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A new generalization of Lucas quaternions with finite operators

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Abstract. In this paper, we introduce a new family of Lucas quaternions by using finite operators. We call these quaternions as Lucas finite operator quaternions. We give some properties and identities of Lucas finite operator quaternions such as Binet-like formula, generating function, exponential generating function, Catalan's identity, Cassini's identity, d'Ocagne's identity and many binomial-sum identities. As an application, we generate Cassini's identity in another form by matrix representations.

1. Introduction

Over the last century, numerous scientists have concentrated on two-dimensional number systems. Hamilton [8] introduced the four dimensional real quaternion algebra as follows:

$$Q=\left\{s_0+s_1\boldsymbol{i}+s_2\boldsymbol{j}+s_3\boldsymbol{k}\ :\ \boldsymbol{i}^2=\boldsymbol{j}^2=\boldsymbol{k}^2=-1,\ \boldsymbol{ijk}=-1,\ s_0,s_1,s_2,s_3\in\mathbb{R}\right\}.$$

The multiplication table for quaternions is given in Table 1. Quaternions are a generalization of complex numbers. They have been studied by scientists from a range of domains such as computer sciences, quantum physics, and control systems.

For $p,q\in\mathbb{Z}$, the Horadam numbers $W_n=W_n(W_0,W_1;p,q)$ are defined by

$$W_n=pW_{n-1}+qW_{n-2},\quad n\geq 2,$$

with the initial values W_0 and W_1 (see [10, 11]).

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TABLE 1. The multiplication table for the basis of Q.

$\cdot$	$\mathbf{1}$	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
$\mathbf{1}$	$\mathbf{1}$	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
$\boldsymbol{i}$	$\boldsymbol{i}$	$-\mathbf{1}$	$\boldsymbol{k}$	$-\boldsymbol{j}$
$\boldsymbol{j}$	$\boldsymbol{j}$	$-\boldsymbol{k}$	$-\mathbf{1}$	$\boldsymbol{i}$
$\boldsymbol{k}$	$\boldsymbol{k}$	$\boldsymbol{j}$	$-\boldsymbol{i}$	$-\mathbf{1}$

Let $\gamma = \dfrac{p+\sqrt{p^2+4q}}{2}$ and $\delta = \dfrac{p-\sqrt{p^2+4q}}{2}$ be the roots of the equation $x^2 - px - q = 0$. Then the Binet's formula of W_n is

$$W_n = A\gamma^n + B\delta^n,$$

where $A = \dfrac{W_1 - \delta W_0}{\gamma - \delta}$ and $B = \dfrac{\gamma W_0 - W_1}{\gamma - \delta}$.

Horadam [9] defined the n-th Lucas quaternion as

$$\mathbb{QL}_n = L_n + L_{n+1}\boldsymbol{i} + L_{n+2}\boldsymbol{j} + L_{n+3}\boldsymbol{k}, \tag{1}$$

where $L_n = L_n(2,1;1,1)$ is the n-th Lucas number defined by

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2.$$

The Binet's formula for the Lucas number L_n is given as

$$L_n = \gamma^n + \delta^n,$$

where $\gamma = \frac{1+\sqrt{5}}{2}$ and $\delta = \frac{1-\sqrt{5}}{2}$. Recently, some works have been done by researchers on Fibonacci and Lucas numbers, which connect with many different areas of science as well as mathematics (see [3, 4, 5]).

Throughout this article, we take $\hat{\gamma} = 1 + \gamma\boldsymbol{i} + \gamma^2\boldsymbol{j} + \gamma^3\boldsymbol{k}$ and $\hat{\delta} = 1 + \delta\boldsymbol{i} + \delta^2\boldsymbol{j} + \delta^3\boldsymbol{k}$, where $\gamma = \frac{1+\sqrt{5}}{2}$ and $\delta = \frac{1-\sqrt{5}}{2}$. In [6], the Binet-like formula for the Lucas quaternion $\mathbb{QL}_n$ is given as

$$\mathbb{QL}_n = \hat{\gamma}\gamma^n + \hat{\delta}\delta^n.$$

We refer to [1, 2, 6, 7, 12, 14, 15, 16, 17] for further information on the Fibonacci and Lucas quaternions.

Let α, β be complex parameters, a, b be real parameters, and $E^a[h](w) = h(w+a)$. Simsek [20] defined an operator such that

$$\mathbb{Y}_{\alpha,\beta}[h;a,b](w) = \alpha E^a[h](w) + \beta E^b[h](w). \tag{2}$$

For any polynomial sequence $h_n(w)$ and $r \geq 1$, r-th finite operator $\mathbb{Y}^{(r)}_{\alpha,\beta}[h_n;a,b](w)$ $\left(or\ \ h_n^{(r)}(w)\right)$ is defined by

$$\mathbb{Y}^{(r)}_{\alpha,\beta}[h_n;a,b](w) = h_n^{(r)}(w) = \mathbb{Y}_{\alpha,\beta}[h_n;a,b](w)\left(\mathbb{Y}^{(r-1)}_{\alpha,\beta}[h_n;a,b](w)\right),$$

where $\mathbb{Y}^{(1)}_{\alpha,\beta}[h_n;a,b](w) = h_n^{(1)}(w) = \alpha h_n(w+a) + \beta h_n(w+b)$. Simsek developed the essential operators utilized in the theory of finite difference techniques for the numerical solution of differential equations for particular cases of $(\alpha,\beta;a,b)$ in (2) as shown in Table 2. These operators have a widespread application in mathematics, physics, and engineering. Simsek constructed novel families of special polynomials and numbers implementing finite operators and scrutinized many of their features. For more details on the finite operators, please see [19, 20].

TABLE 2. Special situations for the finite operator $\mathbb{Y}_{\alpha,\beta}[h;a,b](w)$.

$(\alpha,\beta;a,b)$	Finite Operators
$(1,0;0,0)$	$I(h(w)) = h(w)$, (Identity Operator)
$(1,-1;1,0)$	$\Delta(h(w)) = h(w+1) - h(w)$, (Forward Difference Operator)
$(1,-1;0,-1)$	$\nabla(h(w)) = h(w) - h(w-1)$, (Backward Difference Operator)
$(1/2,-1/2;1,0)$	$M(h(w)) = \frac{1}{2}(h(w+1) - h(w))$, (Means Operator)
$(1,-1;a+b,a)$	$G_{ab}(h(w)) = h(w+a+b) - h(w+a)$, ($a \neq b$, Gould Operator)

Kızılateş [13] used the finite operator to establish Horadam finite operator numbers through implementing it to Horadam sequences. Furthermore, in [21] Terzioğlu et al. founded numerous features associated with Fibonacci finite operator quaternions with the help of matrix representations. In [18], Polatlı implemented the finite operators to the (p,q)-Fibonacci polynomials. Furthermore, Yağmur [22] defined the sequence of Horadam finite operator hybrid numbers and investigated several properties of these hybrid numbers.

Let $\alpha,\beta \in \mathbb{R}$ and $a,b \in \mathbb{Z}$. The r-th Horadam finite operator numbers $W_n^{(r)}$ are defined by

$$\begin{aligned}\Delta^{(r)}_{\alpha,\beta;a,b}(W_n) = W_n^{(r)} &= \alpha\Delta^{(r-1)}_{\alpha,\beta;a,b}(W_{n+a}) + \beta\Delta^{(r-1)}_{\alpha,\beta;a,b}(W_{n+b}) \\ &= \sum_{i=0}^{r}\binom{r}{i}\alpha^{r-i}\beta^{i}W_{n+ib+(r-i)a},\end{aligned} \tag{3}$$

where $W_0^{(r)}$ and $W_1^{(r)}$ are the initial conditions of $W_n^{(r)}$.

For $n \geq 1$, we can obtain by the induction method on r that

$$L_{n+1}^{(r)} = L_n^{(r)} + L_{n-1}^{(r)}, \tag{4}$$

where $L_0^{(r)}$ and $L_1^{(r)}$ are the initial conditions of $L_n^{(r)}$.

In [13], Kızılateş gave the Binet-like formula of $L_n^{(r)}$ such that

$$L_n^{(r)} = A^{(r)}\gamma^n + B^{(r)}\delta^n, \tag{5}$$

where $A^{(r)} = \dfrac{L_1^{(r)} - \delta L_0^{(r)}}{\sqrt{5}}$ and $B^{(r)} = \dfrac{\gamma L_0^{(r)} - L_1^{(r)}}{\sqrt{5}}$.

In view of the earlier recent works, we utilize finite operators to generalize the Lucas quaternions $\mathbb{QL}_n$. These quaternions are referred to as the Lucas finite operator quaternions $\mathbb{QL}_n^{(r)}$. We present some characteristics and identities of $\mathbb{QL}_n^{(r)}$. We find Binet-like formula, generating function, exponential generating function, Catalan's identity, Cassini's identity, d'Ocagne's identity and some binomial-sum identities of $\mathbb{QL}_n^{(r)}$. Then, with the help of matrix representations we show another type of Cassini's identity.

2. Lucas finite operator quaternions

In this section, we introduce the Lucas finite operator quaternions and give Binet-like formula and some other properties for these quaternions.

Definition 1. The *Lucas finite operator quaternions* $\mathbb{QL}_n^{(r)}$ are defined by

$$\mathbb{QL}_n^{(r)} = L_n^{(r)} + L_{n+1}^{(r)}\boldsymbol{i} + L_{n+2}^{(r)}\boldsymbol{j} + L_{n+3}^{(r)}\boldsymbol{k},$$

where $L_n^{(r)}$ is the r-th Lucas finite operator number.

For $r = 1$ and $W_n = L_n$ in Definition 1 and (3), we obtain

$$\begin{aligned} \mathbb{QL}_n^{(1)} &= (\alpha L_{n+a} + \beta L_{n+b}) + (\alpha L_{n+a+1} + \beta L_{n+b+1})\,\boldsymbol{i} \\ &\quad + (\alpha L_{n+a+2} + \beta L_{n+b+2})\,\boldsymbol{j} + (\alpha L_{n+a+3} + \beta L_{n+b+3})\,\boldsymbol{k}. \end{aligned} \tag{6}$$

Now we present some special values of $\mathbb{QL}_n^{(1)}$ for $(\alpha, \beta; a, b)$ in the equation (6) as follows.

(1) For $(1, 0; 0, 0)$, we have the identity operator for Lucas quaternion sequence $I\left(\mathbb{QL}_n^{(1)}\right) = \mathbb{QL}_n$. Hence the Lucas finite operator quaternions are a generalization of the Lucas quaternions in the equation (1).
(2) For $(1, -1; 1, 0)$, we have the forward difference operator for Lucas quaternion sequence $\Delta\left(\mathbb{QL}_n^{(1)}\right) = \mathbb{QL}_{n+1} - \mathbb{QL}_n$.
(3) For $(1, -1; 0, -1)$, we have the backward difference operator for Lucas quaternion sequence $\nabla\left(\mathbb{QL}_n^{(1)}\right) = \mathbb{QL}_n - \mathbb{QL}_{n-1}$.
(4) For $(1/2, -1/2; 1, 0)$, we have the means operator for Lucas quaternion sequence $M\left(\mathbb{QL}_n^{(1)}\right) = \frac{1}{2}\left(\mathbb{QL}_{n+1} - \mathbb{QL}_n\right)$.
(5) For $(1, -1; a + b, a)$ and $ab \neq 0$, we have the Gould operator for Lucas quaternion sequence $G_{ab}\left(\mathbb{QL}_n^{(1)}\right) = \mathbb{QL}_{n+a+b} - \mathbb{QL}_{n+a}$.

The conjugate of the Lucas finite operator quaternion $\mathbb{QL}_n^{(r)}$ is

$$\left(\mathbb{QL}_n^{(r)}\right)^* = L_n^{(r)} - L_{n+1}^{(r)}\boldsymbol{i} - L_{n+2}^{(r)}\boldsymbol{j} - L_{n+3}^{(r)}\boldsymbol{k}. \tag{7}$$

Proposition 1. *For the Lucas finite operator quaternions* $\mathbb{QL}_n^{(r)}$*, we have*

$$\mathbb{QL}_n^{(r)} + \left(\mathbb{QL}_n^{(r)}\right)^* = 2L_n^{(r)}.$$

Proof. By using Definition 1 and (7), we can easily derive

$$\begin{aligned} & \mathbb{QL}_n^{(r)} + \left(\mathbb{QL}_n^{(r)}\right)^* \\ = \; & \left(L_n^{(r)} + L_{n+1}^{(r)}\boldsymbol{i} + L_{n+2}^{(r)}\boldsymbol{j} + L_{n+3}^{(r)}\boldsymbol{k}\right) + \left(L_n^{(r)} - L_{n+1}^{(r)}\boldsymbol{i} - L_{n+2}^{(r)}\boldsymbol{j} - L_{n+3}^{(r)}\boldsymbol{k}\right) \\ = \; & 2L_n^{(r)}. \end{aligned}$$

□

Proposition 2. *The recurrence relation of the Lucas finite operator quaternions* $\mathbb{QL}_n^{(r)}$ *is*

$$\mathbb{QL}_n^{(r)} = \mathbb{QL}_{n-1}^{(r)} + \mathbb{QL}_{n-2}^{(r)}, \quad n \geq 2.$$

Proof. From Definition 1 and (4), we get

$$\begin{aligned} \mathbb{QL}_n^{(r)} &= L_n^{(r)} + L_{n+1}^{(r)}\boldsymbol{i} + L_{n+2}^{(r)}\boldsymbol{j} + L_{n+3}^{(r)}\boldsymbol{k} \\ &= L_{n-1}^{(r)} + L_{n-2}^{(r)} + \left(L_n^{(r)} + L_{n-1}^{(r)}\right)\boldsymbol{i} \\ &\quad + \left(L_{n+1}^{(r)} + L_n^{(r)}\right)\boldsymbol{j} + \left(L_{n+2}^{(r)} + L_{n+1}^{(r)}\right)\boldsymbol{k} \\ &= \mathbb{QL}_{n-1}^{(r)} + \mathbb{QL}_{n-2}^{(r)}. \end{aligned}$$

□

Theorem 1. *The Binet-like formula for the Lucas finite operator quaternions* $\mathbb{QL}_n^{(r)}$ *is*

$$\mathbb{QL}_n^{(r)} = A^{(r)}\gamma^n\hat{\gamma} + B^{(r)}\delta^n\hat{\delta},$$

where $A^{(r)} = \dfrac{L_1^{(r)} - \delta L_0^{(r)}}{\sqrt{5}}$ *and* $B^{(r)} = \dfrac{\gamma L_0^{(r)} - L_1^{(r)}}{\sqrt{5}}$.

Proof. By Definition 1 and (5), we have

$$\begin{aligned} \mathbb{QL}_n^{(r)} &= L_n^{(r)} + L_{n+1}^{(r)}\boldsymbol{i} + L_{n+2}^{(r)}\boldsymbol{j} + L_{n+3}^{(r)}\boldsymbol{k} \\ &= \left(A^{(r)}\gamma^n + B^{(r)}\delta^n\right) + \left(A^{(r)}\gamma^{n+1} + B^{(r)}\delta^{n+1}\right)\boldsymbol{i} \\ &\quad + \left(A^{(r)}\gamma^{n+2} + B^{(r)}\delta^{n+2}\right)\boldsymbol{j} + \left(A^{(r)}\gamma^{n+3} + B^{(r)}\delta^{n+3}\right)\boldsymbol{k} \\ &= A^{(r)}\gamma^n\left(1 + \gamma\boldsymbol{i} + \gamma^2\boldsymbol{j} + \gamma^3\boldsymbol{k}\right) + B^{(r)}\delta^n\left(1 + \delta\boldsymbol{i} + \delta^2\boldsymbol{j} + \delta^3\boldsymbol{k}\right) \\ &= A^{(r)}\gamma^n\hat{\gamma} + B^{(r)}\delta^n\hat{\delta}. \end{aligned}$$

□

Furthermore, by Binet-like formula of $\mathbb{QL}_n^{(r)}$ in Theorem 1, we can derive the following two corollaries.

Corollary 1. *Let n be a positive integer. Then we get*

$$\mathbb{QL}_n^{(r)} = L_1^{(r)} \left(\frac{\gamma^n \hat{\gamma} - \delta^n \hat{\delta}}{\sqrt{5}} \right) + L_0^{(r)} \left(\frac{\gamma^{n-1} \hat{\gamma} - \delta^{n-1} \hat{\delta}}{\sqrt{5}} \right),$$

where $L_0^{(r)}$ and $L_1^{(r)}$ are the initial conditions of the r-th Lucas finite operator numbers $L_n^{(r)}$.

Corollary 2. *Let n be a positive integer. Then we have*

$$\left(\mathbb{QL}_{n+1}^{(r)}\right)^2 + \left(\mathbb{QL}_n^{(r)}\right)^2 = \mathbb{QL}_1^{(r)}\mathbb{QL}_{2n+1}^{(r)} + \mathbb{QL}_0^{(r)}\mathbb{QL}_{2n}^{(r)}.$$

Corollary 3. *Let n be a positive integer. Then we have*

$$\left(\mathbb{QL}_{n+1}^{(r)}\right)^2 - \left(\mathbb{QL}_{n-1}^{(r)}\right)^2 = \mathbb{QL}_1^{(r)}\mathbb{QL}_{2n}^{(r)} + \mathbb{QL}_0^{(r)}\mathbb{QL}_{2n-1}^{(r)}.$$

Proof. From Proposition 2 and Corollary 2, we can derive

$$\begin{aligned}
&\left(\mathbb{QL}_{n+1}^{(r)}\right)^2 - \left(\mathbb{QL}_{n-1}^{(r)}\right)^2 \\
= &\left(\left(\mathbb{QL}_{n+1}^{(r)}\right)^2 + \left(\mathbb{QL}_n^{(r)}\right)^2\right) - \left(\left(\mathbb{QL}_n^{(r)}\right)^2 + \left(\mathbb{QL}_{n-1}^{(r)}\right)^2\right) \\
= &\left(\mathbb{QL}_1^{(r)}\mathbb{QL}_{2n+1}^{(r)} + \mathbb{QL}_0^{(r)}\mathbb{QL}_{2n}^{(r)}\right) - \left(\mathbb{QL}_1^{(r)}\mathbb{QL}_{2n-1}^{(r)} + \mathbb{QL}_0^{(r)}\mathbb{QL}_{2n-2}^{(r)}\right) \\
= &\mathbb{QL}_1^{(r)}\left(\mathbb{QL}_{2n+1}^{(r)} - \mathbb{QL}_{2n-1}^{(r)}\right) + \mathbb{QL}_0^{(r)}\left(\mathbb{QL}_{2n}^{(r)} - \mathbb{QL}_{2n-2}^{(r)}\right) \\
= &\mathbb{QL}_1^{(r)}\mathbb{QL}_{2n}^{(r)} + \mathbb{QL}_0^{(r)}\mathbb{QL}_{2n-1}^{(r)}.
\end{aligned}$$

□

Theorem 2. *The generating function of the Lucas finite operator quaternions $\mathbb{QL}_n^{(r)}$ is*

$$\mathbb{QL}_n^{(r)}(x) = \frac{\mathbb{QL}_0^{(r)} + \left(\mathbb{QL}_1^{(r)} - \mathbb{QL}_0^{(r)}\right)x}{1 - x - x^2}.$$

Proof. Let $\mathbb{QL}_n^{(r)}(x)$ be the generating function for $\mathbb{QL}_n^{(r)}$. That is,

$$\mathbb{QL}_n^{(r)}(x) = \sum_{n=0}^{\infty} \mathbb{QL}_n^{(r)} x^n.$$

Then we get

$$\begin{aligned}
\mathbb{QL}_n^{(r)}(x) &= \mathbb{QL}_0^{(r)} + \mathbb{QL}_1^{(r)}x + \mathbb{QL}_2^{(r)}x^2 + \cdots + \mathbb{QL}_n^{(r)}x^n + \cdots \\
-x\mathbb{QL}_n^{(r)}(x) &= -\mathbb{QL}_0^{(r)}x - \mathbb{QL}_1^{(r)}x^2 - \mathbb{QL}_2^{(r)}x^3 - \cdots - \mathbb{QL}_n^{(r)}x^{n+1} - \cdots
\end{aligned}$$

$$-x^2\mathbb{QL}_n^{(r)}(x) \quad = \quad -\mathbb{QL}_0^{(r)}x^2 - \mathbb{QL}_1^{(r)}x^3 - \mathbb{QL}_2^{(r)}x^4 - \cdots - \mathbb{QL}_n^{(r)}x^{n+2} - \cdots .$$

Using the above equations and Proposition 2, we can find that

$$\begin{aligned}
&\left(1-x-x^2\right)\mathbb{QL}_n^{(r)}(x) \\
= \;& \mathbb{QL}_0^{(r)} + \left(\mathbb{QL}_1^{(r)} - \mathbb{QL}_0^{(r)}\right)x + \sum_{n=2}^{\infty}\left(\mathbb{QL}_n^{(r)} - \mathbb{QL}_{n-1}^{(r)} - \mathbb{QL}_{n-2}^{(r)}\right)x^n \\
= \;& \mathbb{QL}_0^{(r)} + \left(\mathbb{QL}_1^{(r)} - \mathbb{QL}_0^{(r)}\right)x
\end{aligned}$$

and so the proof is completed. □

Theorem 3. *The exponential generating function of the Lucas finite operator quaternions* $\mathbb{QL}_n^{(r)}$ *is*

$$\sum_{n=0}^{\infty}\mathbb{QL}_n^{(r)}\frac{x^n}{n!} = A^{(r)}e^{\gamma x}\hat{\gamma} + B^{(r)}e^{\delta x}\hat{\delta},$$

where $A^{(r)} = \dfrac{L_1^{(r)} - \delta L_0^{(r)}}{\sqrt{5}}$ *and* $B^{(r)} = \dfrac{\gamma L_0^{(r)} - L_1^{(r)}}{\sqrt{5}}$.

Proof. By Binet-like formula of $\mathbb{QL}_n^{(r)}$ in Theorem 1, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathbb{QL}_n^{(r)}\frac{x^n}{n!} &= \sum_{n=0}^{\infty}\left[A^{(r)}\gamma^n\hat{\gamma} + B^{(r)}\delta^n\hat{\delta}\right]\frac{x^n}{n!} \\
&= A^{(r)}\sum_{n=0}^{\infty}\frac{(\gamma x)^n}{n!}\hat{\gamma} + B^{(r)}\sum_{n=0}^{\infty}\frac{(\beta x)^n}{n!}\hat{\delta} \\
&= A^{(r)}e^{\gamma x}\hat{\gamma} + B^{(r)}e^{\delta x}\hat{\delta}.
\end{aligned}$$

□

Theorem 4. *Let* n *be a non-negative integer. Then we have*

$$\sum_{i=0}^{n}\mathbb{QL}_i^{(r)} = \frac{A^{(r)}\gamma^{n+1}\hat{\gamma}}{\gamma-1} + \frac{B^{(r)}\delta^{n+1}\hat{\delta}}{\delta-1} - C^{(r)},$$

where $A^{(r)} = \dfrac{L_1^{(r)} - \delta L_0^{(r)}}{\sqrt{5}}$, $B^{(r)} = \dfrac{\gamma L_0^{(r)} - L_1^{(r)}}{\sqrt{5}}$ *and*

$$C^{(r)} \quad = \quad L_1^{(r)} + \left(L_0^{(r)} + L_1^{(k)}\right)\boldsymbol{i} + \left(L_0^{(r)} + 2L_1^{(r)}\right)\boldsymbol{j} + \left(2L_0^{(r)} + 3L_1^{(r)}\right)\boldsymbol{k}.$$

Proof. By Binet-like formula of $\mathbb{QL}_n^{(r)}$ in Theorem 1, we find that

$$\sum_{i=0}^{n}\mathbb{QL}_i^{(r)} \quad = \quad \sum_{i=0}^{n}\left(A^{(r)}\gamma^i\hat{\gamma} + B^{(r)}\delta^i\hat{\delta}\right)$$

$$
\begin{aligned}
&= \frac{A^{(r)}\left(1-\gamma^{n+1}\right)\hat{\gamma}}{1-\gamma} + \frac{B^{(r)}\left(1-\delta^{n+1}\right)\hat{\delta}}{1-\delta} \\
&= \frac{A^{(r)}\gamma^{n+1}\hat{\gamma}}{\gamma-1} + \frac{B^{(r)}\delta^{n+1}\hat{\delta}}{\delta-1} + \frac{A^{(r)}\hat{\gamma}}{1-\gamma} + \frac{B^{(r)}\hat{\delta}}{1-\delta}.
\end{aligned}
$$

Some basic calculations show that

$$
\sum_{i=0}^{n} \mathbb{QL}_i^{(r)} = \frac{A^{(r)}\gamma^{n+1}\hat{\gamma}}{\gamma-1} + \frac{B^{(r)}\delta^{n+1}\hat{\delta}}{\delta-1} - C^{(r)},
$$

which is desired. □

Theorem 5 (**Catalan's Identity**). *For $n, k \in \mathbb{Z}^+$ such that $n \geq k$, we have*

$$
\mathbb{QL}_{n+k}^{(r)}\mathbb{QL}_{n-k}^{(r)} - \left(\mathbb{QL}_n^{(r)}\right)^2 = (-1)^{n-k}A^{(r)}B^{(r)}\left(\gamma^k-\delta^k\right)\left(\gamma^k\hat{\gamma}\hat{\delta}-\delta^k\hat{\delta}\hat{\gamma}\right),
$$

where $A^{(r)} = \dfrac{L_1^{(r)}-\delta L_0^{(r)}}{\sqrt{5}}$ *and* $B^{(r)} = \dfrac{\gamma L_0^{(r)}-L_1^{(r)}}{\sqrt{5}}$.

Proof. By Binet-like formula of $\mathbb{QL}_n^{(r)}$ in Theorem 1, we obtain

$$
\begin{aligned}
&\mathbb{QL}_{n+k}^{(r)}\mathbb{QL}_{n-k}^{(r)} - \left(\mathbb{QL}_n^{(r)}\right)^2 \\
&= \left(A^{(r)}\gamma^{n+k}\hat{\gamma} + B^{(r)}\delta^{n+k}\hat{\delta}\right)\left(A^{(r)}\gamma^{n-k}\hat{\gamma} + B^{(r)}\delta^{n-k}\hat{\delta}\right) \\
&\quad - \left(A^{(r)}\gamma^{n}\hat{\gamma} + B^{(r)}\delta^{n}\hat{\delta}\right)^2 \\
&= \left(A^{(r)}\right)^2\gamma^{2n}\left(\hat{\gamma}\right)^2 + A^{(r)}B^{(r)}\gamma^{n+k}\delta^{n-k}\hat{\gamma}\hat{\delta} + A^{(r)}B^{(r)}\gamma^{n-k}\delta^{n+k}\hat{\delta}\hat{\gamma} \\
&\quad + \left(B^{(r)}\right)^2\delta^{2n}\left(\hat{\delta}\right)^2 - \left(A^{(r)}\right)^2\gamma^{2n}\left(\hat{\gamma}\right)^2 - A^{(r)}B^{(r)}\gamma^{n}\beta^{n}\hat{\gamma}\hat{\delta} \\
&\quad - A^{(r)}B^{(r)}\gamma^{n}\delta^{n}\hat{\delta}\hat{\gamma} - \left(B^{(r)}\right)^2\delta^{2n}\left(\hat{\delta}\right)^2 \\
&= A^{(r)}B^{(r)}\gamma^{n-k}\delta^{n-k}\left(\gamma^{2k}-\gamma^k\delta^k\right)\hat{\gamma}\hat{\delta} \\
&\quad - A^{(r)}B^{(r)}\gamma^{n-k}\delta^{n-k}\left(\gamma^k\delta^k-\delta^{2k}\right)\hat{\delta}\hat{\gamma} \\
&= A^{(r)}B^{(r)}\gamma^{n-k}\delta^{n-k}\left(\gamma^k-\delta^k\right)\left(\gamma^k\hat{\gamma}\hat{\delta}-\delta^k\hat{\delta}\hat{\gamma}\right) \\
&= (-1)^{n-k}A^{(r)}B^{(r)}\left(\gamma^k-\delta^k\right)\left(\gamma^k\hat{\gamma}\hat{\delta}-\delta^k\hat{\delta}\hat{\gamma}\right).
\end{aligned}
$$

□

In Theorem 5, if we take $k = 1$, then we have Cassini's identity of the Lucas finite operator quaternions $\mathbb{QL}_n^{(r)}$ as follows.

Corollary 4 (**Cassini's Identity**). *For $n \geq 1$, the following equality holds:*

$$\mathbb{QL}_{n+1}^{(r)}\mathbb{QL}_{n-1}^{(r)} - \left(\mathbb{QL}_{n}^{(r)}\right)^2 = (-1)^{n-1}\sqrt{5}A^{(r)}B^{(r)}\left(\gamma\hat{\gamma}\hat{\delta} - \delta\hat{\delta}\hat{\gamma}\right),$$

where $A^{(r)} = \dfrac{L_1^{(r)} - \delta L_0^{(r)}}{\sqrt{5}}$ *and* $B^{(r)} = \dfrac{\gamma L_0^{(r)} - L_1^{(r)}}{\sqrt{5}}$.

Theorem 6 (**d'Ocagne's Identity**). *For $n \in \mathbb{N}$, $k \in \mathbb{Z}^+$ such that $k > n+1$, we get*

$$\mathbb{QL}_{k+1}^{(r)}\mathbb{QL}_{n}^{(r)} - \mathbb{QL}_{k}^{(r)}\mathbb{QL}_{n+1}^{(r)} = (-1)^{n}\sqrt{5}A^{(r)}B^{(r)}\left(\gamma^{k-n}\hat{\gamma}\hat{\delta} - \beta^{k-n}\hat{\delta}\hat{\gamma}\right),$$

where $A^{(r)} = \dfrac{L_1^{(r)} - \delta L_0^{(r)}}{\sqrt{5}}$ *and* $B^{(r)} = \dfrac{\gamma L_0^{(r)} - L_1^{(r)}}{\sqrt{5}}$.

Proof. By Binet-like formula of $\mathbb{QL}_{n}^{(r)}$ in Theorem 1, we find that

$$\begin{aligned}
&\mathbb{QL}_{k+1}^{(r)}\mathbb{QL}_{n}^{(r)} - \mathbb{QL}_{k}^{(r)}\mathbb{QL}_{n+1}^{(r)} \\
=\ & \left(A^{(r)}\gamma^{k+1}\hat{\gamma} + B^{(r)}\delta^{k+1}\hat{\delta}\right)\left(A^{(r)}\gamma^{n}\hat{\gamma} + B^{(r)}\delta^{n}\hat{\delta}\right) \\
& - \left(A^{(r)}\gamma^{k}\hat{\gamma} + B^{(r)}\delta^{k}\hat{\delta}\right)\left(A^{(r)}\gamma^{n+1}\hat{\gamma} + B^{(r)}\delta^{n+1}\hat{\delta}\right) \\
=\ & \left(A^{(r)}\right)^2\gamma^{n+k+1}\left(\hat{\gamma}\right)^2 + A^{(r)}B^{(r)}\gamma^{k+1}\delta^{n}\hat{\gamma}\hat{\delta} + A^{(r)}B^{(r)}\gamma^{n}\delta^{k+1}\hat{\delta}\hat{\gamma} \\
& + \left(B^{(r)}\right)^2\delta^{n+k+1}\left(\hat{\delta}\right)^2 - \left(A^{(r)}\right)^2\gamma^{n+k+1}\left(\hat{\gamma}\right)^2 - A^{(r)}B^{(r)}\gamma^{k}\delta^{n+1}\hat{\gamma}\hat{\delta} \\
& - A^{(r)}B^{(r)}\gamma^{n+1}\delta^{k}\hat{\delta}\hat{\gamma} - \left(B^{(r)}\right)^2\delta^{n+k+1}\left(\hat{\delta}\right)^2 \\
=\ & A^{(r)}B^{(r)}\gamma^{n}\delta^{n}\left(\gamma^{k-n+1} - \gamma^{k-n}\delta\right)\hat{\gamma}\hat{\delta} \\
& - A^{(r)}B^{(r)}\gamma^{n}\delta^{n}\left(\gamma\delta^{k-n} - \delta^{k-n+1}\right)\hat{\delta}\hat{\gamma} \\
=\ & A^{(r)}B^{(r)}\gamma^{n}\delta^{n}\left(\gamma - \delta\right)\left(\gamma^{k-n}\hat{\gamma}\hat{\delta} - \delta^{k-n}\hat{\delta}\hat{\gamma}\right) \\
=\ & (-1)^{n}\sqrt{5}A^{(r)}B^{(r)}\left(\gamma^{k-n}\hat{\gamma}\hat{\delta} - \beta^{k-n}\hat{\delta}\hat{\gamma}\right).
\end{aligned}$$

□

3. Some binomial-sum identities of Lucas finite operator quaternions

In this section, we give binomial-sum properties of Lucas finite operator quaternions $\mathbb{QL}_{n}^{(r)}$ by Binet-like formula of $\mathbb{QL}_{n}^{(r)}$.

Theorem 7. *For non-negative integers n and k, we get*

$$\sum_{i=0}^{n}\binom{n}{i}\mathbb{QL}_{i+k}^{(r)}=\mathbb{QL}_{2n+k}^{(r)}.$$

Proof. By Binet-like formula of $\mathbb{QL}_{n}^{(r)}$ in Theorem 1, we obtain

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}\mathbb{QL}_{i+k}^{(r)} &= \sum_{i=0}^{n}\binom{n}{i}\left(A^{(r)}\gamma^{i+k}\hat{\gamma}+B^{(r)}\delta^{i+k}\hat{\delta}\right)\\ &= A^{(r)}\sum_{i=0}^{n}\binom{n}{i}\gamma^{i}\gamma^{k}\hat{\gamma}+B^{(r)}\sum_{i=0}^{n}\binom{n}{i}\delta^{i}\delta^{k}\hat{\delta}\\ &= A^{(r)}\left(\gamma+1\right)^{n}\gamma^{k}\hat{\gamma}+B^{(r)}\left(\delta+1\right)^{n}\delta^{k}\hat{\delta}\\ &= A^{(r)}\gamma^{2n+k}\hat{\gamma}+B^{(r)}\delta^{2n+k}\hat{\delta}\\ &= \mathbb{QL}_{2n+k}^{(r)}.\end{aligned}$$

□

Theorem 8. *For non-negative integers n and k, we have*

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\mathbb{QL}_{2i+k}^{(r)}=(-1)^{n}\mathbb{QL}_{n+k}^{(r)}.$$

Proof. From Binet-like formula of $\mathbb{QL}_{n}^{(r)}$ in Theorem 1, we find that

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\mathbb{QL}_{2i+k}^{(r)}\\ =\;& \sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\left(A^{(r)}\gamma^{2i+k}\hat{\gamma}+B^{(r)}\delta^{2i+k}\hat{\delta}\right)\\ =\;& A^{(r)}\sum_{i=0}^{n}\binom{n}{i}(-\gamma^{2})^{i}\gamma^{k}\hat{\gamma}+B^{(r)}\sum_{i=0}^{n}\binom{n}{i}(-\delta^{2})^{i}\delta^{k}\hat{\delta}\\ =\;& A^{(r)}\left(1-\gamma^{2}\right)^{n}\gamma^{k}\hat{\gamma}+B^{(r)}\left(1-\delta^{2}\right)^{n}\delta^{k}\hat{\delta}\\ =\;& A^{(r)}\left((-1)^{n}\gamma^{n+k}\hat{\gamma}\right)+B^{(r)}\left((-1)^{n}\delta^{n+k}\hat{\delta}\right)\\ =\;& (-1)^{n}\mathbb{QL}_{n+k}^{(r)}.\end{aligned}$$

□

Theorem 9. *For non-negative integers n and k, we have*

$$\sum_{i=0}^{n}\binom{n}{i}\mathbb{QL}_{2i+k}^{(r)}$$

$$= \begin{cases} 5^{n/2}\mathbb{QL}_{n+k}^{(r)}\ , & if\ n\ is\ even, \\ 5^{(n-1)/2}\left(\mathbb{QL}_{n+k+1}^{(r)}+\mathbb{QL}_{n+k-1}^{(r)}\right)\ , & if\ n\ is\ odd. \end{cases}$$

Proof. From Binet-like formula of $\mathbb{QL}_n^{(r)}$ in Theorem 1, we obtain

$$\begin{aligned} & \sum_{i=0}^{n}\binom{n}{i}\mathbb{QL}_{2i+k}^{(r)} \\ = & \sum_{i=0}^{n}\binom{n}{i}\left(A^{(r)}\gamma^{2i+k}\hat{\gamma}+B^{(r)}\delta^{2i+k}\hat{\delta}\right) \\ = & A^{(r)}\sum_{i=0}^{n}\binom{n}{i}\left(\gamma^2\right)^i\gamma^k\hat{\gamma}+B^{(r)}\sum_{i=0}^{n}\binom{n}{i}\left(\delta^2\right)^i\delta^k\hat{\delta} \\ = & A^{(r)}\left(\gamma^2+1\right)^n\gamma^k\hat{\gamma}+B^{(r)}\left(\delta^2+1\right)^n\delta^k\hat{\delta}. \end{aligned} \tag{8}$$

If n is even, i.e., $n=2t$ where t is a non-negative integer, using (8) we have

$$\begin{aligned} \sum_{i=0}^{n}\binom{n}{i}\mathbb{QL}_{2i+k}^{(r)} &= A^{(r)}\left(\gamma^2+1\right)^{2t}\gamma^k\hat{\gamma}+B^{(r)}\left(\delta^2+1\right)^{2t}\delta^k\hat{\delta} \\ &= A^{(r)}5^t\gamma^{2t+k}\hat{\gamma}+B^{(r)}5^t\delta^{2t+k}\hat{\delta} \\ &= 5^t\left(A^{(r)}\gamma^{2t+k}\hat{\gamma}+B^{(r)}\delta^{2t+k}\hat{\delta}\right) \\ &= 5^{n/2}\mathbb{QL}_{n+k}^{(r)}. \end{aligned}$$

If n is odd, i.e., $n=2t+1$, where t is a non-negative integer, using (8) we have

$$\begin{aligned} \sum_{i=0}^{n}\binom{n}{i}\mathbb{QL}_{2i+k}^{(r)} &= A^{(r)}\left(\gamma^2+1\right)^{2t+1}\gamma^k\hat{\gamma}+B^{(r)}\left(\delta^2+1\right)^{2t+1}\delta^k\hat{\delta} \\ &= A^{(r)}5^t\left(\gamma^2+1\right)\gamma^{2t+k}\hat{\gamma}+B^{(r)}5^t\left(\delta^2+1\right)\delta^{2t+k}\hat{\delta} \\ &= 5^t\left(A^{(r)}\gamma^{2t+2+k}\hat{\gamma}+B^{(r)}\delta^{2t+2+k}\hat{\delta}\right) \\ &\quad +5^t\left(A^{(r)}\gamma^{2t+k}\hat{\gamma}+B^{(r)}\delta^{2t+k}\hat{\delta}\right) \\ &= 5^{(n-1)/2}\left(\mathbb{QL}_{n+k+1}^{(r)}+\mathbb{QL}_{n+k-1}^{(r)}\right). \end{aligned}$$

□

4. Matrix representations of Lucas finite operator quaternions

In this section, we construct the matrix representation of the Lucas finite operator quaternions. We define two matrices $\boldsymbol{S}$ and $\boldsymbol{L^{(r)}}$ as

$$\boldsymbol{S}=\begin{bmatrix}1&1\\1&0\end{bmatrix}\quad \text{and}\quad \boldsymbol{L^{(r)}}=\begin{bmatrix}\mathbb{QL}_2^{(r)}&\mathbb{QL}_1^{(r)}\\\mathbb{QL}_1^{(r)}&\mathbb{QL}_0^{(r)}\end{bmatrix}. \tag{9}$$

In light of our conclusion, we provide the following theorem.

Theorem 10. *For $n\in\mathbb{N}$, we have*

$$\boldsymbol{S}^n\boldsymbol{L^{(r)}}=\begin{bmatrix}\mathbb{QL}_{n+2}^{(r)}&\mathbb{QL}_{n+1}^{(r)}\\\mathbb{QL}_{n+1}^{(r)}&\mathbb{QL}_n^{(r)}\end{bmatrix}.$$

Proof. We prove the theorem by the induction method on n. For $n=0$, the equality holds. Suppose that the hypothesis is true for $n=i$. That is,

$$\boldsymbol{S}^i\boldsymbol{L^{(r)}}=\begin{bmatrix}\mathbb{QL}_{i+2}^{(r)}&\mathbb{QL}_{i+1}^{(r)}\\\mathbb{QL}_{i+1}^{(r)}&\mathbb{QL}_i^{(r)}\end{bmatrix}. \tag{10}$$

For $n=i+1$, by (10) and Proposition 2 we find that

$$\begin{aligned}\boldsymbol{S}^{i+1}\boldsymbol{L^{(r)}} &= \boldsymbol{S}\boldsymbol{S}^i\boldsymbol{L^{(r)}}\\ &= \begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}\mathbb{QL}_{i+2}^{(r)}&\mathbb{QL}_{i+1}^{(r)}\\\mathbb{QL}_{i+1}^{(r)}&\mathbb{QL}_i^{(r)}\end{bmatrix}\\ &= \begin{bmatrix}\mathbb{QL}_{i+2}^{(r)}+\mathbb{QL}_{i+1}^{(r)}&\mathbb{QL}_{i+1}^{(r)}+\mathbb{QL}_i^{(r)}\\\mathbb{QL}_{i+2}^{(r)}&\mathbb{QL}_{i+1}^{(r)}\end{bmatrix}\\ &= \begin{bmatrix}\mathbb{QL}_{i+3}^{(r)}&\mathbb{QL}_{i+2}^{(r)}\\\mathbb{QL}_{i+2}^{(r)}&\mathbb{QL}_{i+1}^{(r)}\end{bmatrix}.\end{aligned}$$

As a result, the proof is completed. □

In the following corollary, we obtain Cassini's identity of Lucas finite operator quaternions by applying the matrices given above.

Corollary 5. *For $n\in\mathbb{Z}^+$, we have*

$$\mathbb{QL}_{n+1}^{(r)}\mathbb{QL}_{n-1}^{(r)}-\left(\mathbb{QL}_n^{(r)}\right)^2=(-1)^{(n-1)}\left[\mathbb{QL}_2^{(r)}\mathbb{QL}_0^{(r)}-\left(\mathbb{QL}_1^{(r)}\right)^2\right].$$

Proof. Using (9) and Theorem 10, we obtain

$$\begin{bmatrix}1&1\\1&0\end{bmatrix}^{n-1}\begin{bmatrix}\mathbb{QL}_2^{(r)}&\mathbb{QL}_1^{(r)}\\\mathbb{QL}_1^{(r)}&\mathbb{QL}_0^{(r)}\end{bmatrix}=\begin{bmatrix}\mathbb{QL}_{n+1}^{(r)}&\mathbb{QL}_n^{(r)}\\\mathbb{QL}_n^{(r)}&\mathbb{QL}_{n-1}^{(r)}\end{bmatrix}. \tag{11}$$

If we consider the determinant on both sides of (11), then we find that

$$\mathbb{QL}_{n+1}^{(r)}\mathbb{QL}_{n-1}^{(r)} - \left(\mathbb{QL}_{n}^{(r)}\right)^2 = (-1)^{(n-1)}\left[\mathbb{QL}_{2}^{(r)}\mathbb{QL}_{0}^{(r)} - \left(\mathbb{QL}_{1}^{(r)}\right)^2\right].$$

□

5. Conclusions

In this work, we describe Lucas finite operator quaternions by implementing finite operators to Lucas quaternions. Lucas quaternions have been generalized to provide these new quaternions. Moreover, we obtain many algebraic properties of Lucas finite operator quaternions including Binet-like formula, generating function, exponential generating function, Catalan's identity, Cassini's identity, d'Ocagne's identity and numerous binomial-sum properties. We offer two new matrices $\boldsymbol{S}$ and $\boldsymbol{L}^{(r)}$. Finally, with these matrices we construct a matrix whose entries are Lucas finite operator quaternions and attain the Cassini's identity.

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