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On the construction of iterated collocation-type approximations for linear fractional differential equations

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Abstract. The present paper is concerned with the numerical solution of initial value problems for linear Caputo-type fractional differential equations. Some regularity results are presented and, using a reformulated integral equation approach, a high-order collocation method and its iterated version are constructed. Global superconvergence results of the iterated version are studied. Numerical examples confirming the theoretical results are also given.

1. Introduction

Differential equations with derivatives of fractional (non-integer) order have been shown to be a promising tool in high-accuracy modeling of many diverse real-life processes (see, e.g., [21, 23, 26]). Since an analytical solution to a fractional differential equation is rarely possible, the numerical analysis of fractional differential equations and operators has been a widely developing field in the last decade, see, for example, [1, 2, 5, 7–16, 19, 20, 22, 27, 28]. In the present paper we report some recent results regarding a collocation-type approximation and its iterated version for solving linear fractional differential equations with initial conditions. For simplicity of presentation we restrict ourselves to differential equations with a single Caputo-type fractional derivative of order less than one. Our main results are given by Theorem 3 below.

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{N} = \{1, 2, \dots\}$. Let $C^k(\Omega)$ be the set of all k times ($k \geq 0$) continuously differentiable functions on $\Omega \subset \mathbb{R}$, $C^0(\Omega) = C(\Omega)$. In

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particular, by $C[0,b]$ we denote the Banach space of continuous functions u on $[0,b]$ with the norm $\|u\|_\infty = \max_{0\le t\le b}|u(t)|$. For Banach spaces X and Y, by $\mathcal{L}(X,Y)$ we denote the space of linear bounded operators from X to Y with the norm $\|A\|_{\mathcal{L}(X,Y)} = \sup\{\|Ax\|_Y : \|x\|_X < 1\}$ for $A \in \mathcal{L}(X,Y)$.

Let $\alpha \in (0,1)$. We consider the following initial value problem:

$$(D^\alpha_{Cap} y)(t) + a(t)y(t) = f(t), \quad 0 \le t \le b, \quad b > 0, \tag{1}$$

$$y(0) = y_0, \tag{2}$$

where $y_0 \in \mathbb{R}$, $a, f \in C[0,b]$ are some given continuous functions, and $D^\alpha_{Cap} y$ denotes the Caputo fractional derivative of order α of an unknown function $y \in C[0,b]$, defined as

$$(D^\alpha_{Cap} y)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha}(y(s) - y(0))ds, \quad 0 < t \le b.$$

Here Γ is the Euler gamma function: $\Gamma(x) = \int_0^\infty e^{-s}s^{x-1}ds$, $x > 0$. The necessary and sufficient conditions for the existence of a continuous Caputo fractional derivative are given in [25]. Note that for any solution $y \in C[0,b]$ of (1)–(2) we have that $D^\alpha_{Cap} y \in C[0,b]$.

For $\delta > 0$, we denote by J^δ the Riemann–Liouville integral operator $J^\delta : L_\infty(0,b) \to C[0,b]$, defined by

$$\left(J^\delta u\right)(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} u(s)ds, \ t > 0, \ u \in L_\infty(0,b). \tag{3}$$

Here by $L_\infty(0,b)$ we denote the Banach space of all essentially bounded measurable functions $u : (0,b) \to \mathbb{R}$ with the norm

$$\|u\|_{L_\infty(0,b)} = \inf_{\mu(\Omega)=0} \sup_{t\in(0,b)\setminus\Omega} |u(t)| < \infty,$$

where $\mu(\Omega)$ is the Lebesgue measure of a (measurable) set $\Omega \subset (0,b)$. Note that the operator J^δ is a compact (linear) operator from $L_\infty(0,b)$ to $C[0,b]$ (see, for example, [4]). Note also that $D^\alpha_{Cap} J^\alpha u = u$ for $u \in C[0,b]$ [6].

To study the regularity of an exact solution y of (1)–(2), we introduce the following class of weighted functions $C^{m,\nu}(0,b]$, first studied by Vainikko in [24].

For given $b \in \mathbb{R}, b > 0$, $m \in \mathbb{N}$ and $\nu < 1$, by $C^{m,\nu}(0,b]$ we denote the set of continuous functions $u : [0,b] \to \mathbb{R}$ which are m times continuously differentiable in $(0,b]$ such that for $i = 1, ..., m$ the following estimates hold:

$$\left|u^{(i)}(t)\right| \le c \begin{cases} 1, & \text{if } i < 1-\nu, \\ 1 + |\log t|, & \text{if } i = 1-\nu, \\ t^{1-\nu-i}, & \text{if } i > 1-\nu, \end{cases}$$

where $t \in (0, b]$, $c = c(u)$ is a positive constant independent of t. Equipped with the norm

$$\|u\|_{C^{m,\nu}(0,b]} := \|u\|_\infty + \sum_{i=1}^{m} \sup_{0<t\le b} \omega_{i-1+\nu}(t) \left| u^{(i)}(t) \right|, \quad u \in C^{m,\nu}(0,b],$$

where, for $t > 0$,

$$\omega_\rho(t) := \begin{cases} 1, & \text{if } \rho < 0, \\ \frac{1}{1+|\log t|}, & \text{if } \rho = 0, \\ t^\rho, & \text{if } \rho > 0, \end{cases}$$

the set $C^{m,\nu}(0,b]$ becomes a Banach space. Note that

$$C^q[0,b] \subset C^{q,\nu}(0,b] \subset C^{m,\mu}(0,b] \subset C[0,b], \quad q \ge m \ge 1, \quad 0 < \nu \le \mu < 1.$$

Using this class of functions, we can formulate the following existence, uniqueness and regularity result (see, e.g., [27]).

Theorem 1. *Let $\alpha \in (0,1)$ and $a, f \in C[0,b]$. Then (1)–(2) has a unique solution $y \in C[0,b]$. Moreover, if $a, f \in C^{q,\mu}(0,b]$, $q \in \mathbb{N}$, $\mu < 1$, then $y \in C^{q,\nu}(0,b]$, where*

$$\nu = \max\{1-\alpha, \mu\}.$$

Note that problem (1)–(2) can be reformulated as a Volterra integral equation. Indeed, let y be the exact solution of (1)–(2). Then (cf. [6])

$$y(t) = (J^\alpha z)(t) + y_0, \quad 0 \le t \le b,$$

where $z(t) := (D^\alpha_{Cap} y)(t)$, $0 \le t \le b$. Since $z(t) = f(t) - a(t)y(t)$, we obtain

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s) - a(s)y(s)] ds, \quad 0 \le t \le b, \tag{4}$$

or, in operator form,

$$y = Ty + g, \tag{5}$$

where

$$(Ty)(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} a(s) y(s) ds, \quad 0 \le t \le b, \tag{6}$$

$$g(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad 0 \le t \le b. \tag{7}$$

Clearly, operator T is compact as an operator from $L_\infty(0,b)$ to $C[0,b]$ and $g \in C[0,b]$. Thus, if $y \in C[0,b]$ is the exact solution of (1)–(2), then y satisfies the integral equation (5). It is easy to see that the converse also holds: if $y \in C[0,b]$ is a solution of (5) (that is, y satisfies (4)), then y is also a solution of (1)–(2).

Thus, to find a numerical solution to (1)–(2), it is sufficient to construct an approximation method for the integral equation (5).

2. Collocation method

In order to take into account the possible non-smoothness of the exact solution $y = y(t)$ of (5) at the origin $t = 0$, we introduce on the interval $[0, b]$ a graded grid $\Pi_N = \{t_0, \dots, t_N\}$. More exactly, for given $N \in \mathbb{N}$, we divide the underlying interval of integration $[0, b]$ into N subintervals $[t_{j-1}, t_j]$, $j = 1, \dots, N$, with grid points

$$t_j = b\left(\frac{j}{N}\right)^r, \; j = 0, \dots, N. \tag{8}$$

The real number $r \geq 1$ characterizes the non-uniformity of the grid. If $r = 1$, then the grid is uniform; if $r > 1$, then the grid points are more densely clustered near the left endpoint 0.

Next, for a given integer $k \geq 0$, we introduce

$$S_k^{(-1)}(\Pi_N) := \big\{u : u\big|_{[t_{j-1},t_j]} \in \pi_k, \, j = 1, \dots, N\big\}.$$

Here $u\big|_{[t_{j-1},t_j]}$ is the restriction of $u : [0, b] \to \mathbb{R}$ onto the subinterval $[t_{j-1}, t_j] \subset [0, b]$ and π_k denotes the set of polynomials of degree not exceeding k. Note that the elements of $S_k^{(-1)}(\Pi_N)$ may have jump discontinuities at the interior points $t_1, \dots, t_{N-1}$ of the grid Π_N. We choose $m \in \mathbb{N}$ points $\eta_1, \dots, \eta_m$ in the interval $[0, 1]$ so that

$$0 \leq \eta_1 < \dots < \eta_m \leq 1 \tag{9}$$

and define in every subinterval $[t_{j-1}, t_j] \subset [0, b]$ the collocation points

$$t_{jp} = t_{j-1} + \eta_p(t_j - t_{j-1}), \quad p = 1, \dots, m, \quad j = 1, \dots, N. \tag{10}$$

We find approximations $y_N \in S_{m-1}^{(-1)}(\Pi_N)$ to the solution y of (5) by assuming that the following conditions hold:

$$y_N(t_{jp}) = (Ty_N)(t_{jp}) + g(t_{jp}), \quad p = 1, \dots, m, \quad j = 1, \dots, N, \tag{11}$$

where T and g are defined by (6) and (7), respectively. If $\eta_1 = 0$, then by $y_N(t_{j1})$ we denote the right limit $\lim_{t \to t_{j-1}, t > t_{j-1}} y_N(t)$. If $\eta_m = 1$, then by $y_N(t_{jm})$ we denote the left limit $\lim_{t \to t_j, t < t_j} y_N(t)$. The collocation conditions (11) have an operator equation representation

$$y_N = \mathcal{P}_{N,m} T y_N + \mathcal{P}_{N,m} g. \tag{12}$$

Here $\mathcal{P}_{N,m} : C[0, b] \to S_{m-1}^{(-1)}(\Pi_N)$ is defined by

$$(\mathcal{P}_{N,m}u)(t_{jp}) = u(t_{jp}), \quad p = 1, \dots, m, \; j = 1, \dots, N, \quad u \in C[0, b], \tag{13}$$

where Π_N and $\{t_{jp}\}$ are given by (8) and (10), respectively. If $\eta_1 = 0$, then by $(\mathcal{P}_{N,m}u)(t_{j1})$ we denote the right limit $\lim_{t \to t_{j-1}, t > t_{j-1}} (\mathcal{P}_{N,m}u)(t)$. If $\eta_m = 1$, then by $(\mathcal{P}_{N,m}u)(t_{jm})$ we denote the left limit $\lim_{t \to t_j, t < t_j} (\mathcal{P}_{N,m}u)(t)$. It follows from [4, 24] that $\mathcal{P}_{N,m} \in \mathcal{L}(C[0, b], L_\infty(0, b))$ and that the norms of

$\mathcal{P}_{N,m}$ are uniformly bounded: there exists a constant $c > 0$ which does not depend on N such that $\|\mathcal{P}_{N,m}\|_{\mathcal{L}(C[0,b],L_\infty(0,b))} \le c$ for all $N \in \mathbb{N}$.

The collocation conditions (11) form a system of equations whose exact form is determined by the choice of a basis in the space $S_{m-1}^{(-1)}(\Pi_N)$. We will use the Lagrange fundamental polynomial representation:

$$y_N(t) = \sum_{l=1}^{N} \sum_{k=1}^{m} c_{lk} \varphi_{lk}(t), \quad t \in [0,b], \tag{14}$$

where

$$\varphi_{lk}(t) = \begin{cases} 0, & \text{if} \quad t \notin [t_{l-1}, t_l], \\ \prod\limits_{i=1, i\neq k}^{m} \frac{t - t_{li}}{t_{lk} - t_{li}}, & \text{if} \quad t \in [t_{l-1}, t_l], \end{cases} \qquad k = 1, \dots, m, \quad l = 1, \dots, N.$$

Note that $y_N(t_{lk}) = c_{lk}$, $k = 1, \dots, m$, $l = 1, \dots, N$, and using representation (14) the collocation conditions (11) form the following system of linear algebraic equations with respect to the unknown values $\{c_{jp}\}$:

$$c_{jp} = \sum_{l=1}^{N} \sum_{k=1}^{m} (T\varphi_{lk})(t_{jp}) c_{lk} + g(t_{jp}), \quad p = 1, \dots, m, \quad j = 1, \dots, N.$$

After solving this system of equations, we find the approximation y_N to y, the solution of (1)–(2), by the formula (14).

Note that this algorithm can be used also in the case if $\eta_1 = 0$ and $\eta_m = 1$. In this case we have that $t_j = t_{jm} = t_{j+1,1}$, $c_{jm} = c_{j+1,1} = y_N(t_j)$ and hence in the system with respect to $\{c_{jp}\}$ there are $(m-1)N + 1$ equations and unknowns.

It follows from [27] that the following convergence result holds (cf. [3]).

Theorem 2. (***i***) *Let $\alpha \in (0,1)$ and $a, f \in C[0,b]$. Let $N, m \in \mathbb{N}$ and assume that the grid points* (8) *and the collocation points* (10) *(with collocation parameters $\eta_1, \dots, \eta_m$ satisfying* (9)*) are used.*

Then there exists an integer N_0 such that for all $N \ge N_0$ equation (12) *possesses a unique solution $y_N \in S_{m-1}^{(-1)}(\Pi_N)$ and*

$$\sup_{0 \le t \le b} |y(t) - y_N(t)| \to 0 \quad \text{as} \quad N \to \infty,$$

where y is the solution of (1)–(2).

(***ii***) *If, in addition to (i), we assume that $a, f \in C^{m,\mu}(0,b]$, where $\mu < 1$, then for all $N \ge N_0$ the following error estimate holds:*

$$\sup_{0 \le t \le b} |y(t) - y_N(t)| \le c \begin{cases} N^{-r(1-\nu)}, & \text{if } 1 \le r \le \frac{m}{1-\nu}, \\ N^{-m}, & \text{if } r \ge \frac{m}{1-\nu}. \end{cases} \tag{15}$$

Here $\nu = \max\{1-\alpha, \mu\}$, $r \ge 1$ is the grading parameter of the grid (8) *and c is a constant independent of N.*

3. Iterated approximations

Theorem 2 shows that if functions a and f are smooth enough, and if the grading parameter r is chosen to be sufficiently large, the expected convergence order of the proposed numerical method is of order $O(N^{-m})$. This convergence order can be improved by using an iterated method, assuming a little more smoothness of a and f, together with a more precise choice of collocation parameters $\eta_1, \ldots, \eta_m$. More precisely, let y_N be the solution of (12), with $N \geq N_0$ (see Theorem 2). Denote

$$y_N^{it} := Ty_N + g, \quad N \geq N_0, \tag{16}$$

where T and g are defined by (6) and (7), respectively. Note that $y_N^{it} \in C[0,b]$.

To study the convergence properties of the iterated method, we prove Lemma 3 below. To this end we will use some ideas of [18] and Lemmas 1 and 2 (see [24] and [17], respectively).

Lemma 1. *Let* $u \in C^{m,\nu}(0,b]$, $m \in \mathbb{N}$ *and* $\nu \in (0,1)$. *Let* $N \in \mathbb{N}$ *and let* $\mathcal{P}_{N,m} : C[0,b] \to S_{m-1}^{(-1)}(\Pi_N)$ *be defined by the formula* (13). *Then*

$$\sup_{t_{j-1}<t<t_j} |u(t) - (\mathcal{P}_{N,m}u)(t)| \leq c\,(t_j - t_{j-1})^m t_j^{1-\nu-m}, \; j = 1, ..., N.$$

Here the positive constant c *is independent of* j *and* N.

Lemma 2. *Let* $\gamma < 0$ *and* β *be real numbers and let* $N \geq 2$ *be a natural number. Then, for all* $l \in \mathbb{N}$ *satisfying* $2 \leq l \leq N$, *the following estimate holds:*

$$\sum_{j=1}^{l-1} j^\beta (l-j)^\gamma \leq c \begin{cases} 1, & \text{if } \beta + \gamma < -1 \text{ and } \beta < 0, \\ N^\beta, & \text{if } \beta \geq 0 \text{ and } \gamma < -1, \\ N^{\beta+\gamma+1}, & \text{if } \beta + \gamma \geq -1 \text{ and } \gamma > -1, \end{cases}$$

where c *is a positive constant that does not depend on* l *and* N.

Lemma 3. *Let* $y \in C^{m+1,\nu}(0,b]$, $a \in C^{1,\mu}(0,b]$, *where* $m \in \mathbb{N}$, $\nu \in (0,1)$ *and* $\mu < 1$. *Let* $N \in \mathbb{N}$ *and assume that the grid points* (8) *and the collocation parameters* $\eta_1, \ldots, \eta_m$ *satisfying* (9) *are used. Moreover, assume that the collocation parameters are chosen so that the quadrature approximation*

$$\int_0^1 F(x)\,dx \approx \sum_{k=1}^m w_k\, F(\eta_k) \tag{17}$$

with appropriate weights $\{w_k\}$ *is exact for all polynomials* F *of degree* m. *Finally, let* $\alpha \in (0,1)$ *and let* J^α *and* $\mathcal{P}_{N,m}$ *be defined by* (3) *and* (13), *respectively. Then*

$$\|J^\alpha(a(y - \mathcal{P}_{N,m}y))\|_\infty \leq c \begin{cases} N^{-r(1+\alpha-\nu)}, & \text{if } 1 \leq r < \frac{m+\alpha}{1+\alpha-\nu}, \\ N^{-m-\alpha}, & \text{if } r \geq \frac{m+\alpha}{1+\alpha-\nu}. \end{cases} \tag{18}$$

Here $r \geq 1$ is the grading parameter of the grid (8) *and c is a constant independent of N.*

Proof. Without loss of generality we assume that $a \in C^{1,\mu}(0,b]$ where $\mu \in (0,1)$ (if $\mu \leq 0$, then $a \in C^{1,\mu}(0,b]$ implies $a \in C^{1,\varepsilon}(0,b]$ for all $0 < \varepsilon < 1$). Fix $t \in (0,b]$ and let $k \in \{0,1,...,N-1\}$ be such that $t \in (t_k, t_{k+1}]$. Let

$$A_j(t) := \int_{t_{j-1}}^{t_j} (t-s)^{\alpha-1} a(s)(y(s) - (\mathcal{P}_{N,m}y)(s))ds, \; j = 1,...,k \; (k \geq 1), \tag{19}$$

$$A_{k+1}(t) := \int_{t_k}^{t} (t-s)^{\alpha-1} a(s)(y(s) - (\mathcal{P}_{N,m}y)(s))ds.$$

Then

$$\left(J^\alpha(a(y - \mathcal{P}_{N,m}y))\right)(t) = \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{k+1} A_j(t). \tag{20}$$

Throughout this proof we use the notation $c, c_1, \ldots$ to denote positive constants that can in different places have different values, but which do not depend on N, k and t. Let

$$h_j := t_j - t_{j-1}, \quad j = 1,...,k+1.$$

Since $y \in C^{m+1,\nu}(0,b] \subset C^{m,\nu}(0,b]$ and $a \in C^{1,\mu}(0,b] \subset C[0,b]$, it follows from Lemma 1 that

$$|A_{k+1}(t)| \leq c_1 \, h_{k+1}^{m+\alpha} t_{k+1}^{1-\nu-m}.$$

Due to

$$t_j = bj^r N^{-r}, \quad 0 < h_j = t_j - t_{j-1} \leq brj^{r-1}N^{-r}, \; j = 1,...,N, \tag{21}$$

we obtain

$$\begin{aligned} |A_{k+1}(t)| &\leq c_2 N^{-r(1+\alpha-\nu)}(k+1)^{r(1+\alpha-\nu)-(m+\alpha)} \\ &\leq c \begin{cases} N^{-r(1+\alpha-\nu)}, & \text{if } 1 \leq r \leq \frac{m+\alpha}{1+\alpha-\nu}, \\ N^{-m-\alpha}, & \text{if } \; r > \frac{m+\alpha}{1+\alpha-\nu}. \end{cases} \end{aligned} \tag{22}$$

Next, let $k \geq 1$. Then, on the basis of (19) we can write

$$A_k(t) = \int_{t_{k-1}}^{t_k} (t-s)^{\alpha-1} a(s)(y(s) - (\mathcal{P}_{N,m}y)(s))ds$$

and, due to Lemma 1 and $a \in C[0,b]$, we have that

$$|A_k(t)| \leq c_1 h_k^m t_k^{1-\nu-m} (t - t_{k-1})^\alpha.$$

Since

$$t - t_{k-1} \leq h_k + h_{k+1} = \left(1 + \frac{h_{k+1}}{h_k}\right) h_k$$

and $h_{k+1}/h_k \to 1$, as $k \to \infty$, we have that there exists a constant $c_2 > 0$ (which is independent of k) such that $t - t_{k-1} \le c_2 h_k$. Using this, we obtain in a similar way as for the estimate $|A_{k+1}(t)|$ that

$$|A_k(t)| \le c \begin{cases} N^{-r(1+\alpha-\nu)}, & \text{if } 1 \le r \le \frac{m+\alpha}{1+\alpha-\nu}, \\ N^{-m-\alpha}, & \text{if } r > \frac{m+\alpha}{1+\alpha-\nu}. \end{cases} \tag{23}$$

If $k \ge 2$, we still need to estimate $A_j(t)$, $j = 1, ..., k-1$. Let us consider the following function ψ_t:

$$\psi_t(s) = a(s)(t-s)^{\alpha-1}, \ 0 \le s < t, \tag{24}$$

where $t \in (0, b]$ was fixed earlier. In every subinterval (t_{j-1}, t_j), $j = 1, ..., k-1$, it holds that

$$\psi_t(s) = \psi_t(t_j) + \psi_t'(\xi(s))(s - t_j), \quad s \in (t_{j-1}, t_j), \quad \xi(s) \in (s, t_j).$$

Let

$$B_0(t) := \sum_{j=1}^{k-1} \psi_t(t_j) \int_{t_{j-1}}^{t_j} (y(s) - (\mathcal{P}_{N,m} y)(s)) ds,$$

$$B_1(t) := \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} \psi_t'(\xi(s))(s - t_j)(y(s) - (\mathcal{P}_{N,m} y)(s)) ds.$$

Then

$$\sum_{j=1}^{k-1} A_j(t) = B_0(t) + B_1(t).$$

We now introduce a new collocation parameter $\eta_{m+1} \in [0,1]$ such that $\eta_{m+1} \neq \eta_p$, $p = 1, \dots, m$. Without loss of generality we can assume that $0 \le \eta_1 < \cdots < \eta_{m+1} \le 1$. Using $\eta_1, \dots, \eta_{m+1}$ we also define new collocation points

$$t_{jl} = t_{j-1} + \eta_l (t_j - t_{j-1}), \ j = 1, ..., N, \ l = 1, ..., m+1,$$

and introduce an operator $\mathcal{P}_{N,m+1} : C[0,b] \to S_m^{-1}(\Pi_N)$ by conditions (cf. (13))

$$(\mathcal{P}_{N,m+1} u)(t_{jl}) = u(t_{jl}), \quad j = 1, ..., N, \quad l = 1, ..., m+1, \quad u \in C[0,b].$$

Since the quadrature approximation (17) is exact for all polynomials of degree m, we have

$$\int_{t_{j-1}}^{t_j} (\mathcal{P}_{N,m} u)(s) ds = \int_{t_{j-1}}^{t_j} (\mathcal{P}_{N,m+1} u)(s) ds, \ j = 1, ...k-1, \ u \in C[0,b].$$

Thus, for all $u \in C[0,b]$ and $j = 1, ..., k-1$, we have

$$\int_{t_{j-1}}^{t_j} (u(s) - (\mathcal{P}_{N,m} u)(s)) ds = \int_{t_{j-1}}^{t_j} (u(s) - (\mathcal{P}_{N,m+1} u)(s)) ds.$$

Due to $t - t_j \geq t_k - t_j \geq (k-j)h_j$, $j = 1, ..., k-1$ and $a \in C[0,b]$, we have that $|\psi_t(t_j)| \leq c_1(k-j)^{\alpha-1}h_j^{\alpha-1}$, and therefore

$$
\begin{aligned}
|B_0(t)| &\leq c_1 \sum_{j=1}^{k-1} (k-j)^{\alpha-1} h_j^{\alpha-1} \left| \int_{t_{j-1}}^{t_j} (y(s) - (\mathcal{P}_{N,m}y)(s))\, ds \right| \\
&= c_1 \sum_{j=1}^{k-1} (k-j)^{\alpha-1} h_j^{\alpha-1} \left| \int_{t_{j-1}}^{t_j} (y(s) - (\mathcal{P}_{N,m+1}y)(s))\, ds \right| .
\end{aligned}
$$

Now, by applying Lemma 1 with operator $\mathcal{P}_{N,m+1}$ instead of the operator $\mathcal{P}_{N,m}$, we obtain

$$
|B_0(t)| \leq c_2 \sum_{j=1}^{k-1} (k-j)^{\alpha-1} h_j^{\alpha+m+1} t_j^{-\nu-m}.
$$

From (21) it follows

$$
|B_0(t)| \leq c_3 N^{-r(1+\alpha-\nu)} \sum_{j=1}^{k-1} (k-j)^{\alpha-1} j^{r(1+\alpha-\nu)-(m+\alpha+1)}.
$$

Since

$$
\sum_{j=1}^{k-1} (k-j)^{\alpha-1} j^{r(1+\alpha-\nu)-(m+\alpha+1)} \leq \sum_{j=1}^{k-1} (k-j)^{\alpha-1} j^{r(1+\alpha-\nu)-(m+\alpha+1)} j^{-\alpha+1}, \tag{25}
$$

it follows from Lemma 2 that

$$
|B_0(t)| \leq c \begin{cases} N^{-r(1+\alpha-\nu)}, & \text{if } 1 \leq r < \frac{m+\alpha}{1+\alpha-\nu}, \\ N^{-m-\alpha}, & \text{if } r \geq \frac{m+\alpha}{1+\alpha-\nu}. \end{cases} \tag{26}
$$

Next we estimate $B_1(t)$. It follows from (24) and $a \in C^{1,\mu}(0,b]$, $0 < \mu < 1$, that

$$
\psi_t'(s) = a'(s)(t-s)^{\alpha-1} - (\alpha-1)a(s)(t-s)^{\alpha-2}, \quad 0 < s < t.
$$

Since $t - s \geq t - t_j \geq (k-j)h_j$ and $|a'(\xi(s))| \leq c\, s^{-\mu}$ $(0 < s < \xi(s) < t)$, then there exists $c_1 > 0$ such that

$$
|\psi_t'(\xi(s))| \leq c_1 \left((k-j)^{\alpha-1} h_j^{\alpha-1} s^{-\mu} + (k-j)^{\alpha-2} h_j^{\alpha-2} \right), \quad \xi(s) \in (s, t_j).
$$

Therefore

$$
|B_1(t)| \leq c_1 \sum_{j=1}^{k-1} (k-j)^{\alpha-1} h_j^{\alpha-1} \int_{t_{j-1}}^{t_j} s^{-\mu} (t_j - s) |y(s) - (\mathcal{P}_{N,m}y)(s)|\, ds
$$

$$+ c_1 \sum_{j=1}^{k-1} (k-j)^{\alpha-2} h_j^{\alpha-2} \int_{t_{j-1}}^{t_j} (t_j - s)|y(s) - (\mathcal{P}_{N,m}y)(s)|\, ds. \quad (27)$$

Due to Lemma 1 we see that

$$S_1 := c_1 \sum_{j=1}^{k-1} (k-j)^{\alpha-1} h_j^{\alpha-1} \int_{t_{j-1}}^{t_j} s^{-\mu}(t_j - s)|y(s) - (\mathcal{P}_{N,m}y)(s)|\, ds$$
$$\leq c_2 \sum_{j=1}^{k-1} (k-j)^{\alpha-1} h_j^{\alpha-1+m} t_j^{1-\nu-m} \int_{t_{j-1}}^{t_j} s^{-\mu}(t_j - s) ds.$$

Note that for $j = 1$ we have

$$\int_0^{t_1} s^{-\mu}(t_1 - s) ds = t_1^{2-\mu} \int_0^1 s^{-\mu}(1-s) ds \leq c_3 t_1^{2-\mu} = c_3 h_1^2 t_1^{-\mu}$$

and for $j \geq 2$ we obtain

$$\int_{t_{j-1}}^{t_j} s^{-\mu}(t_j - s) ds = h_j^2 \int_0^1 (h_j s + t_{j-1})^{-\mu}(1-s) ds$$
$$\leq h_j^2 t_{j-1}^{-\mu} \int_0^1 (1-s) ds \leq c_4 h_j^2 t_j^{-\mu}.$$

Therefore, with the help of (21), we see that

$$S_1 \leq c_5 \sum_{j=1}^{k-1} (k-j)^{\alpha-1} h_j^{\alpha-1+m} h_j^2 t_j^{1-\nu-m-\mu}$$
$$\leq c_6 N^{-r(1+\alpha-\nu)} \sum_{j=1}^{k-1} (k-j)^{\alpha-1} j^{r(1+\alpha-\nu)-(1+\alpha+m)} \left(\frac{j}{N}\right)^{r(1-\mu)}$$
$$\leq c_6 N^{-r(1+\alpha-\nu)} \sum_{j=1}^{k-1} (k-j)^{\alpha-1} j^{r(1+\alpha-\nu)-(1+\alpha+m)}.$$

Due to (25) and Lemma 2 we now obtain

$$S_1 \leq c \begin{cases} N^{-r(1+\alpha-\nu)}, & \text{if } 1 \leq r < \frac{m+\alpha}{1+\alpha-\nu}, \\ N^{-m-\alpha}, & \text{if } r \geq \frac{m+\alpha}{1+\alpha-\nu}. \end{cases} \quad (28)$$

Finally, we estimate the remaining terms in (27). Due to Lemmas 1 and 2 it follows

$$S_2 := c_1 \sum_{j=1}^{k-1} (k-j)^{\alpha-2} h_j^{\alpha-2} \int_{t_{j-1}}^{t_j} (t_j - s)|y(s) - (\mathcal{P}_{N,m}y)(s)|\, ds$$

$$\leq c_2 N^{-r(1+\alpha-\nu)} \sum_{j=1}^{k-1} (k-j)^{\alpha-2} j^{r(1+\alpha-\nu)-(\alpha+m)}$$

$$\leq c \begin{cases} N^{-r(1+\alpha-\nu)}, & \text{if } 1 \leq r < \frac{m+\alpha}{1+\alpha-\nu}, \\ N^{-m-\alpha}, & \text{if } r \geq \frac{m+\alpha}{1+\alpha-\nu}. \end{cases} \tag{29}$$

Thus, from (28) and (29) we get the estimate

$$|B_1(t)| \leq S_1 + S_2 \leq c \begin{cases} N^{-r(1+\alpha-\nu)}, & \text{if } 1 \leq r < \frac{m+\alpha}{1+\alpha-\nu}, \\ N^{-m-\alpha}, & \text{if } r \geq \frac{m+\alpha}{1+\alpha-\nu}. \end{cases} \tag{30}$$

Since

$$\sum_{j=1}^{k+1} A_j(t) = B_0(t) + B_1(t) + A_k(t) + A_{k+1}(t),$$

the equality (20) along with the estimates (22), (23), (26) and (30) yields the estimate (18). □

Based on Theorem 1, Theorem 2 and Lemma 3, we can now prove the following result.

Theorem 3. *Let $\alpha \in (0,1)$ and $m \in \mathbb{N}$. Assume that $a, f \in C^{m+1,\mu}(0,b]$, where $\mu < 1$. Let $N \in \mathbb{N}$ and assume that the grid points* (8) *and the collocation points $\eta_1, \dots, \eta_m$ satisfying* (9) *are used. Moreover, assume that the collocation parameters are chosen so that the quadrature approximation*

$$\int_0^1 F(x)\,dx \approx \sum_{k=1}^{m} w_k\, F(\eta_k)$$

with appropriate weights $\{w_k\}$ is exact for all polynomials F of degree m.

Then there exists $N_0 \in \mathbb{N}$ so that for $N \geq N_0$ the approximation y_N^{it} defined by (16) *is unique and the error estimate*

$$\|y - y_N^{it}\|_\infty \leq c \begin{cases} N^{-r(1+\alpha-\nu)}, & \text{if } 1 \leq r < \frac{m+\alpha}{1+\alpha-\nu}, \\ N^{-m-\alpha}, & \text{if } r \geq \frac{m+\alpha}{1+\alpha-\nu}, \end{cases} \tag{31}$$

holds. Here y is the exact solution of problem (1)–(2), *$\nu = \max\{1-\alpha, \mu\}$, $r \geq 1$ is the grading parameter of the grid* (8) *and c is a constant independent of N.*

Proof. It follows from Theorem 2 that there exists an $N_0 \in \mathbb{N}$ such that the iterated approximate solution y_N^{it} is defined for $N \geq N_0$. Note that due to $y_N = \mathcal{P}_{N,m} T y_n + \mathcal{P}_{N,m} g$ we have that

$$\mathcal{P}_{N,m} y_N^{it} = \mathcal{P}_{N,m}(T y_N + g) = \mathcal{P}_{N,m} T y_N + \mathcal{P}_{N,m} g = y_N, \quad N \geq N_0.$$

Therefore,

$$y_N^{it} = T \mathcal{P}_{N,m} y_N^{it} + g, \quad N \geq N_0.$$

From Theorem 2, we know that there exists the inverse $(I-\mathcal{P}_{N,m}T)^{-1}$ when $N \geq N_0$. One can verify that if $N \geq N_0$, then there also exists the inverse $(I-T\mathcal{P}_{N,m})^{-1}$ and

$$(I-T\mathcal{P}_{N,m})^{-1} = I + T(I-\mathcal{P}_{N,m}T)^{-1}\mathcal{P}_{N,m}. \tag{32}$$

Due to $T \in \mathcal{L}(L_\infty(0,b), C[0,b])$ and $\mathcal{P}_{N,m} \in \mathcal{L}(C[0,b], L_\infty(0,b))$, we have that $(I-\mathcal{P}_{N,m}T) \in \mathcal{L}(L_\infty(0,b), L_\infty(0,b))$. Therefore also $(I-\mathcal{P}_{N,m}T)^{-1} \in \mathcal{L}(L_\infty(0,b), L_\infty(0,b))$, when $N \geq N_0$, and, consequently, $(I-T\mathcal{P}_{N,m})^{-1} \in \mathcal{L}(C[0,b], C[0,b])$, when $N \geq N_0$.

It can be shown (see, e.g., [27]) that

$$\left\|(I-\mathcal{P}_{N,m}T)^{-1}\right\|_{\mathcal{L}(L_\infty(0,b),L_\infty(0,b))} \leq c_1, \quad N \geq N_0, \tag{33}$$

where c_1 is a positive constant which does not depend on N. Therefore, since the norms of $\mathcal{P}_{N,m}$ are uniformly bounded (see Section 2), we get from (32) and (33) that there exists a constant $c_2 > 0$ (independent of N) such that

$$\left\|(I-T\mathcal{P}_{N,m})^{-1}\right\|_{\mathcal{L}(C[0,b],C[0,b])} \leq c_2, \quad N \geq N_0. \tag{34}$$

Note that due to $y = Ty + g$ and $y_N^{it} = T\mathcal{P}_{N,m}y_N^{it} + g$, it follows that

$$(I-T\mathcal{P}_{N,m})\left(y_N^{it} - y\right) = T(\mathcal{P}_{N,m}y - y).$$

Therefore, from (34) it follows that

$$\left\|y_N^{it} - y\right\|_\infty \leq c_2\|T(\mathcal{P}_{N,m}y - y)\|_\infty = c_2\left\|J^\alpha(a(y-\mathcal{P}_{N,m}y))\right\|_\infty, \quad N \geq N_0.$$

This together with Lemma 3 yields the estimate (31). □

Remark 1. In a more restrictive case of $a, f \in C^{m+1}[0,b]$ we obtain from Theorem 3, by taking $\nu = 1-\alpha$, that

$$\|y - y_N^{it}\|_\infty \leq c \begin{cases} N^{-2\alpha r}, & \text{if } 1 \leq r < \frac{m+\alpha}{2\alpha}, \\ N^{-m-\alpha}, & \text{if } r \geq \frac{m+\alpha}{2\alpha}. \end{cases}$$

This result for $r = 1$ and $r \geq \frac{m}{\alpha} > \frac{m+\alpha}{2\alpha}$ is also obtained in [28].

4. Examples

4.1. Example 1.

Consider the following initial value problem:

$$(D_{Cap}^{\frac{1}{2}}y)(t) + (t^{\frac{3}{4}} - 1)y(t) = t^{\frac{5}{4}} + t^{\frac{3}{4}} - t^{\frac{1}{2}} + \Gamma\left(\frac{3}{2}\right) - 1, \quad 0 \leq t \leq 1; \quad y(0) = 1. \tag{35}$$

We see that (35) is a special problem of (1)–(2) with

$$\alpha = \frac{1}{2},\ a(t) = t^{\frac{3}{4}} - 1,\ f(t) = t^{\frac{5}{4}} + t^{\frac{3}{4}} - t^{\frac{1}{2}} + \Gamma\left(\frac{3}{2}\right) - 1,\ y_0 = 1,\ b = 1.$$

Note that $a \in C^{m,\frac{1}{4}}(0,1] \subset C^{m,\frac{1}{2}}(0,1]$, $f \in C^{m,\frac{1}{2}}(0,1]$ for all $m \in \mathbb{N}$. Therefore, by Theorem 1, it follows that the solution $y \in C^{m,\nu}(0,b]$, where

$$\nu = \max\left\{1-\frac{1}{2},\frac{1}{2}\right\} = \frac{1}{2}.$$

We apply the collocation method (see Section 2) and its iterated version (see Section 3) with $m = 2$, using as collocation parameters the shifted Gauss–Legendre points

$$\eta_1 = \frac{3-\sqrt{3}}{6},\ \eta_2 = 1-\eta_1,$$

which satisfy the conditions set for collocation parameters in Theorem 2 and Theorem 3. It follows from Theorem 2 that for sufficiently large $N \in \mathbb{N}$ we have

$$\sup_{0\le t\le b}|y(t)-y_N(t)| \le c_0 \begin{cases} N^{-0.5\,r}, & \text{if} \quad 1 \le r < 4, \\ N^{-2}, & \text{if} \quad r \ge 4, \end{cases} \tag{36}$$

where

$$y(t) = t^{\frac{1}{2}} + 1, \quad 0 \le t \le 1,$$

is the exact solution of problem (35) and c_0 is a positive constant independent of N. Similarly, it follows from Theorem 3 that

$$\|y-y_N^{it}\|_\infty = \max_{0\le t\le b}|y(t)-y_N^{it}(t)| \le c_1 \begin{cases} N^{-r}, & \text{if} \quad 1 \le r < \frac{5}{2}, \\ N^{-2.5}, & \text{if} \quad r \ge \frac{5}{2}, \end{cases} \tag{37}$$

where c_1 is a positive constant independent of N.

In Table 1 and Table 2 some results of numerical experiments for different values of the parameter r are presented. The actual numerical error ε_N for $\sup_{0\le t\le b}|y(t)-y_N(t)|$ and ε_N^{it} for $\|y-y_N^{it}\|_\infty$ are calculated as follows:

$$\varepsilon_N := \max_{j=1,\dots,N}\ \max_{k=0,\dots,10} |y(\tau_{jk}) - y_N(\tau_{jk})|,$$

$$\varepsilon_N^{it} := \max_{j=1,\dots,N}\ \max_{k=0,\dots,10} |y(\tau_{jk}) - y_N^{it}(\tau_{jk})|,$$

where

$$\tau_{jk} := t_{j-1} + k(t_j - t_{j-1})/10,\ k = 0,\dots,10,\ j = 1,\dots,N,$$

with $\{t_j\}$ defined by (8). The ratios

$$\varrho_N := \frac{\varepsilon_{N/2}}{\varepsilon_N}, \quad \varrho_N^{it} := \frac{\varepsilon_{N/2}^{it}}{\varepsilon_N^{it}},$$

characterizing the observed convergence rates, are also presented. Due to (36), the ratios ϱ_N for the non-iterated collocation method for $r = 1$, $r = 2$ and $r = 4$ ought to be approximately $2^{0.5} \approx 1.41$, $2^1 = 2$ and $2^2 = 4$, respectively. Due to (37), the ratios ϱ_N^{it} for the iterated method for $r = 1$, $r = 2$ and $r = 2.5$ ought to be approximately $2^1 = 2$, $2^2 = 4$ and $2^{2.5} \approx 5.67$, respectively. All these ratios are also given in the last rows of Table 1 and

Table 2. As we can see from Table 1 and Table 2, the numerical results are in good agreement with the theoretical estimates given by Theorem 2 and Theorem 3 (the estimates (15) and (31)).

TABLE 1. Numerical results for problem (35) using collocation method with $m = 2$, $\eta_1 = \frac{3-\sqrt{3}}{6}$, $\eta_2 = 1 - \eta_1$.

	$r = 1$		$r = 2$		$r = 4$	
N	ε_N	ϱ_N	ε_N	ϱ_N	ε_N	ϱ_N
4	$1.62 \cdot 10^{-1}$		$7.81 \cdot 10^{-2}$		$3.12 \cdot 10^{-2}$	
8	$1.12 \cdot 10^{-1}$	1.44	$3.84 \cdot 10^{-2}$	2.03	$7.19 \cdot 10^{-3}$	4.34
16	$7.81 \cdot 10^{-2}$	1.43	$1.91 \cdot 10^{-2}$	2.02	$1.87 \cdot 10^{-3}$	3.84
32	$5.47 \cdot 10^{-2}$	1.43	$9.50 \cdot 10^{-3}$	2.01	$4.68 \cdot 10^{-4}$	4.01
64	$3.84 \cdot 10^{-2}$	1.42	$4.74 \cdot 10^{-3}$	2.00	$1.17 \cdot 10^{-4}$	4.00
128	$2.71 \cdot 10^{-2}$	1.42	$2.37 \cdot 10^{-3}$	2.00	$2.92 \cdot 10^{-5}$	4.00
		≈ 1.41		2		4

TABLE 2. Numerical results for problem (35) using the iterated method with $m = 2$, $\eta_1 = \frac{3-\sqrt{3}}{6}$, $\eta_2 = 1 - \eta_1$.

	$r = 1$		$r = 2$		$r = 2.5$	
N	ε_N^{it}	ϱ_N^{it}	ε_N^{it}	ϱ_N^{it}	ε_N^{it}	ϱ_N^{it}
4	$1.05 \cdot 10^{-2}$		$2.48 \cdot 10^{-3}$		$2.07 \cdot 10^{-3}$	
8	$5.09 \cdot 10^{-3}$	2.07	$6.00 \cdot 10^{-4}$	4.13	$3.59 \cdot 10^{-4}$	5.77
16	$2.48 \cdot 10^{-3}$	2.05	$1.48 \cdot 10^{-4}$	4.07	$6.23 \cdot 10^{-5}$	5.76
32	$1.22 \cdot 10^{-3}$	2.04	$3.66 \cdot 10^{-5}$	4.03	$1.09 \cdot 10^{-5}$	5.71
64	$6.00 \cdot 10^{-4}$	2.03	$9.10 \cdot 10^{-6}$	4.02	$1.92 \cdot 10^{-6}$	5.68
128	$2.97 \cdot 10^{-4}$	2.02	$2.27 \cdot 10^{-6}$	4.01	$3.39 \cdot 10^{-7}$	5.67
		2		4		≈ 5.67

We also briefly highlight the case when the chosen collocation parameters do not satisfy the quadrature condition set in Theorem 3, by applying both methods for $m = 2$ with collocation parameters

$$\eta_1 = 0.1,\ \eta_2 = 0.9.$$

The obtained results for the collocation method and its iterated version are shown in Table 3 and Table 4, respectively. The last row of Table 3 shows the theoretical estimates of the collocation method for $r = 1$, $r = 2$ and $r = 4$, respectively (see Theorem 2). Correspondingly, the last row of Table 4 shows the theoretical estimates of the iterated version for $r = 1$, $r = 2$ and $r = 2.5$, respectively (see Theorem 3). We see that for these collocation parameters for $r \geq 2.5$ the iterated method does not attain the convergence order $O(N^{-2.5})$ predicted by Theorem 3. This shows that the collocation parameter assumption of Theorem 3 can not be relaxed.

TABLE 3. Numerical results for problem (35) using the collocation method with $m = 2$, $\eta_1 = 0.1$, $\eta_2 = 0.9$.

	$r = 1$		$r = 2$		$r = 4$	
N	ε_N	ϱ_N	ε_N	ϱ_N	ε_N	ϱ_N
4	$1.25 \cdot 10^{-1}$		$6.07 \cdot 10^{-2}$		$2.18 \cdot 10^{-2}$	
8	$8.68 \cdot 10^{-2}$	1.44	$3.00 \cdot 10^{-2}$	2.03	$5.89 \cdot 10^{-3}$	3.70
16	$6.07 \cdot 10^{-2}$	1.43	$1.49 \cdot 10^{-2}$	2.01	$1.60 \cdot 10^{-3}$	3.68
32	$4.26 \cdot 10^{-2}$	1.43	$7.43 \cdot 10^{-3}$	2.01	$4.09 \cdot 10^{-4}$	3.92
64	$3.00 \cdot 10^{-2}$	1.42	$3.71 \cdot 10^{-3}$	2.00	$1.03 \cdot 10^{-4}$	3.97
128	$2.11 \cdot 10^{-2}$	1.42	$1.85 \cdot 10^{-3}$	2.00	$2.60 \cdot 10^{-5}$	3.96
		≈ 1.41		2		4

TABLE 4. Numerical results for problem (35) using the iterated method with $m = 2$, $\eta_1 = 0.1$, $\eta_2 = 0.9$.

	$r = 1$		$r = 2$		$r = 2.5$	
N	ε_N^{it}	ϱ_N^{it}	ε_N^{it}	ϱ_N^{it}	ε_N^{it}	ϱ_N^{it}
4	$9.71 \cdot 10^{-3}$		$4.07 \cdot 10^{-3}$		$4.21 \cdot 10^{-3}$	
8	$5.46 \cdot 10^{-3}$	1.78	$1.21 \cdot 10^{-3}$	3.38	$1.21 \cdot 10^{-3}$	3.50
16	$2.88 \cdot 10^{-3}$	1.90	$3.35 \cdot 10^{-4}$	3.60	$3.17 \cdot 10^{-4}$	3.80
32	$1.48 \cdot 10^{-3}$	1.95	$8.68 \cdot 10^{-5}$	3.86	$8.01 \cdot 10^{-5}$	3.96
64	$7.47 \cdot 10^{-4}$	1.98	$2.21 \cdot 10^{-5}$	3.92	$2.00 \cdot 10^{-5}$	4.00
128	$3.75 \cdot 10^{-4}$	1.99	$5.58 \cdot 10^{-6}$	3.97	$5.00 \cdot 10^{-6}$	4.01
		2		4		≈ 5.67

4.2. Example 2. Consider the following problem:

$$(D^{\frac{2}{3}}_{Cap}y)(t)+(t^{\frac{3}{4}}+2)y(t) = t^{\frac{17}{12}}+2t^{\frac{3}{4}}+2t^{\frac{2}{3}}+\Gamma\left(\frac{5}{3}\right)+4, \quad 0 \le t \le 1; \quad y(0) = 2. \tag{38}$$

This is a special case of (1)–(2) with

$$\alpha = \frac{2}{3},\ a(t) = t^{\frac{3}{4}} + 2,\ f(t) = t^{\frac{17}{12}} + 2t^{\frac{3}{4}} + 2t^{\frac{2}{3}} + \Gamma\left(\frac{5}{3}\right) + 4,\ y_0 = 2,\ b = 1.$$

Note that here $a, f \in C^{m,\frac{1}{3}}(0, 1]$ for all $m \in \mathbb{N}$. Therefore, by Theorem 1 it follows that $y \in C^{m,\nu}(0, b]$, where

$$\nu = \max\left\{1 - \frac{2}{3}, \frac{1}{3}\right\} = \frac{1}{3}.$$

We apply the collocation method and its iterated version for $m = 3$, using the shifted Gauss–Legendre points

$$\eta_1 = \frac{5-\sqrt{15}}{10},\ \eta_2 = \frac{1}{2},\ \eta_3 = \frac{5+\sqrt{15}}{10}. \tag{39}$$

The numerical results for the collocation method with these collocation parameters are given in Table 5. It follows from Theorem 2 that for sufficiently large $N \in \mathbb{N}$, we have

$$\sup_{0\le t\le b} |y(t) - y_N(t)| \le c_3 \begin{cases} N^{-\frac{2}{3}r}, & \text{if } 1 \le r \le \frac{9}{2}, \\ N^{-3}, & \text{if } r \ge \frac{9}{2}, \end{cases}$$

where $y(t) = t^{\frac{2}{3}} + 2, \quad 0 \le t \le 1$, is the exact solution of problem (38) and c_3 is a positive constant independent of N. Correspondingly, we expect for $r = 1$, $r = 2$ and $r = 4.5$ the convergence order to be $2^{\frac{2}{3}} \approx 1.59$, $2^{\frac{4}{3}} \approx 2.52$ and $2^3 = 8$, respectively. These values are given in the last row of Table 5.

Similarly, the numerical results for the iterated version with collocation parameters (39) are given in Table 6. It follows from Theorem 3 that, for sufficiently large $N \in \mathbb{N}$, we have

$$\|y - y_N^{it}\|_\infty = \max_{0\le t\le b} |y(t) - y_N^{it}(t)| \le c_4 \begin{cases} N^{-\frac{4}{3}r}, & \text{if } 1 \le r \le \frac{11}{4}, \\ N^{-\frac{11}{3}}, & \text{if } r \ge \frac{11}{4}, \end{cases}$$

where c_4 is a positive constant not dependent on N. In the last row of Table 6 are given the corresponding values for $r = 1$, $r = 2$ and $r = 2.75$, respectively.

As we can see, the numerical results are in accord with the theoretical estimates given by Theorem 2 and Theorem 3.

TABLE 5. Numerical results for problem (38) using collocation method with $m = 3$, $\eta_1 = \frac{5-\sqrt{15}}{10}, \eta_2 = \frac{1}{2}, \eta_3 = \frac{5+\sqrt{15}}{10}$.

	$r = 1$		$r = 2$		$r = 4.5$	
N	ε_N	ϱ_N	ε_N	ϱ_N	ε_N	ϱ_N
4	$3.67 \cdot 10^{-2}$		$1.51 \cdot 10^{-2}$		$4.29 \cdot 10^{-3}$	
8	$2.37 \cdot 10^{-2}$	1.55	$6.09 \cdot 10^{-3}$	2.48	$5.94 \cdot 10^{-4}$	7.22
16	$1.51 \cdot 10^{-2}$	1.57	$2.43 \cdot 10^{-3}$	2.50	$7.53 \cdot 10^{-5}$	7.89
32	$9.61 \cdot 10^{-3}$	1.57	$9.68 \cdot 10^{-4}$	2.51	$9.43 \cdot 10^{-6}$	7.99
64	$6.09 \cdot 10^{-3}$	1.58	$3.85 \cdot 10^{-4}$	2.52	$1.18 \cdot 10^{-6}$	8.00
128	$3.85 \cdot 10^{-3}$	1.58	$1.53 \cdot 10^{-4}$	2.52	$1.47 \cdot 10^{-7}$	8.00
		≈ 1.59		≈ 2.52		8

TABLE 6. Numerical results for problem (38) using the iterated method with $m = 3$, $\eta_1 = \frac{5-\sqrt{15}}{10}$, $\eta_2 = \frac{1}{2}$, $\eta_3 = \frac{5+\sqrt{15}}{10}$.

	$r = 1$		$r = 2$		$r = 2.75$	
N	ε_N^{it}	ϱ_N^{it}	ε_N^{it}	ϱ_N^{it}	ε_N^{it}	ϱ_N^{it}
4	$1.45 \cdot 10^{-3}$		$2.57 \cdot 10^{-4}$		$1.50 \cdot 10^{-4}$	
8	$6.19 \cdot 10^{-4}$	2.35	$4.25 \cdot 10^{-5}$	6.04	$1.30 \cdot 10^{-5}$	11.54
16	$2.57 \cdot 10^{-4}$	2.41	$6.84 \cdot 10^{-6}$	6.22	$1.23 \cdot 10^{-6}$	10.57
32	$1.05 \cdot 10^{-4}$	2.45	$1.09 \cdot 10^{-6}$	6.30	$9.31 \cdot 10^{-8}$	13.19
64	$4.25 \cdot 10^{-5}$	2.47	$1.72 \cdot 10^{-7}$	6.33	$7.41 \cdot 10^{-9}$	12.56
128	$1.71 \cdot 10^{-5}$	2.49	$2.70 \cdot 10^{-8}$	6.34	$6.24 \cdot 10^{-10}$	11.87
		≈ 2.52		≈ 6.35		≈ 12.70

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