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On the algebraic property of locally convex topological algebras

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Abstract. By a Fréchet algebra, we mean a complete metrizable locally convex topological algebra. The boundedness of a set in Fréchet algebras is of course a topological property, but the uniform bound of a uniformly bounded set is an algebraic property, since it depends on the choice of seminorms generating the same topology on Fréchet algebra. In this paper, we show that if S is a bounded subsemigroup of a Fréchet algebra $(A, (p_n)_{n\in\mathbb{N}})$, then there is an equivalent family of seminorms $(t_n)_{n\in\mathbb{N}}$ on A, such that $t_n(s) \leq 1$ $(s \in S, n \in \mathbb{N})$. In the rest of this paper, we get a result by using this fact, and we also have a discussion on continuous inverse algebras.

1. Introduction

By a topological algebra, we mean an associative algebra A over the field $\mathbb{C}$ of complex numbers, for which the ring multiplication in A is separately continuous. In 1939, the notion of topological algebras had their first foundation in the presented work by Gelfand on normed rings. By development of topological rings and topological vector spaces, the theory of normed and Banach algebras got increasingly important and research in general topological algebras became unavoidable. It is important to explore how far we can extend beyond normed and Banach algebras while retaining their distinctive features. After studying general topological algebras, exactly by introduction of locally multiplicatively convex topological algebras by R. Arens and independently by E. A. Michael, the necessity for such an extension has been clear [9, 10].

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A topological vector space is said to be normable if and only if its origin has a convex bounded neighborhood [13]. This concept leads to generalization of normed algebras: locally bounded algebras and locally convex algebras.

A locally bounded topological algebra which is a topological algebra whose underlying topological vector space is a locally bounded space, was introduced in the early 60's, and regularly studied by Żelazko [15]. If A is a locally bounded algebra, then the following assertions are equivalent:

(i) A has jointly continuous multiplication;
(ii) A is an α-normed algebra, with $0 < \alpha \leq 1$.

It is proved that its topology can be generated by a submultiplicative α-norm if A is also complete [11, 15].

A locally convex algebra is a topological algebra A, whose underlying space is locally convex. If A is a complete metrizable locally convex topological algebra, since the multiplication will be jointly continuous, its topology can be generated by a family of seminorms $(p_n)_{n\in\mathbb{N}}$ such that

$$p_n(x) \leq p_{n+1}(x), \quad p_n(xy) \leq p_{n+1}(x)p_{n+1}(y) \quad \forall x, y \in A. \tag{1}$$

For more details, see [17, 16, 15].

In a topological algebra A, a seminorm p is said to be submultiplicative if, for every $x, y \in A$, we have

$$p(xy) \leq p(x)p(y).$$

By a locally multiplicatively convex topological algebra (abbreviated by locally m-convex or LMC-algebra), we mean a topological algebra A for which its topology can be generated by a separating family of submultiplicative seminorms. It is clear that every LMC-algebra has jointly continuous multiplication. A well known characterization for locally convex topological spaces is given in Theorem 1.37 of [13].

The concepts of locally convex topological algebras and LMC-algebras are so close. Obviously, every LMC-algebra is a locally convex topological algebra with jointly continuous multiplication. Moreover, we can find some more topological algebras that become an LMC-algebra, for example see Chapter 2, Section 5, of [10].

We are interested in investigating which theorems from a complete metrizable LMC-algebra can be generalized to a locally convex algebra by applying condition (1). Some properties of a complete metrizable locally convex topological algebra change by choosing a suitable equivalent family of seminorms, generating the same topology. These properties are not, in fact, topological properties but algebraic properties. In our main result, Theorem 2, we extend an algebraic property of LMC-algebras for complete metrizable locally convex topological algebras.

In Section 3, we have a discussion on the property of continuous inverse algebras in complete metrizable locally convex topological algebras with unit elements. Continuous inverse algebras have some important applications in functional analysis, operator theory, and quantum physics.

We recall that, the unital algebra A is said to have a continuous inversion property, if the map $x \to x^{-1}$ is continuous on a neighborhood of the unit element e of A. A unital topological algebra A with jointly continuous multiplication, which has the continuous inversion property, is called a continuous inverse algebra.

In Theorem 4, we observe that, if there exists a unital commutative complete metrizable locally convex topological algebra A which is not an LMC-algebra, then the group G of invertible elements in this algebra is not open and, if the unit element of A is an interior point of G, then G is not closed. Next, in Theorem 5, we give a condition on a unital Fréchet algebra A for which the inverse map is continuous on G and then we achieve a nice corollary.

In the final section, we close our discussion by some more notes on Fréchet algebras.

2. Bounded subsemigroups in locally convex algebras

In this section, at first we give a simple characterization of a metrizable LMC-algebra, and then we find a suitable bound for a bounded subsemigroup of a complete metrizable locally convex topological algebra.

Theorem 1. *Let $(A, (p_n)_{n\in\mathbb{N}})$ be a metrizable LMC-algebra. Then there exists a family $(q_n)_{n\in\mathbb{N}}$ of submultiplicative seminorms, generating the same topology on A, such that*

$$q_n(x) \leq q_{n+1}(x), \quad q_n(xy) \leq q_{n+1}(x)q_{n+1}(y) \quad \forall x, y \in A \text{ and } n \in \mathbb{N}.$$

Proof. Define

$$q_1(x) = p_1(x),$$
$$q_n(x) = \max\{q_{n-1}(x), p_n(x)\} \text{ for } n \geq 2,$$

for each $x \in A$. It is obvious that $(q_n)_{n\in\mathbb{N}}$ is a separating family of seminorms on A, so $(A, (q_n))$ is a metrizable locally convex topological algebra. Now, we prove that $(A, (q_n))$ is an LMC-algebra. It is easy to check that $q_n(x) \leq q_{n+1}(x)$ for all $n \in \mathbb{N}$. Obviously $q_1(xy) \leq q_1(x)q_1(y)$. Suppose $q_n(xy) \leq q_n(x)q_n(y)$. Then we have:

$$\begin{aligned} q_{n+1}(xy) &= \max\{q_n(xy), p_{n+1}(xy)\} \\ &\leq \max\{q_n(x)q_n(y), p_{n+1}(x)p_{n+1}(y)\} \\ &\leq q_{n+1}(x)q_{n+1}(y). \end{aligned}$$

Now it is clear that $q_n(xy) \leq q_{n+1}(x)q_{n+1}(y)$, for every $x, y \in A$ and $n \in \mathbb{N}$. Let τ_p and τ_q be the topologies generated by the families of (p_n) and (q_n) respectively. It is obvious that $\tau_q \subseteq \tau_p$.

By induction, we show that $\tau_p \subseteq \tau_q$. We have

$$V\big(p_1,\epsilon\big) = \{x \in A : p_1(x) \leq \epsilon\} = V\big(q_1,\epsilon\big),$$
$$V\big(p_1,\epsilon\big) \bigcap V\big(p_2,\epsilon\big) \subset V\big(q_2,\epsilon\big).$$

By the induction hypothesis, suppose $V\big(p_1,\epsilon\big) \bigcap ... \bigcap V\big(p_n,\epsilon\big) \subset V\big(q_n,\epsilon\big)$. Let $x \in V\big(p_1,\epsilon\big) \bigcap ... \bigcap V\big(p_n,\epsilon\big) \bigcap V\big(p_{n+1},\epsilon\big)$. Thus

$$x \in V\big(p_1,\epsilon\big) \bigcap ... \bigcap V\big(p_n,\epsilon\big), x \in V\big(p_{n+1},\epsilon\big).$$

So $x \in V\big(q_{n+1},\epsilon\big)$, i.e. $\tau_p \subseteq \tau_q$. □

In [6], (Theorem 1, Section 4, Chapter I), Bonsall and Duncan prove that if S is a bounded subsemigroup of a normed algebra $(A, \|.\|)$, then there is an algebra-norm p on A, equivalent to $\|.\|$ such that $p(s) \leq 1$ $(s \in S)$. In [2], Ansari-Piri proves the same fact for LMC-algebras: Let A be an LMC-algebra with a family $(p_\alpha)_{\alpha \in I}$ of submultiplicative seminorms, generating the topology on A, and let $S \subseteq A$ be a bounded subsemigroup. Then there exists an equivalent family $(q_\alpha)_{\alpha\in I}$ of submultiplicative seminorms, generating the same topology on A, with $q_\alpha(s) \leq 1$ $(\alpha \in I, s \in S)$.

Now, we prove this theorem for a complete metrizable locally convex topological algebra (not necessarily LMC-algebra). In the next theorem, we discuss the uniform bound of a bounded subsemigroup of a Fréchet algebra.

Definition 1. Let A be a complete metrizable locally convex topological algebra. Then $S \subseteq A$ is *uniformly bounded* if its topology can be generated by a sequence $(p_n)_{n\in\mathbb{N}}$ of seminorms and there exists $M > 0$ such that

$$\forall n \in \mathbb{N}, \forall s \in S \quad p_n(s) \leq M.$$

The definition for uniformly power bounded sets in general topological algebras is given in [2].

Theorem 2. *Let $(A,(p_n)_{n\in\mathbb{N}})$ be a complete metrizable locally convex topological algebra and S be a bounded subsemigroup of A. It is possible to choose a sequence of seminorms $(t_n)_{n\in\mathbb{N}}$ generating the same topology such that $(A,(t_n)_{n\in\mathbb{N}})$ be a complete locally convex one and S become a uniformly bounded subsemigroup, namely $t_n(s) \leq 1$ for all $n \in \mathbb{N}$ and all $s \in S$.*

Proof. Without loss of generality, we suppose $e \in S \subseteq A$. Since S is a bounded subsemigroup, we have

$$\forall n \in \mathbb{N}, \exists M_n, \forall s \in S, \quad p_n(s) \leq M_n.$$

As we mentioned in (1), the topology of A can be generated by a family (p_n) of seminorms with two conditions $p_n(x) \leq p_{n+1}(x)$ and $p_n(xy) \leq p_{n+1}(x)p_{n+1}(y)$ for each $x, y \in A$ and $n \in \mathbb{N}$.

Define

$$r_n(x) = \sup\{p_n(sx) : s \in S\}.$$

Then (r_n) is a seminorm on A and the inequalities $r_n(x) \leq r_{n+1}(x)$ and $r_n(xy) \leq r_{n+1}(x)r_{n+1}(y)$ hold, because $e \in S$. We prove that the topology, generated by (r_n) is equivalent to the topology, generated by (p_n). We know that $p_n(sx) \leq r_n(x)$ for each $s \in S$. Since $e \in S$, $p_n(x) \leq r_n(x)$ and $p_n(sx) \leq p_{n+1}(s)p_{n+1}(x) \leq M_{n+1}p_{n+1}(x)$ for all $s \in S$. Thus $r_n(x) \leq M_{n+1}p_{n+1}(x)$ for each $x \in X$. Therefore $p_n(x) \leq r_n(x) \leq M_{n+1}p_{n+1}(x)$ for each $x \in X$.

Now, suppose that $(a_i)_{i\in\mathbb{N}}$ is a Cauchy sequence with respect to $(A,(r_n))$, so

$$\forall n \in \mathbb{N}, \forall \epsilon > 0, \exists N, \forall i,j \in \mathbb{N} \quad i > j > N \quad r_n(a_i - a_j) < \epsilon.$$

Thus $p_n(a_i - a_j) < \epsilon$ and therefore (a_i) is a Cauchy sequence with respect to $(A,(p_n))$. Now, since $(A,(p_n))$ is complete, there is $a \in A$ such that $a_i \to a$ in $(A,(p_n))$. Take arbitrary elements $n \in \mathbb{N}$ and $\epsilon > 0$. Then

$$\exists N_0, i > N_0 \Rightarrow a_i - a \in V\big(p_{n+1}, \frac{\epsilon}{M_{n+1}}\big).$$

So we have that

$$r_n(a_i - a) \leq M_{n+1}p_{n+1}(a_i - a) < M_{n+1}\frac{\epsilon}{M_{n+1}} = \epsilon.$$

This means that $(A,(r_n))$ is complete.

Define

$$t_n(x) = \sup\{r_n(xb) : b \in A, r_{n+1}(b) \leq 1\}. \tag{2}$$

Then (t_n) is a seminorm on A. We prove that the topology, generated by (t_n), is equivalent to the topology, generated by (r_n). Let $b \in A$ and $r_{n+1}(b) \leq 1$. Then

$$r_n(xb) \leq r_{n+1}(x)r_{n+1}(b) \leq r_{n+1}(x),$$

therefore $t_n(x) \leq r_{n+1}(x)$.

If $r_{n+1}(e) = 0$, then $r_n(x) \leq r_{n+1}(e)r_{n+1}(x) = 0 \leq t_n(x)$.

If $r_{n+1}(e) \neq 0$, then

$$r_n(x) = r_n\big(x\frac{e}{r_{n+1}(e)}\big)r_{n+1}(e) \leq t_n(x)r_{n+1}(e).$$

Put $K_n = max\{1, r_{n+1}(e)\}$. We have that $r_n(x) \leq K_n t_n(x)$. Suppose (x_i) is a Cauchy sequence with respect to $(A,(t_n))$. Then

$$\forall n \in \mathbb{N}, \forall \epsilon > 0, \exists N_0, \forall i,j \in \mathbb{N} \quad i,j > N_0 \quad x_i - x_j \in V\big(t_n, \frac{\epsilon}{K_n}\big).$$

Thus $r_n(x_i - x_j) \leq K_n\frac{\epsilon}{K_n} = \epsilon$ and therefore (x_i) is a Cauchy sequence with respect to $(A,(r_n))$. Now, since $(A,(r_n))$ is complete, there is $x \in A$ such that $x_i \to x$ in $(A,(r_n))$.

Take arbitrary elements $n \in \mathbb{N}$ and $\epsilon > 0$. Then

$$\exists N_0, i > N_0 \Rightarrow \quad x_i - x \in V\big(r_{n+1}, \epsilon\big),$$

so $t_n(x_i - x) \leq r_{n+1}(x_i - x) \leq \epsilon$. Thus $(A, (t_n))$ is complete. Let $s \in S, b \in A$ and $r_{n+1}(b) \leq 1$. Then (2) holds and

$$\begin{aligned} r_n(sb) &= \sup\{p_n(z(sb)) : z \in S\} \\ &\leq \sup\{p_n(wb) : w \in S\} = r_n(b) \leq r_{n+1}(b). \end{aligned}$$

Therefore $t_n(s) \leq 1$ for each $s \in S$. □

In [7], Theorem 7.3, Dixon proves that in a commutative normed algebra $(A, \|.\|)$ with a bounded approximate identity $(e_n)_{n\in\mathbb{N}}$, there is an equivalent algebra norm on A for which there is an approximate identity $(f_n)_{n\in\mathbb{N}}$ of norm 1.

In [2], Theorem 5.6, Ansari-Piri proves that in a locally convex topological algebra A with a semiaccurate uniformly power bounded abelian approximate identity $(e_n)_{n\in\mathbb{N}}$ with a uniform bound $K > 0$, there exists an approximate identity $(f_n)_{n\in\mathbb{N}}$ of the same type of uniform bound $1 + \xi$ for each $\xi > 0$.

Here we give a better result for complete metrizable locally convex topological algebras. First, we recall a definition and a lemma.

Definition 2. An approximate identity $(e_\lambda)_{\lambda\in\Lambda}$ in a topological algebra A is said to be *semiaccurate* if for each $\lambda \in \Lambda$ there exists $\mu(\lambda) \in \Lambda$ such that $\gamma \in \Lambda$ with $\gamma \geq \mu(\lambda)$ implies $e_\gamma e_\lambda = e_\lambda$.

Lemma 1 (see [2]). *Let A be a topological algebra with a semiaccurate sequential approximate identity $(e_n)_{n\in\mathbb{N}}$. Then A has a semiaccurate sequential approximate identity $(f_n)_{n\in\mathbb{N}}$ such that $f_j f_i = f_i$ for all $i, j \in \mathbb{N}$ with $i < j$.*

Theorem 3. *Let $(A, (p_n)_{n\in\mathbb{N}})$ be a complete metrizable locally convex topological algebra and $(e_k)_{k\in\mathbb{N}}$ be a bounded semiaccurate abelian approximate identity. Then there is an equivalent family of seminorms $(t_n)_{n\in\mathbb{N}}$ and there is a uniformly bounded semiaccurate approximate identity $(f_k)_{k\in\mathbb{N}}$ such that $t_n(f_k) \leq 1$ for each k, n.*

Proof. By Lemma 1, there is a bounded semiaccurate abelian approximate identity (f_k) such that $f_j f_i = f_i$ for each $i, j \in \mathbb{N}$ with $i < j$. Put $S = \{f_1, f_2, ...\}$. Then S is a bounded subsemigroup of A. Now by Theorem 2, there is an equivalent family of seminorms (t_n) such that $t_n(s) \leq 1$ for each $s \in S$. □

3. Continuous inverse algebras

In this section, we explore the characteristics of continuous inverse algebras within complete metrizable locally convex topological algebras that include a unit element.

Theorem 4. *Let A be a unital commutative complete metrizable locally convex topological algebra which is not an LMC-algebra. Then the group G of invertible elements of A is not open, and if the unit element of A is an interior point of G, then G is not closed.*

Proof. Suppose G is the set of invertible elements of A. In [14], Chapter VIII, page 115, it is proved that every complete metrizable Q-algebra is a continuous inverse algebra, where by Q-algebra we mean an algebra whose set of invertible elements is open. So if G is open, then A is a continuous inverse algebra. In [14], Chapter VIII, Proposition 3, Turpin proves that every commutative locally convex continuous inverse algebra is an LMC-algebra. Therefore G cannot be an open set.

Assume G is closed. According to the continuity of multiplication in A, the maps

$$l_x : a \to xa, r_x : a \to ax \quad x, a \in A$$

are continuous. Since $G \subseteq A$, l_a and r_a are also continuous for each $a \in G$. Now, since G is closed and the induced topology from A on G is complete, by Proposition 2.1.8 in [5], the map $x \to x^{-1}$ is continuous on G, hence is continuous on a neighborhood of unit element $e \in A$, i.e. A is a continuous inverse algebra. Therefore, by Proposition 3, Chapter VIII in [14], A is an LMC-algebra which is a contradiction. □

Theorem 5. *Let A be a unital complete metrizable locally convex topological algebra and G be the group of invertible elements. If G is a bounded set, then the inverse map is continuous on G.*

Proof. Suppose that the topology of A is generated by a family $(p_i)_{i\in\mathbb{N}}$ of seminorms satisfying condition (1). Since G is bounded,

$$\forall i \in \mathbb{N}, \exists M_i, \forall g \in G, \quad p_i(g) \leq M_i.$$

If $a, b \in G$, then

$$p_i(b^{-1} - a^{-1}) \leq p_{i+2}(b^{-1})p_{i+2}(a - b)p_{i+2}(a^{-1}).$$

Let $V\big(p_i, \epsilon\big)$ be an arbitrary neighbourhood of zero. If $a - b \in V\big(p_{i+2}, \frac{\epsilon}{M_{i+2}^2}\big)$, then $p_i(b^{-1} - a^{-1}) < \epsilon$. □

Let A be a locally convex topological algebra with a unit element and the inverse map be continuous on the set of invertible elements of A. In [11], page 58, it is proved that the spectrum $Sp_A(a)$ of every element $a \in A$ is a non-empty subset of $\mathbb{C}$.

Corollary 1. *Let A be a unital complete metrizable locally convex topological algebra for which its group G of invertible elements is bounded. Then $Sp_A(a)$ is non-empty for each $a \in A$.*

4. Some more notes on Fréchet algebras

As we mentioned, every LMC-algebra is a locally convex topological algebra with jointly continuous multiplication.

Consider $L^\omega = \bigcap_{n\in\mathbb{N}} L^n[0,1]$, which consists of all equivalent classes of complex valued functions on the interval $[0,1]$. With a topology generated by a sequence of seminorms (p_n) such that

$$p_n(f) = \Big(\int_0^1 |f(t)|^n\Big)^{\frac{1}{n}}, \quad f \in L^n[0,1],$$

L^ω is a complete metrizable locally convex topological algebra for which the map $x \to x^{-1}$ is not continuous at unit element e, see [9], page 79, and hence is not continuous on a neighborhood of the unit element e of L^ω, namely L^ω has not the continuous inversion property, for more details see [4, 9, 10, 14, 17].

Since every LMC-algebra has the continuous inversion property, and the continuity is a topological property, by no equivalent sequence of seminorms, generated the same topology, it can be an LMC-algebra.

There are some other properties for LMC-algebras which may not hold for locally convex topological algebras. Here we recall two questions, which we believe to be open.

1. Does the series $\Sigma\frac{x^n}{n!}$ converge for each element x in a complete metrizable locally convex topological algebra?

2. Is it possible to choose the sequence of seminorms for a unital complete metrizable locally convex topological algebra $(A,(p_n))$ (with the unit element e) such that $p_n(e) = 1$? In [16], this question stands as an open problem, for a unital locally convex topological algebra.

In Theorem 2, if A has a unit element e, we have proved that it is possible to choose the sequence (p_n) such that $p_n(e) \leq 1$, for all $n \in \mathbb{N}$. If for each $n \in \mathbb{N}$, we suppose $p_n(e) \geq 1$, is it possible to choose (t_n) generating the same topology with $t_n(e) = 1$?

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