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From Yamabe to almost contact metric structure

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Abstract. The investigation of Yamabe solitons within almost contact metric manifolds has garnered significant interest recently, producing notable findings. This paper aims to explore the inverse problem: constructing almost contact metric structures on a three-dimensional Riemannian manifold endowed with an almost Yamabe soliton. Subsequently, we provide the techniques required to characterize the nature of these structures, accompanied by concrete examples.

1. Introduction

A connected Riemannian manifold (M^n, g), where $n > 2$, is deemed to be a Yamabe soliton if it possesses a vector field U satisfying the condition

$$(\mathcal{L}_U g) + 2(r - \lambda)g = 0, \tag{1}$$

where $(\mathcal{L}_U g)$ denotes the Lie derivative of the metric g with respect to the vector field U and λ is a real number [4]. The classification of such solitons depends on the behavior of λ. A Yamabe soliton is categorized as shrinking, steady or expanding, depending on whether $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$, respectively.

If λ is a smooth function on M, the metric g satisfying equation (1) is termed an almost Yamabe soliton. Moreover, if the vector field U is a Killing vector field, meaning $\mathcal{L}_U g = 0$, the soliton is referred to as trivial.

The study of Yamabe solitons has been conducted on 3-dimensional Sasakian, Kenmotsu, and cosymplectic manifolds, as detailed in [7, 8, 9]. These manifolds are normal; however, there is ongoing research on Yamabe solitons on non-normal almost contact metric manifolds [2].

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Considering these studies, an intriguing question arises: can we explore the reverse scenario, namely, the construction of almost contact metric structures starting from an almost Yamabe soliton?

This paper aims to investigate alternative methods for constructing almost contact metric structures on three-dimensional Riemannian manifolds equipped with an almost Yamabe soliton. Our results are divided into two parts. Firstly, we construct almost contact metric structures solely from a unit vector field, and secondly, we analyze the characteristics of these structures in the presence of an almost Yamabe soliton on a Riemannian manifold.

2. Preliminaries

An almost contact structure on an odd-dimensional Riemannian manifold (M^{2n+1}, g) is denoted by the triple (φ, ξ, η), where φ is a tensor field of type $(1, 1)$, ξ is a vector field, and η is a differential 1-form. These components are subject to the following conditions:

$$\begin{cases} \eta(\xi) = 1, \\ \varphi^2 X = -X + \eta(X)\xi, \\ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \end{cases} \tag{2}$$

for any vector field X, Y on M. In particular, in an almost contact metric manifold we also have

$$\varphi\xi = 0 \quad \text{and} \quad \eta \circ \varphi = 0. \tag{3}$$

We call the data $(M^{2n+1}, \varphi, \xi, \eta, g)$ an *almost contact metric manifold*. We associate to this later a fundamental 2-form Φ, expressed by

$$\Phi(X, Y) = g(X, \varphi Y).$$

We say that (φ, ξ, η, g) is *normal* if

$$N^{(1)} = N_\varphi + 2\mathrm{d}\eta \otimes \xi = 0,$$

and is *integrable* if

$$N_\varphi = 0,$$

where N_φ is the Nijenhuis tensor, expressed by

$$N_\varphi(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y].$$

In [5], the author proves that (φ, ξ, η, g) is a trans-Sasakian structure of type (α, β) if and only if it is normal and

$$\mathrm{d}\eta = \alpha\Phi \quad \text{and} \quad \mathrm{d}\Phi = 2\beta\eta \wedge \Phi, \tag{4}$$

where d denotes the exterior derivative and α, β are smooth functions defined by

$$\alpha = \frac{1}{2n}\delta\Phi(\xi) \quad \text{and} \quad \beta = \frac{1}{2n}\mathrm{div}\xi,$$

where δ is the co-differential of g. It is well known that the trans-Sasakian condition may be expressed as an almost contact metric structure satisfying

$$\big(\nabla_X\varphi\big)Y = \alpha\big(g(X,Y)\xi - \eta(Y)X\big) + \beta\big(g(\varphi X,Y)\xi - \eta(Y)\varphi X\big).$$

In this paper, we investigate 3-dimensional almost contact metric manifolds. Olszak [6] demonstrated that an almost contact metric structure is trans-Sasakian of type (α,β) if and only if

$$\nabla_X\xi = -\alpha\varphi X - \beta\varphi^2 X, \tag{5}$$

in conformity with

$$\nabla_{\varphi X}\xi = \varphi\nabla_X\xi. \tag{6}$$

Clearly,

$$\begin{cases} (1) : \alpha - \mathit{Sasakian} \Leftrightarrow \text{trans-Sasakian of type } (\alpha, 0), \\ (2) : \beta - \mathit{Kenmotsu} \Leftrightarrow \text{trans-Sasakian of type } (0, \beta), \\ (3) : \mathit{cosymplectic} \Leftrightarrow \text{trans-Sasakian of type } (0, 0). \end{cases}$$

A trans-Sasakian manifold of type (α,β) is designated as a proper trans-Sasakian manifold when both functions α and β are non-zero.

Another significant class of almost contact metric structures has been investigated recently in [1]. This class, referred to as non-normal integrable almost contact metric structures, is termed the *generalized C_{12}-structure*. The authors demonstrate that (φ,ξ,η,g) qualifies as a generalized C_{12}-structure if and only if it is integrable, i.e., $N_\varphi = 0$ and

$$\mathrm{d}\eta = \eta\wedge\omega \quad \text{and} \quad \mathrm{d}\Phi = 2\beta\eta\wedge\Phi,$$

where $\omega = \nabla_\xi\eta$ and $\omega^\sharp = \psi = \nabla_\xi\xi$, that is, $\omega(X) = g(\psi, X)$ for all vector field X on M. It is well known that the generalized-C_{12} condition can be expressed as an almost contact metric structure satisfying

$$(\nabla_X\varphi)Y = \beta\big(g(\varphi X,Y)\xi - \eta(Y)\varphi X\big) - \eta(X)\big(\omega(\varphi Y)\xi + \eta(Y)\varphi\psi\big).$$

In the three-dimensional case, [1] provides two elegant characterizations of generalized C_{12}-manifolds.

Theorem 1. *Let (M^3,φ,ξ,η,g) be a 3-dimensional almost contact metric manifold. M is a generalized C_{12}-manifold if and only if*

$$\nabla_X\xi = -\beta\varphi^2X + \eta(X)\psi,$$

where $\psi = \nabla_\xi\xi$. Or, equivalently,

$$\nabla_{\varphi X}\xi = \beta\varphi X.$$

Typically, the vector field ψ is not normalized. Hence, for 3-dimensional generalized-C_{12} manifolds, we introduce $V = e^{-\rho}\psi$, where $e^\rho = \|\psi\|$. This formulation readily establishes $\{\xi, V, \varphi V\}$ as an orthonormal frame. This set is commonly referred to as the "fundamental basis".

As per [1], the components of the Levi-Civita connection, associated with the fundamental basis, are presented as follows.

Theorem 2. *For any three-dimensional generalized-C_{12} manifold, the covariant derivatives of the fundamental basis $\{\xi, V, \varphi V\}$ are given by:*

(1) $\nabla_\xi \xi = e^\rho V$,
(2) $\nabla_V \xi = \beta V$,
(3) $\nabla_{\varphi V} \xi = \beta \varphi V$,
(4) $\nabla_\xi V = -e^\rho \xi + \kappa e^{-2\rho} \varphi V$,
(5) $\nabla_V V = -\beta \xi + \varphi V(\rho) \varphi V$,
(6) $\nabla_{\varphi V} V = a \varphi V$,
(7) $\nabla_\xi \varphi V = -\kappa e^{-2\rho} V$,
(8) $\nabla_V \varphi V = -\varphi V(\rho) V$,
(9) $\nabla_{\varphi V} \varphi V = -\beta \xi - aV$.

Here, $\kappa = g(\nabla_\xi \psi, \varphi \psi)$ and $a = e^{-\rho} \operatorname{div} \psi + e^\rho - V(\rho)$, where $\psi = \nabla_\xi \xi$ and $\rho = \log \|\psi\|$.

Employing this theorem, routine calculations lead to

$$\begin{cases} R(\xi, V)\xi = (\beta^2 + \xi(\beta) + e^{2\rho} - V(e^\rho))V - \varphi V(e^\rho)\varphi V, \\ R(\xi, \varphi V)\xi = -\varphi V(e^\rho)V + (\beta^2 + \xi(\beta) - ae^\rho)\varphi V, \\ R(V, \varphi V)\xi = -\varphi V(\beta)V + V(\beta)\varphi V, R(V, \varphi V)V = \varphi V(\beta)\xi + b\varphi V, \end{cases}$$

where $b = a^2 + \beta^2 + V(a) - \varphi V(\varphi V(\rho)) + (\varphi V(\rho))^2$.

The Ricci tensor S, which is defined for all vector fields X, Y on M by

$$S(X, Y) = g(R(X, \xi)\xi, Y) + g(R(X, V)V, Y) + g(R(X, \varphi V)\varphi V, Y),$$

is fully characterized by

$$\begin{cases} S(\xi, \xi) = \operatorname{div}\psi - 2(\beta^2 + \xi(\beta)), \\ S(\xi, V) = -V(\beta), \\ S(\xi, \varphi V) = -\varphi V(\beta), \\ S(V, V) = -b - \beta^2 - \xi(\beta) - e^{2\rho} + V(e^\rho), \\ S(V, \varphi V) = \varphi V(e^\rho), \\ S(\varphi V, \varphi V) = -b - \beta^2 - \xi(\beta) + ae^\rho. \end{cases} \tag{7}$$

According to (7), the scalar curvature r can be expressed as

$$r = 2\operatorname{div}\psi - 2b - 4(\beta^2 + \xi(\beta)).$$

3. Construction of 3-dimensional almost contact metric structures

In this section, we will elucidate a clear method for constructing an almost contact metric structure. This approach is straightforward and practical, grounded in linear algebra and fundamental definitions of structural elements.

Let (M, g) be a 3-dimensional oriented Riemannian manifold. For every local orthonormal frame $e_{i\{1\leq i\leq 3\}}$, we define a unit vector field by

$$\xi = \sum_{i=1}^{3} \xi^i e_i, \tag{8}$$

where $\xi^i \in \mathcal{C}^\infty(M)$ and $\sum_{i=1}^{3}(\xi^i)^2 = 1$. Accordingly, the g-dual of ξ is represented by the differential 1-form defined as

$$\eta = \sum_{i=1}^{3} \xi^i \theta^i, \tag{9}$$

where ${\theta^i}_{\{1\leq i\leq 3\}}$ is the dual co-frame.

Note: We will adopt Einstein's summation convention (i.e., whenever an index is repeated, it is considered a dummy index). Specifically, equations (8) and (9) become

$$\xi = \xi^i e_i \qquad \text{and} \qquad \eta = \xi^i \theta^i.$$

Now, let us begin by defining φ. We denote it as

$$\varphi e_i = \sum_{j=1}^{3} \varphi_i^j e_j,$$

where φ_i^j are functions on M that need to be determined. With the help of (2), we have

$$g(\varphi e_i, e_j) = -g(e_i, \varphi e_j) \qquad \text{and} \qquad g(\varphi e_i, \varphi e_j) = g(e_i, e_j) - \eta(e_i)\eta(e_j).$$

We obtain the system

$$\begin{cases} \varphi_i^j = -\varphi_j^i, \\ \varphi_i^a \varphi_j^a = \delta_{ij} - \xi^i \xi^j. \end{cases}$$

By observing that i and j are fixed, one readily derives from this arrangement

$$\varphi_i^i = 0, \qquad \varphi_i^a \varphi_i^a = 1 - (\xi^i)^2 \qquad \text{and} \qquad \varphi_i^a \varphi_j^a = -\xi^i \xi^j, \quad \text{for} \quad i \neq j.$$

For each $i, j, k \in \{1, 2, 3\}$ such that $i \neq j$, $i \neq k$, and $j \neq k$, the first two equations above produce the following result:

$$\begin{cases} (\varphi_i^j)^2 + (\varphi_i^k)^2 = 1 - (\xi^i)^2, \\ (\varphi_j^i)^2 + (\varphi_j^k)^2 = 1 - (\xi^j)^2, \\ (\varphi_k^i)^2 + (\varphi_k^j)^2 = 1 - (\xi^k)^2. \end{cases}$$

Subtracting the second equation from the first and considering $\varphi_i^j = -\varphi_j^i$, we find

$$(\varphi_i^k)^2 - (\varphi_j^k)^2 = 2\big((\xi^j)^2 - (\xi^i)^2\big).$$

From the third equation in the aforementioned system, we obtain

$$\begin{aligned} 2(\varphi_i^k)^2 &= 1 + (\xi^j)^2 - \big((\xi^k)^2 + (\xi^i)^2\big) \\ &= 1 + (\xi^j)^2 - \big(1 - (\xi^j)^2\big) \\ &= 2(\xi^j)^2, \end{aligned}$$

which gives

$$\varphi_i^k = \epsilon \xi^j, \quad \text{where} \quad \epsilon = \pm 1.$$

It is worth noting that φ is entirely determined by ξ. Based on these observations, we state the following theorem.

Theorem 3. *Let (M^3, g) be a 3-dimensional oriented Riemannian manifold. If ξ is a global unit vector field written on the local orthonormal frame $e_{i\{1\leq i\leq 3\}}$ in the form $\xi = \xi^i e_i$, then there exists an almost contact metric structure (φ, ξ, η, g), where*

$$\varphi = \epsilon \begin{pmatrix} 0 & -\xi^3 & \xi^2 \\ \xi^3 & 0 & -\xi^1 \\ -\xi^2 & \xi^1 & 0 \end{pmatrix}.$$

Proof. It is easy to check that the conditions in (2) are fulfilled. □

4. From Yamabe to almost contact metric structure

Suppose (g, U, λ) forms a Yamabe soliton on M. This implies

$$\mathcal{L}_U g + 2(r - \lambda) = 0. \tag{10}$$

We are aware that the potential vector field U is defined globally on M, yet it may not necessarily be unitary. We therefore define ξ as

$$\xi = \frac{1}{f} U, \qquad \text{where} \qquad f = \|U\|. \tag{11}$$

Subsequent to Theorem (3), an almost contact metric structure (φ, ξ, η, g) emerges.

Now, putting $X = \xi$ in (10), we have

$$(\mathcal{L}_{f\xi} g)(\xi, Y) + 2(r - \lambda) g(\xi, Y) = 0. \tag{12}$$

Using the definition of Lie derivative, we get

$$\begin{aligned}(\mathcal{L}_{f\xi}g)(\xi,Y) &= g(\nabla_\xi f\xi,Y)+g(\nabla_Y f\xi,\xi)\\ &= \xi(f)\eta(Y)+fg(\nabla_\xi\xi,Y)+Y(f)+2(r-\lambda)\eta(Y),\end{aligned}$$

where $g(\nabla_Y\xi,\xi)=0$. Substituting in (12) gives rise to

$$\nabla_\xi\xi=-\frac{1}{f}\Big(\big(2(r-\lambda)+\xi(f)\big)\xi+\mathrm{grad}f\Big). \tag{13}$$

Assume (M,φ,ξ,η,g) constitutes a normal almost contact metric manifold. It is noteworthy that in three dimensions, any normal almost contact metric manifold is trans-Sasakian of type (α,β). Consequently, we observe $\nabla_\xi\xi=0$, and equation (13) leads to

$$\mathrm{grad}f=\big(2(\lambda-r)-\xi(f)\big)\xi. \tag{14}$$

Hence, f is a function that depends solely on the direction of ξ.

To determine α and β, employing a φ-basis $\{\xi,e,e\varphi\}$, we have

$$\begin{aligned}2\alpha &= -g(\nabla_e\xi,\varphi e)-g(\nabla_{\varphi e}\xi,\varphi^2 e)\\ &= g(\varphi\nabla_e\xi,e)+g(\nabla_{\varphi e}\xi,e)\\ &= 2g(\nabla_{\varphi e}\xi,e),\end{aligned}$$

which results in

$$\alpha=g(\nabla_{\varphi e}\xi,e). \tag{15}$$

With the use of (5), we have

$$\begin{aligned}2\beta=\mathrm{div}\xi &= g(\nabla_e\xi,e)-g(\nabla_{\varphi e}\xi,\varphi e)\\ &= g(\varphi\nabla_e\xi,e)+\beta,\end{aligned}$$

which gives

$$\beta=g(\nabla_e\xi,e). \tag{16}$$

Conversely, by setting $(X,Y)=(e,\varphi e)$ in (10) and using (5), (6) we obtain

$$\begin{aligned}0 &= (\mathcal{L}_U g)(e,e)+2(r-\lambda)\\ &= 2fg(\nabla_e\xi,e)+2(r-\lambda).\end{aligned}$$

With the help of (16), we obtain

$$\beta=\frac{\lambda-r}{f}. \tag{17}$$

From (15) and (17), it can be deduced that a 3-dimensional Riemannian manifold (M^3,g), admitting (g,U,λ) as an almost Yamabe soliton, can be endowed with a proper trans-Sasakian structure of type (α,β). Nevertheless, in this scenario, since f depends exclusively on the direction of ξ, we will demonstrate, using two distinct methods, that f must be constant.

Our first method is analytical, rewriting formula (14) as

$$\mathrm{d}f(X) = \big(2(\lambda - r) - \xi(f)\big)\eta(X) = \sigma\eta(X), \tag{18}$$

where we have set $\sigma = 2(\lambda - r) - \xi(f)$. Taking the exterior derivative of (14) and using formula (4) yields

$$\begin{aligned} 0 = (\mathrm{d}^2 f)(X,Y) &= \big(\mathrm{d}\sigma \wedge \eta\big)(X,Y) + \sigma \mathrm{d}\eta(X,Y) \\ &= \big(\mathrm{d}\sigma \wedge \eta\big)(X,Y) + \alpha\sigma\Phi(X,Y). \end{aligned} \tag{19}$$

Substituting $(X,Y) = (e, \varphi e)$ in (19), we get

$$\alpha\sigma\Phi(e,\varphi e) = 0,$$

i.e.,

$$\alpha\sigma = \alpha\big(2(\lambda - r) - \xi(f)\big) = 0. \tag{20}$$

On the other hand, taking $X = \xi$ in (18) provides

$$\xi(f) = \lambda - r. \tag{21}$$

Combining formulas (20) and (21) gives

$$\begin{cases} \alpha = 0 \quad \text{and} \quad \xi(f) = \lambda - r, \\ \text{or} \\ \alpha \neq 0 \quad \text{and} \quad \xi(f) = 0. \end{cases}$$

From a geometric standpoint, supposing that (φ, ξ, η, g) conforms to the condition of normality in a 3-dimensional setting, it aligns with a trans-Sasakian structure, inherently non-integrable by definition. The tangent bundle TM can thus be delineated into the split

$$TM = \{\xi\} \oplus \mathcal{D},$$

where $\mathcal{D} = \ker\eta$, as per Frobenius's Theorem [3], the non-integrability of $\mathcal{D}$ implies the existence of at least two distinct vector fields X and Y belonging to $\mathcal{D}$ such that $[X,Y] \in \{\xi\}$. Given that f represents a smooth function contingent solely upon the direction of ξ, we can express:

$$\xi(f) = [X,Y](f) = X\big(Y(f)\big) - Y\big(X(f)\big) = 0.$$

Thus, f remains constant.

Hence, we discern two distinctive scenarios meriting investigation. The initial scenario involves (φ, ξ, η, g) being normal but non-integrable. This arises from the presence of an α-Sasakian structure with $\alpha \neq 0$. But it is well know that for an α-Sasakian structure ξ is Killing, and since f is constant, we have

$$\mathcal{L}_{f\xi} g = \mathcal{L}_U g = 0.$$

This contradicts the existence of a proper almost Yamabe soliton, i.e., $\mathcal{L}_U g \neq 0$. Based on these facts, we give the following result.

Proposition 1. *Any 3-dimensional oriented Riemannian manifold (M^3, g) endowed with a trivial almost Yamabe soliton (g, U, λ) admits an α-Sasakian structure such that*

$$\xi = \frac{U}{\|U\|}, \qquad \eta = \xi^\flat, \qquad \varphi = \epsilon \begin{pmatrix} 0 & -\xi^3 & \xi^2 \\ \xi^3 & 0 & -\xi^1 \\ -\xi^2 & \xi^1 & 0 \end{pmatrix} \qquad \textit{and} \qquad r = \lambda.$$

Example 1. Let $M = \{(x, y, z) \in \mathbb{R}^3 \ / \ z > 0\}$ and $\{e_1, e_2, e_3\}$ be the frame of vector fields on M given by

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = -\frac{2\sigma}{z}\frac{\partial}{\partial y}, \qquad e_3 = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial z},$$

where $\sigma = \sigma(y)$ is a non zero function on M. We define a Riemannian metric g by

$$g = \left(\begin{array}{ccc} 1 & 0 & -\frac{y}{z} \\ 0 & \frac{1}{4\sigma^2} & 0 \\ -\frac{y}{z} & 0 & \frac{1+y^2}{z^2} \end{array} \right).$$

Let ∇ be the Riemannian connection of g, then we have

$$[e_2, e_3] = -2\sigma e_1.$$

By using the Koszul formula for the Riemannian metric g, the nonzero components of the Levi-Civita connection corresponding to g are given by

$$\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \sigma e_3, \quad \nabla_{e_1} e_3 = \nabla_{e_3} e_1 = -\sigma e_2, \quad \nabla_{e_3} e_2 = -\nabla_{e_2} e_3 = \sigma e_1.$$

The non-vanishing curvature tensor R components are computed as

$$\begin{aligned} R(e_1, e_2)e_1 &= -\sigma^2 e_2, & R(e_1, e_2)e_2 &= \sigma^2 e_1 + 2\sigma\sigma' e_3, \\ R(e_1, e_2)e_3 &= -2\sigma\sigma' e_2, & R(e_1, e_3)e_1 &= -\sigma^2 e_3, \\ R(e_1, e_3)e_3 &= \sigma^2 e_1, & R(e_2, e_3)e_1 &= 2\sigma\sigma' e_2, \\ R(e_2, e_3)e_2 &= -2\sigma\sigma' e_1 + \sigma^2 e_3, & R(e_2, e_3)e_3 &= -3\sigma^2 e_2. \end{aligned}$$

The Ricci curvature S components and the scalar curvature r are computed as

$$S(e_1, e_1) = -S(e_2, e_2) = -S(e_3, e_3) = 2\sigma^2, \ S(1, 3) = 2\sigma\sigma' \text{ and } r = -2\sigma^2.$$

For $U = e_1$, one can easily show that $\mathcal{L}_U g = 0$. That is, U is a Killing vector field which implies that (g, U, λ) is a trivial almost Yamabe soliton with $\lambda = -2\sigma^2$. So, taking $\xi = e_1$, we note that $\nabla_\xi \xi = 0$. From (3) we get

$$\varphi e_1 = 0, \qquad \varphi e_2 = \epsilon e_3 \qquad \text{and} \qquad \varphi e_3 = -\epsilon e_2,$$

which gives

$$\varphi = \epsilon \left(\begin{array}{ccc} 0 & -\frac{y}{2\sigma} & 0 \\ 0 & 0 & \frac{2\sigma}{z} \\ 0 & -\frac{z}{2\sigma} & 0 \end{array} \right),$$

then (φ, ξ, η, g) is an almost contact metric structure on M with $\eta = dx - \frac{y}{z}dz$. One can easily get

$$f = 1, \qquad \alpha = -\sum_{i=1}^{3} g(\nabla_{e_i}\xi, \varphi e_i) = -2\sigma \quad \text{and} \quad \beta = 0,$$

which implies that (φ, ξ, η, g) is an α-Sasakian structure where $\alpha = -\epsilon\sigma$.

The second scenario pertains to the instance where (φ, ξ, η, g) exhibits normality and integrability, corresponding to (M^3, g) being β-Kenmotsu. Subsequently

$$\xi(f) = \lambda - r,$$

and employing computations from equation (17), we obtain

$$\beta = \frac{\lambda - r}{f} = \xi(\ln f).$$

In the subsequent portion of this investigation, we focus on the second scenario, namely when $(M^3, \varphi, \xi, \eta, g)$ represents an integrable yet non-normal almost contact metric manifold, indicating $\nabla_\xi \xi \neq 0$. Specifically, we aim to develop a generalized C_{12}-structure, originating from an almost Yamabe soliton on a 3-dimensional oriented Riemannian manifold (M^3, g).

Suppose (g, U, λ) constitutes a Yamabe soliton, yielding an almost contact metric structure (φ, ξ, η, g) through equation (11) and Theorem (3), which we shall presume to be a generalized C_{12}-structure. From equation (10) and through extensive direct computations, we derive

$$\begin{aligned} 0 &= X(f)\eta(Y) + Y(f)\eta(X) + fe^{\rho}\big(\eta(X)\omega(Y) + \eta(Y)\omega(X)\big) \\ &\quad - 2f\beta g(\varphi^2 X, Y) + 2(r - \lambda)g(X, Y), \end{aligned}$$

or equivalently,

$$X(f)\xi + \eta(X)\text{grad}f + fe^{\rho}\big(\eta(X)V + \omega(X)\xi\big) - 2f\beta\varphi^2 X + 2(r - \lambda)X = 0. \tag{22}$$

Taking $X = \xi$ in (22) yields

$$\text{grad}f = \big(2(\lambda - r) - \xi(f)\big)\xi - fe^{\rho}V, \tag{23}$$

taking the inner product of equation (23) with respect to ξ yields

$$\xi(f) = \lambda - r.$$

For β, with the help of a φ-basis $\{\xi, e, \varphi e\}$, it is adequate to substitute $X = Y = e$ into equation (22), yielding

$$\beta = \frac{\lambda - r}{f} = \xi(\ln f).$$

By amalgamating the antecedent propositions, we can formulate the ensuing result.

Theorem 4. *Let* (g, U, λ) *be an almost Yamabe soliton on a* 3*-dimensional oriented Riemannian manifold* (M^3, g) *and denote* $f = \|U\|$*. Then* (φ, ξ, η, g) *is an almost contact metric structure on* M *such that*

$$\xi = \frac{1}{f}U, \qquad \eta = \xi^\flat, \qquad \varphi = \epsilon \begin{pmatrix} 0 & -\xi^3 & \xi^2 \\ \xi^3 & 0 & -\xi^1 \\ -\xi^2 & \xi^1 & 0 \end{pmatrix},$$

and

(1) if $\mathrm{grad} f = \xi(f)\xi$*, where* $\xi(f) = \lambda - r$*, then* (φ, ξ, η, g) *is a* β*-Kenmotsu structure;*
(2) if $\mathrm{grad} f \neq \xi(f)\xi$ *and* $\xi(f) = \lambda - r$*, then* (φ, ξ, η, g) *is a generalized* C_{12}*-structure.*

Moreover, in both cases $\beta = \xi(\ln f)$ *and the scalar curvature* $r = \lambda - \xi(f)$*.*

For the sake of illustration we give the following example.

Example 2. Let $M = \{(x, y, z) \in \mathbb{R}^3 \ / \ z > 0\}$ and $\{e_1, e_2, e_3\}$ be the frame of vector fields on M given by

$$e_1 = \frac{1}{z}\frac{\partial}{\partial x}, \qquad e_2 = \frac{1}{z}\frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z}.$$

We define a Riemannian metric g by

$$g = z^2(dx^2 + dy^2) + dz^2.$$

Let ∇ be the Riemannian connection of g, then we have

$$[e_1, e_3] = \frac{1}{z}e_1, \qquad [e_2, e_3] = \frac{1}{z}e_2.$$

By using the Koszul formula for the Riemannian metric g, the nonzero components of the Levi-Civita connection corresponding to g are given by:

$$\nabla_{e_1}e_1 = -\frac{1}{z}e_3, \qquad \nabla_{e_1}e_3 = \frac{1}{z}e_1, \qquad \nabla_{e_2}e_2 = -\frac{1}{z}e_3, \qquad \nabla_{e_2}e_3 = \frac{1}{z}e_2.$$

The non-vanishing curvature tensor R components are computed as

$$R(e_1, e_2)e_1 = -\frac{1}{z^2}e_2, \qquad R(e_1, e_2)e_2 = -\frac{1}{z^2}e_1.$$

The Ricci curvature S components and the scalar curvature r are computed as

$$S(e_1, e_1) = S(e_2, e_2) = -\frac{2}{z^2}, \qquad S(e_3, e_3) = 0 \quad \text{and} \quad r = -\frac{2}{z^2}.$$

For $U = ze_3$, the nonzero component of $\mathcal{L}_U g$ is

$$(\mathcal{L}_U g)(e_1, e_1) = (\mathcal{L}_U g)(e_2, e_2) = (\mathcal{L}_U g)(e_3, e_3) = 2.$$

Now, we can easily see that (g, U, λ) is an almost Yamabe soliton with $\lambda = -1 - \frac{2}{z^2}$. So, taking $\xi = e_3$ we note that $\nabla_\xi \xi = 0$. From (3) we get

$$\varphi = \epsilon \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then (φ, ξ, η, g) is an almost contact metric structure on M with $\eta = dz$. One can easily get

$$f = z, \qquad \alpha = -\sum_{i=1}^{3} g(\nabla_{e_i}\xi, \varphi e_i) = 0 \qquad \text{and} \qquad \beta = \frac{1}{2}\sum_{i=1}^{3} g(\nabla_{e_i}\xi, e_i) = \frac{1}{z},$$

which allows us to conclude that (φ, ξ, η, g) is a trans-Sasakian structure of type $(0, \frac{1}{z})$, i.e., β-Kenmotsu structure.

For the second case, let us take $U = z(e_2 + e_3)$, the nonzero component of $\mathcal{L}_U g$ is

$$(\mathcal{L}_U g)(e_1, e_1) = (\mathcal{L}_U g)(e_2, e_2) = (\mathcal{L}_U g)(e_3, e_3) = 2.$$

Now, we can easily see that (g, U, λ) is an almost Yamabe soliton with $\lambda = -1 - \frac{2}{z^2}$. Note that the value of λ is the same in the first case even though the vector field U is different.

So, taking $\xi = \frac{1}{\sqrt{2}}(e_2 + e_3)$ we note that

$$\begin{aligned} \nabla_\xi \xi &= \frac{1}{2}\big(\nabla_{e_2}e_2 + \nabla_{e_2}e_3 + \nabla_{e_3}e_2 + \nabla_{e_3}e_3\big) \\ &= \frac{1}{2z}(e_2 - e_3). \end{aligned}$$

From (3) we get

$$\varphi = \frac{\epsilon}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}_{|\{e_1, e_2, e_3\}},$$

that is

$$\varphi = \frac{\epsilon}{\sqrt{2}} \begin{pmatrix} 0 & -1 & \frac{1}{z} \\ 1 & 0 & 0 \\ -z & 0 & 0 \end{pmatrix}_{|\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}},$$

then (φ, ξ, η, g) is an almost contact metric structure on M with $\eta = \frac{1}{\sqrt{2}}(zdy + dz)$. One easily can get

$$f = \sqrt{2}z, \quad \alpha = -\sum_{i=1}^{3} g(\nabla_{e_i}\xi, \varphi e_i) = 0 \quad \text{and} \quad \beta = \frac{1}{2}\sum_{i=1}^{3} g(\nabla_{e_i}\xi, e_i) = \frac{1}{\sqrt{2}z},$$

which allows us to conclude that (φ, ξ, η, g) is a generalized C_{12}-structure with $\beta = \frac{1}{\sqrt{2}z}$.

5. Open question

Given that the presence of a Yamabe soliton (g, U, λ) on an odd-dimensional oriented Riemannian manifold guarantees the existence of a global vector field known as the "potential vector field", and recognizing that this vector field is generally non-unitary, we have derived a unit vector ξ using the formula $\xi = \frac{1}{\|U\|}U$.

It is feasible to designate U as a characteristic vector field with an appropriate deformation of the metric, such that U becomes a unit vector field. For instance, we can define $\tilde{g} = \frac{1}{\|U\|^2}g$.

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