aligned}
& d_V(\vartheta(x)u) \\
= & d_V(\sigma(x) \cdot_{\hat{\mathfrak{J}}} u) \\
= & d_V(\sigma(x)) \cdot_{\hat{\mathfrak{J}}} u + \sigma(x) \cdot_{\hat{\mathfrak{J}}} d_V(\sigma(u)) + \lambda\sigma(x) \cdot_{\hat{\mathfrak{J}}} u \\
= & \vartheta(d(x))u + \vartheta(x)d_V(u) + \lambda\vartheta(x)u,
\end{aligned}$$

Hence, $(V, \vartheta, d_V)$ is a representation over $(\mathfrak{J}, \cdot, d)$. □

We further define linear maps $\varsigma : \mathfrak{J} \times \mathfrak{J} \to V$ and $\varpi : \mathfrak{J} \to V$ by

$$\begin{aligned}
\varsigma(x, y) &= \sigma(x) \cdot_{\hat{\mathfrak{J}}} \sigma(y) - \sigma(x \cdot y), \\
\varpi(x) &= \hat{d}(\sigma(x)) - \sigma(d(x)), \quad \forall x, y \in \mathfrak{J}.
\end{aligned}$$

We transfer the modified $\lambda$-differential Jacobi–Jordan algebra structure on $\hat{\mathfrak{J}}$ to $\mathfrak{J} \oplus V$ by endowing $\mathfrak{J} \oplus V$ with a multiplication $\cdot_\varsigma$ and the modified $\lambda$-differential operator $d_\varpi$ defined by

$$(x + u) \cdot_\varsigma (y + v) = x \cdot y + \vartheta(x)v + \vartheta(y)u + \varsigma(x, y), \tag{14}$$

$$d_\varpi(x + u) = d(x) + \varpi(x) + d_V(u), \ \forall x, y \in \mathfrak{J}, \ u, v \in V. \tag{15}$$

**Proposition 5.4.** *The triple* $(\mathfrak{J} \oplus V, \cdot_\varsigma, d_\varpi)$ *is a modified* $\lambda$*-differential Jacobi–Jordan algebra if and only if* $(\varsigma, \varpi)$ *is a* 2*-cocycle of the modified* $\lambda$*-differential Jacobi–Jordan algebra* $(\mathfrak{J}, \cdot, d)$ *with the coefficient in* $(V; \theta, d_V)$.

*Proof.* The triple $(\mathfrak{J} \oplus V, \cdot_\varsigma, d_\varpi)$ is a modified $\lambda$-differential Jacobi–Jordan algebra if and only if

$$\varsigma(x, y) - \varsigma(y, x) = 0,$$

$$\vartheta(x)\varsigma(y, z) + \vartheta(y)\varsigma(z, x) + \vartheta(z)\varsigma(x, y) + \varsigma(x, y \cdot z) + \varsigma(y, z \cdot x) + \varsigma(z, x \cdot y) = 0 \tag{16}$$

$$\varpi(x \cdot y) + d_V(x \cdot y) = -\varsigma(x)(\varpi(y)) - \varsigma(y)\varpi(x) + \varsigma(d(x), y) + \varsigma(x, d(y)) + \lambda\varsigma(x, y), \tag{17}$$

for any $x, y \in \mathfrak{J}$. Using equations (16) and (17), we get $\partial_2\varsigma = 0$ and $\partial_1\varpi - \delta\varsigma = 0$, respectively. Therefore, $\phi_2(\varsigma, \varpi) = (-\partial_2\varsigma, \partial_1\varpi - \delta\varsigma) = 0$, that is, $(\varsigma, \varpi)$ is a 2-cocycle.

Conversely, if $(\varsigma, \varpi)$ satisfies equations (16) and (17), one can easily check that $(\mathfrak{J} \oplus V, \cdot_{\varsigma}, d_{\varpi})$ is a modified $\lambda$-differential Jacobi–Jordan algebra. □

Next we are ready to classify abelian extensions of a modified $\lambda$-differential Jacobi–Jordan algebra.

**Theorem 5.5.** *Abelian extensions of a modified $\lambda$-differential Jacobi–Jordan algebra $(\mathfrak{J}, \cdot, d)$ by $(V, \cdot_V, d_V)$ are classified by the second cohomology group $\mathcal{H}^2_{mDJJ^\lambda}(\mathfrak{J}, V)$ of $(\mathfrak{J}, \cdot, d)$ with coefficients in the representation $(V; \vartheta, d_V)$ constructed using the section $\sigma$. Moreover, its cohomology class does not depend on the choice of the section $\sigma$.*

*Proof.* Let $(\hat{\mathfrak{J}}, \cdot_{\hat{\mathfrak{J}}}, d_{\hat{\mathfrak{J}}})$ be an abelian extension of $(\mathfrak{J}, \cdot, d)$ by $(V, \cdot_V, d_V)$. We choose a section $\sigma : \mathfrak{J} \to \hat{\mathfrak{J}}$ to obtain a 2-cocycle $(\varsigma, \varpi)$ by Proposition 5.4. First, we show that the cohomology class of $(\varsigma, \varpi)$ is independent of the choice of $\sigma$. Let $\sigma_1, \sigma_2 : \mathfrak{J} \to \hat{\mathfrak{J}}$ be two distinct sections providing 2-cocycles $(\varsigma_1, \varpi_1)$ and $(\varsigma_2, \varpi_2)$ respectively. Define a linear map $\xi : \mathfrak{J} \to V$ by $\xi(x) = \sigma_1(a) - \sigma_2(x)$. Then

$$\begin{aligned}
\varsigma_1(x, y) &= \sigma_1(x) \cdot_{\hat{\mathfrak{J}}_1} \sigma_1(y) - \sigma_1(x \cdot y)\\
&= (\sigma_2(x) + \xi(x)) \cdot_{\hat{\mathfrak{J}}_1} (\sigma_2(y) + \xi(y)) - (\sigma_2(x \cdot y) + \xi(x \cdot y))\\
&= \sigma_2(x) \cdot_{\hat{\mathfrak{J}}_2} \sigma_2(y) - \vartheta(x)\xi(y) - \vartheta(y)\xi(x) - \sigma_2(x \cdot y) - \xi(x \cdot y)\\
&= (\sigma_2(x) \cdot_{\hat{\mathfrak{J}}_2} \sigma_2(y) - \sigma_2(x \cdot y)) - \vartheta(x)\xi(y) - \vartheta(y)\xi(x) - \xi(x \cdot y)\\
&= \varsigma_2(x, y) + d_1\xi(x, y)
\end{aligned}$$

and

$$\begin{aligned}
\varpi_1(a) &= \hat{d}(\sigma_1(x)) - \sigma_1(d(x))\\
&= \hat{d}(\sigma_2(x) + \xi(x)) - (\sigma_2(d(x)) + \xi(d(x)))\\
&= (\hat{d}(\sigma_2(x)) - \sigma_2(d(x))) + \hat{d}(\xi(x)) - \xi(d(x))\\
&= \varpi_2(x) + d_V(\xi(x)) - \xi(d(x))\\
&= \xi_2(a) - \delta\xi(x),
\end{aligned}$$

i.e., $(\varsigma_1, \varpi_1) - (\varsigma_2, \varpi_2) = \phi_1(\xi) \in \mathfrak{C}^2_{mDJJ^\lambda}(\mathfrak{J}, V)$. So $(\varsigma_1, \varpi_1)$ and $(\varsigma_2, \varpi_2)$ are in the same cohomology class in $\mathcal{H}^2_{mDJJ^\lambda}(\mathfrak{J}, V)$.

Next, assume that $(\hat{\mathfrak{J}}_1, \cdot_{\hat{\mathfrak{J}}_1}, \hat{d}_1)$ and $(\hat{\mathfrak{J}}_2, \cdot_{\hat{\mathfrak{J}}_2}, \hat{d}_2)$ are two equivalent abelian extensions of $(\mathfrak{J}, \cdot, d)$ by $(V, \cdot_V, d_V)$ with the associated isomorphism $\zeta : (\hat{\mathfrak{J}}_1, \cdot_{\hat{\mathfrak{J}}_1}, \hat{d}_1) \to (\hat{\mathfrak{J}}_2, \cdot_{\hat{\mathfrak{J}}_2}, \hat{d}_2)$. Let $\sigma_1$ be a section of $(\hat{\mathfrak{J}}_1, \cdot_{\hat{\mathfrak{J}}_1}, \hat{d}_1)$. As $p_2 \circ \zeta = p_1$, we get

$$p_2 \circ (\zeta \circ \sigma_1) = p_1 \circ \sigma_1 = id_{\mathfrak{J}}.$$

That is, $\zeta \circ \sigma_1$ is a section of $(\hat{\mathfrak{J}}_2, \cdot_{\hat{\mathfrak{J}}_2}, \hat{d}_2)$. Denote $\sigma_2 := \zeta \circ \sigma_1$. Since $\zeta$ is an isomorphism of modified $\lambda$-differential Jacobi–Jordan algebras such that $\zeta|_V = id_V$, we have

$$\begin{aligned}\varsigma_2(x,y) &= \sigma_2(x)\cdot_{\hat{\mathfrak{J}}_2}\sigma_2(y)-\sigma_2(x\cdot y)\\ &= (\zeta\circ\sigma_1)(x)\cdot_{\hat{\mathfrak{J}}_2}(\zeta\circ\sigma_1)(y)-(\zeta\circ\sigma_1)(x\cdot y)\\ &= \zeta(\sigma_1(x)\cdot_{\hat{\mathfrak{J}}_1}\sigma_1(y)-\sigma_1(x\cdot y))\\ &= \sigma_1(x)\cdot_{\hat{\mathfrak{J}}_1}\sigma_1(y)-\sigma_1(x,y)\\ &= \varsigma_1(x,y)\end{aligned}$$

and

$$\begin{aligned}\varpi_2(x) &= \hat{d}_2(\sigma_2(x))-\sigma_2(d(x)) = \hat{d}_2(\zeta(\sigma_1(x)))-\zeta(\sigma_1(d(x)))\\ &= \zeta(\hat{d}_1(\sigma_1(x))-\sigma_1(d(a)))\\ &= \hat{d}_1(\sigma_1(x))-\sigma_1 d(a)\\ &= \varpi_1(x).\end{aligned}$$

So, all equivalent abelian extensions give rise to the same element in $\mathcal{H}^2_{mDJJ^\lambda}(\mathfrak{J},V)$.

Conversely, given two cohomologous 2-cocycles $(\varsigma_1,\varpi_1)$ and $(\varsigma_2,\varpi_2)$ in $\mathcal{H}^2_{mDJJ^\lambda}(\mathfrak{J},V)$, we can construct two abelian extensions $(\mathfrak{J}\oplus V,\cdot_{\zeta_1},d_{\varpi_1})$ and $(\mathfrak{J}\oplus V,\cdot_{\zeta_2},d_{\varpi_2})$ via equations (14) and (15). Then, there is a linear map $\xi:\mathfrak{J}\to V$ such that

$$(\varsigma_1,\varpi_1) = (\varsigma_2,\varpi_2)+\phi_1(\xi).$$

Define a linear map $\zeta_\xi:\mathfrak{J}\oplus V\to\mathfrak{J}\oplus V$ by $\zeta_\xi(x+u) := x+\xi(x)+u,\ x\in\mathfrak{J}, u\in V$. Then, $\zeta_\xi$ is an isomorphism of these two abelian extensions. □

## 6. T*-extensions of modified λ-differential Jacobi–Jordan algebras

In this section, we consider T*-extensions of modified λ-differential Jacobi–Jordan algebras by the second cohomology groups with the coefficient in a dual adjoint representation.

Let $(\mathfrak{J},\cdot,d)$ be a modified λ-differential Jacobi–Jordan algebra and $\mathfrak{J}^*$ be the dual space of $\mathfrak{J}$. By Example 2.19, $(\mathfrak{J}^*;ad^*,-d^*)$ is a representation of $(\mathfrak{J},\cdot,d)$. Suppose that $(f,g)\in\mathfrak{C}^2_{mDJJ^\lambda}(\mathfrak{J},\mathfrak{J}^*)$. Define a trilinear map $\cdot_f:\otimes^2(\mathfrak{J}\oplus\mathfrak{J}^*)\to\mathfrak{J}\oplus\mathfrak{J}^*$ and a linear map $d_g:\mathfrak{J}\oplus\mathfrak{J}^*\to\mathfrak{J}\oplus\mathfrak{J}^*$ respectively by

$$(x+\alpha)\cdot_f(y+\beta) = x\cdot y+ad^*(x)\beta+ad^*(y)\alpha+f(x,y), \tag{18}$$

$$d_g(x+\alpha) = d(x)-d^*(\alpha)+g(x),\ \forall x,y\in\mathfrak{J},\ \alpha,\beta\in\mathfrak{J}^*. \tag{19}$$

Similarly to Proposition 2.18, we have the following result.

**Proposition 6.1.** *With the above notations,* $(\mathfrak{J}\oplus\mathfrak{J}^*,\cdot_f,d_g)$ *is a modified λ-differential Jacobi–Jordan algebra if and only if* $(f,g)$ *is a 2-cocycle in the cohomology of the modified λ-differential Jacobi–Jordan algebra* $(\mathfrak{J},\cdot,d)$ *with the coefficient in the representation* $(\mathfrak{J}^*;ad^*,-d^*)$.

**Definition 6.2.** The modified $\lambda$-differential Jacobi–Jordan algebra $(\mathfrak{J}\oplus\mathfrak{J}^*, \cdot_{\mathrm{f}}, \mathrm{d}_{\mathrm{g}})$ is called the $\mathrm{T}^*$-*extension* of the modified $\lambda$-differential Jacobi–Jordan algebra $(\mathfrak{J}, \cdot, \mathrm{d})$. Denote the $\mathrm{T}^*$-extension by $\mathrm{T}^*_{(\mathrm{f},\mathrm{g})}(\mathfrak{J}) = (\mathrm{T}^*(\mathfrak{J}) = \mathfrak{J}\oplus\mathfrak{J}^*, \cdot_{\mathrm{f}}, \mathrm{d}_{\mathrm{g}})$.

**Definition 6.3.** A modified $\lambda$-differential Jacobi–Jordan algebra $(\mathfrak{J}, \cdot, \mathrm{d})$ is called *metrised* if it has a non-degenerate symmetric bilinear form $\varpi_{\mathfrak{J}}$ satisfying

$$\varpi_{\mathfrak{J}}(\mathrm{x}, \mathrm{y}\cdot\mathrm{z}) - \varpi_{\mathfrak{J}}(\mathrm{x}\cdot\mathrm{y}, \mathrm{z}) = 0, \tag{20}$$

$$\varpi_{\mathfrak{J}}(\mathrm{d}(\mathrm{x}), \mathrm{y}) + \varpi_{\mathfrak{J}}(\mathrm{x}, \mathrm{d}(\mathrm{y})) = 0, \quad \forall \mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathfrak{J}. \tag{21}$$

We may also say that $(\mathfrak{J}, \cdot, \mathrm{d}, \varpi_{\mathfrak{J}})$ is a metrised modified $\lambda$-differential Jacobi–Jordan algebra.

Define a bilinear map $\varpi : \otimes^2\mathrm{T}^*(\mathfrak{J}) \to \mathfrak{J}$ by

$$\varpi(\mathrm{x}+\alpha, \mathrm{y}+\beta) = \alpha(\mathrm{y}) + \beta(\mathrm{x}), \quad \forall \mathrm{x}, \mathrm{y} \in \mathfrak{J}, \alpha, \beta \in \mathfrak{J}^*. \tag{22}$$

**Proposition 6.4.** *With the above notations,* $(\mathrm{T}^*_{(\mathrm{f},\mathrm{g})}(\mathfrak{J}), \varpi)$ *is a metrised modified $\lambda$-differential Jacobi–Jordan algebra if and only if*

$$\mathrm{f}(\mathrm{y},\mathrm{z})(\mathrm{x}) - \mathrm{f}(\mathrm{x},\mathrm{y})(\mathrm{z}) = 0, \quad \mathrm{g}(\mathrm{x})(\mathrm{y}) + \mathrm{g}(\mathrm{y})(\mathrm{x}) = 0, \quad \forall \mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathfrak{J}.$$

*Proof.* For any $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathfrak{J}$, $\alpha, \beta, \gamma \in \mathfrak{J}^*$, using equations (6), (18)-(22) we have

$$\begin{aligned}
&\varpi(\mathrm{x}+\alpha, (\mathrm{y}+\beta)\cdot_{\mathrm{f}}(\mathrm{z}+\gamma)) - \varpi((\mathrm{x}+\alpha)\cdot_{\mathrm{f}}(\mathrm{y}+\beta), \mathrm{z}+\gamma)\\
&= \varpi(\mathrm{x}+\alpha, \mathrm{y}\cdot\mathrm{z} + \mathrm{ad}^*(\mathrm{y})\gamma + \mathrm{ad}^*(\mathrm{z})\beta + \mathrm{f}(\mathrm{y},\mathrm{z}))\\
&\quad -\varpi(\mathrm{x}\cdot\mathrm{y} + \mathrm{ad}^*(\mathrm{x})\beta + \mathrm{ad}^*(\mathrm{y})\alpha + \mathrm{f}(\mathrm{x},\mathrm{y}), \mathrm{z}+\gamma)\\
&= \alpha(\mathrm{y}\cdot\mathrm{z}) + \mathrm{ad}^*(\mathrm{y})\gamma(\mathrm{x}) + \mathrm{ad}^*(\mathrm{z})\beta(\mathrm{x}) + \mathrm{f}(\mathrm{y},\mathrm{z})(\mathrm{x})\\
&\quad -\gamma(\mathrm{x}\cdot\mathrm{y}) - \mathrm{ad}^*(\mathrm{x})\beta(\mathrm{z}) - \mathrm{ad}^*(\mathrm{y})\alpha(\mathrm{z}) - \mathrm{f}(\mathrm{x},\mathrm{y})(\mathrm{z})\\
&= \alpha(\mathrm{y}\cdot\mathrm{z}) + \gamma(\mathrm{y}\cdot\mathrm{x}) + \beta(\mathrm{z}\cdot\mathrm{x}) + \mathrm{f}(\mathrm{y},\mathrm{z})(\mathrm{x})\\
&\quad -\gamma(\mathrm{x}\cdot\mathrm{y}) - \beta(\mathrm{x}\cdot\mathrm{z}) - \alpha(\mathrm{y}\cdot\mathrm{z}) - \mathrm{f}(\mathrm{x},\mathrm{y})(\mathrm{z})\\
&= \mathrm{f}(\mathrm{y},\mathrm{z})(\mathrm{x}) - \mathrm{f}(\mathrm{x},\mathrm{y})(\mathrm{z})\\
&= 0,\\
&\varpi(\mathrm{d}_{\mathrm{g}}(\mathrm{x}+\alpha), \mathrm{y}+\beta) + \varpi(\mathrm{x}+\alpha, \mathrm{d}_{\mathrm{g}}(\mathrm{y}+\beta))\\
&= \varpi(\mathrm{d}(\mathrm{x}) - \mathrm{d}^*(\alpha) + \mathrm{g}(\mathrm{x}), \mathrm{y}+\beta) + \varpi(\mathrm{x}+\alpha, \mathrm{d}(\mathrm{y}) - \mathrm{d}^*(\beta) + \mathrm{g}(\mathrm{y}))\\
&= -\mathrm{d}^*(\alpha)(y) + g(x)(y) + \beta(\mathrm{d}(x)) + \alpha(\mathrm{d}(y)) - \mathrm{d}^*(\beta)(x) + g(y)(x)\\
&= -\alpha(\mathrm{d}(\mathrm{y})) + \mathrm{g}(\mathrm{x})(\mathrm{y}) + \beta(\mathrm{d}(\mathrm{x})) + \alpha(\mathrm{d}(\mathrm{y})) - \beta(\mathrm{d}(\mathrm{x})) + \mathrm{g}(\mathrm{y})(\mathrm{x})\\
&= \mathrm{g}(\mathrm{x})(\mathrm{y}) + \mathrm{g}(\mathrm{y})(\mathrm{x})\\
&= 0.
\end{aligned}$$

Thus, we obtain the result. □

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