ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 29, Number 1, June 2025
Available online at https://ojs.utlib.ee/index.php/ACUTM

A comparative study of compactness on topological spaces and pretopological spaces

Devangi Sojitra, Minakshi Biswas Hathiwala, and Gautam Hathiwala

Abstract. Pretopological spaces are generalized topological spaces that can serve as valuable tools for structuring intricate systems. Nevertheless, there remains a considerable amount of work within the realm of pretopology theory that needs to be done. The concepts of closure or neighborhood can be harnessed to establish the foundations of pretopology. This paper delves into the new definitions of compactness within pretopological space, offering both an exploration of these ideas and a comparative analysis alongside topological space.

1. Introduction

Topology is one of the important areas of mathematics. Continuity, compactness, and connectedness are some of the important concepts that we study in general topology. Convergence and limits can be the preliminary concepts to define a topology, whereas, filters and nets are fundamental notions of convergence theory. Usually, textbooks (see [25],[47], [58]) on general topology start with open sets to define a topology. However, a topological space can be constructed equivalently by closure function, interior function, or neighborhood systems (see [17]). Pretopological spaces are weaker versions of topological spaces in which certain properties are not satisfied. The topological concepts of continuity, compactness and connectedness can be extended in a pretopology. These notions can be equivalently defined in terms of closure, interior and neighborhood. Stadler and Stadler (see [62]) defined connectedness and separation axioms of pretopological spaces in terms of

Received October 23, 2024.
2020 *Mathematics Subject Classification.* 54A20, 54A35.
Key words and phrases. Clopen space, closure, compactness, door space, neighbourhood, pretopology.
https://doi.org/10.12697/ACUTM.2025.29.02
Corresponding author: Devangi Sojitra

neighborhood filters. For a pretopology, closure and interior are also equivalent to adherence and inherence, respectively, in convergence theory (see [56]). Below we present a short review of the development of the idea of pretopology. Prior to this, Table 1 describes a few applications of pretopological spaces in many different fields.

Research area	**References**
Pattern classification	[1], [10], [11], [20], [22], [23], [36]
Clustering	[9], [28], [35], [40], [65]
Economic analysis	[4], [60]
Image analysis	[7], [24], [33], [42], [43], [46], [50], [51]
Pollution model	[2], [32]
Social networks	[8], [38], [39]
Stock market	[48], [49]
Structural analysis	[34], [57]
Text exploration	[14], [15]
Information system	[5]
Biology	[62], [63], [64]

TABLE 1. Applications of pretopological spaces.

Listing (see [41]) introduced the word topology in 1847, though his works were not fully inclined to modern day topology. The development of topology started with the works of Henri Poincare who defined the term homeomorphism while discussing the qualitative aspects of differential equations in his remarkable paper "Analysis Situs" [52]. The inadequacy of sequences lead to the theory of filters (see [66]). The convergence of filters involved in convergence theory can form the base for the topology on a set. In this context, a lot of works on convergence theory has been contributed by Fréchet [21], Hausdorff [26], Choquet [13] and Dolecki [18]. Another approach of defining topology is the use of closure spaces. In fact, modern day topological spaces emerged out of closure spaces, with the definition of metrics by Hausdorff [27]. Kuratowski [31] generalised the idea of closure in Euclidean space to topological spaces. Many of the properties of closure spaces were explored by Čech [12] and Stadler and Stadler [61]. Choquet [13] defined pretopology and pseudotopology as variants of topology in terms of filters. Pretopological spaces can be considered to be closure spaces in which the closure operator is not idempotent. The approaches that involve the use of neighborhood as well as interior operators, in defining a topology and hence, a pretopology goes hand in hand with closure and convergence (see [30], [61]). The equivalences of all these approaches are available in the literature (see [61]). Though the idea of pretopology dates back to mid 20th century, very little work has been done. The theory of pretopology needs to be explored before we find its associated applications without any technical glitches.

In this paper, we discuss the property of compactness in pretopological space. We define fully compact space and observe that among all versions of compactness, the idea of fully compact space is the strongest.

2. Preliminaries

Definition 1. Suppose that $\mathfrak{P}$ is a set which is non-empty. For each $\mathfrak{p} \in \mathfrak{P}$, the family $\mathfrak{N}(\mathfrak{p})$ of subsets of $\mathfrak{P}$ will be called a *neighborhood system* of $\mathfrak{p}$, if the following conditions are true:

(N1) $\mathfrak{P} \in \mathfrak{N}(\mathfrak{p}), \forall \mathfrak{p} \in \mathfrak{P}$;
(N2) $J \in \mathfrak{N}(\mathfrak{p}), J \subseteq J' \Rightarrow J' \in \mathfrak{N}(\mathfrak{p})$;
(N3) $J \in \mathfrak{N}(\mathfrak{p}) \Rightarrow \mathfrak{p} \in J$;
(N4) $J, J' \in \mathfrak{N}(\mathfrak{p}) \Rightarrow J \cap J' \in \mathfrak{N}(\mathfrak{p})$.

The pair $(\mathfrak{P}, \mathfrak{N})$ is called a *pretopological space*. The system $\mathfrak{N}(\mathfrak{p})$ may satisfy an additional condition

(N5) $\forall J \in \mathfrak{N}(\mathfrak{p})$, there exist $J' \in \mathfrak{N}(\mathfrak{p})$ such that $J' \subseteq J$ and $J' \in \mathfrak{N}(\mathfrak{s})$; for every $\mathfrak{s} \in J'$.

If $\mathfrak{N}(\mathfrak{p})$ satisfies (N1)–(N5) for all $\mathfrak{p} \in \mathfrak{P}$ then $(\mathfrak{P}, \mathfrak{N})$ is called a *topological space* ([12, pp. 242, 250, 251], [29, p. 56], [62, pp. 578, 579]). Here, $\mathfrak{N}$ is a mapping $\mathfrak{N} : \mathfrak{P} \to P(\mathfrak{P})$, where $P(\mathfrak{P})$ denotes the power set of $\mathfrak{P}$.

Definition 2. Let $(\mathfrak{P}, \mathfrak{N})$ be a pretopological space and $\mathfrak{N}(\mathfrak{p})$ be the neighborhood system of each $\mathfrak{p} \in \mathfrak{P}$, then we will call each member J of $\mathfrak{N}(\mathfrak{p})$ a *neighborhood* of $\mathfrak{p}$.

To distinguish the definition of topological spaces from that of pretopological spaces, we will call the neighborhood of $\mathfrak{p}$ in a pretopological space as a P-neighbourhood of $\mathfrak{p}$ over the pretopological space $(\mathfrak{P}, \mathfrak{N})$. Here, if $J \subseteq \mathfrak{P}$ then P-neighborhood of J is denoted as $\mathfrak{N}(J)$ and it is defined as $\mathfrak{N}(J) = \bigcap_{\mathfrak{s} \in J} \mathfrak{N}(\mathfrak{s})$.

Example 1. Let $\mathfrak{P} = \mathbb{N} = \{1, 2, 3, 4, ...\}$. For $\mathfrak{p} \in \mathfrak{P}$, let the P-neighbourhood be defined as $\mathfrak{N}(\mathfrak{p}) = \{J \subseteq \mathbb{N} | \{\mathfrak{p}, \mathfrak{p}+1\} \subseteq J\}$. Now we will see that $\mathfrak{N}(\mathfrak{p})$ satisfies (N1)–(N4) but not (N5).

(N1) Clearly, $\mathfrak{P} = \mathbb{N} \in \mathfrak{N}(\mathfrak{p})$.

(N2) For any $J \in \mathfrak{N}(\mathfrak{p})$ and $J \subseteq J'$, we have $J' \in \mathfrak{N}(\mathfrak{p})$ because $\{\mathfrak{p}, \mathfrak{p}+1\}$ is the smallest P-neighborhood of $\mathfrak{p}$ and $\mathfrak{N}(\mathfrak{p})$ contains all the super sets of $\{\mathfrak{p}, \mathfrak{p}+1\}$.

(N3) Let $J \in \mathfrak{N}(\mathfrak{p})$. Clearly, $\{\mathfrak{p}, \mathfrak{p}+1\} \subseteq J$ and $\mathfrak{p} \in \{\mathfrak{p}, \mathfrak{p}+1\}$. Therefore, $\mathfrak{p} \in J$.

(N4) It is easy to see that, if $J, J' \in \mathfrak{N}(\mathfrak{p})$, then $J \cap J' \in \mathfrak{N}(\mathfrak{p})$.

(N5) Let $J' = \{2, 3, 4\} \in \mathfrak{N}(2)$. Possible subsets of J', which are in $\mathfrak{N}(2)$

are $K = \{2,3\}$ and $L = \{2,3,4\}$. Clearly, for $3 \in K = \{2,3\}$, $K \notin \mathfrak{N}(3)$ as $\mathfrak{N}(3) = \{J \subseteq \mathbb{N} | \{3,4\} \subseteq J\}$. Moreover, $4 \in L = \{2,3,4\}$, $L \notin \mathfrak{N}(4)$ as $\mathfrak{N}(4) = \{J \subseteq \mathbb{N} | \{4,5\} \subseteq J\}$.

Hence, $\mathfrak{N}(\mathfrak{p})$ satisfy (N1)–(N4) but not (N5). Therefore, $(\mathfrak{P}, \mathfrak{N})$ is a pretopological space.

Example 2. Let $\mathfrak{P} = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$. Suppose that P-neighborhood system of each point of $\mathfrak{P}$ is defined as follows:
$\mathfrak{N}(\mathfrak{p}_1) = \{\{\mathfrak{p}_1\}, \{\mathfrak{p}_1, \mathfrak{p}_2\}, \{\mathfrak{p}_1, \mathfrak{p}_3\}, \mathfrak{P}\}$,
$\mathfrak{N}(\mathfrak{p}_2) = \{\{\mathfrak{p}_2, \mathfrak{p}_3\}, \mathfrak{P}\}$,
$\mathfrak{N}(\mathfrak{p}_3) = \{\{\mathfrak{p}_1, \mathfrak{p}_3\}, \mathfrak{P}\}$.

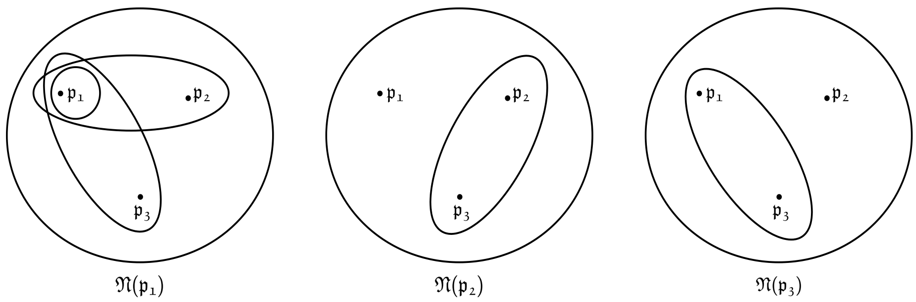

FIGURE 1. P-neighborhood system on $\mathfrak{P}$.

One can easily check that (N1)–(N4) are satisfied. For (N5), $\{\mathfrak{p}_2, \mathfrak{p}_3\} \in \mathfrak{N}(\mathfrak{p}_2)$ and $\{\mathfrak{p}_2, \mathfrak{p}_3\} \subseteq \{\mathfrak{p}_2, \mathfrak{p}_3\}$. We see that $\mathfrak{p}_3 \in \{\mathfrak{p}_2, \mathfrak{p}_3\}$, but $\{\mathfrak{p}_2, \mathfrak{p}_3\} \notin \mathfrak{N}(\mathfrak{p}_3)$. Hence, (N5) is not satisfied. Therefore $(\mathfrak{P}, \mathfrak{N})$ is a pretopological space but not a topological space.

A topology on a set can be defined in another way as follows.

Definition 3. Suppose that $\mathfrak{P}$ is a set which is non-empty. Let $P(\mathfrak{P})$ be a collection of all subsets of $\mathfrak{P}$. Define a mapping $\mathfrak{c} : P(\mathfrak{P}) \to P(\mathfrak{P})$ for which the following conditions are true:

(c1) $\mathfrak{c}(\phi) = \phi$;
(c2) $S \subseteq S' \Rightarrow \mathfrak{c}(S) \subseteq \mathfrak{c}(S')$;
$\mathfrak{c}(S \cap S') \subseteq \mathfrak{c}(S) \cap \mathfrak{c}(S')$;
$\mathfrak{c}(S) \cup \mathfrak{c}(S') \subseteq \mathfrak{c}(S \cup S')$;
(c3) $S \subseteq \mathfrak{c}(S)$;
(c4) $\mathfrak{c}(S \cup S') \subseteq \mathfrak{c}(S) \cup \mathfrak{c}(S')$.

The pair $(\mathfrak{P}, \mathfrak{c})$ will be called a *pretopological space*. If $\mathfrak{c}$ satisfies one more condition (c5) then $(\mathfrak{P}, \mathfrak{c})$ will be called a topological space ([12, pp. 237, 250], [62, pp. 578, 579]):

(c5) $\mathfrak{c}(\mathfrak{c}(S)) = \mathfrak{c}(S)$.

To distinguish between the definitions of the topological space and of the pretopological space, we will now call the operator $\mathfrak{c}$ a P-closure over the pretopological space $(\mathfrak{P}, \mathfrak{c})$.

Remark 1. In literature (see [16]), there exists a different classification of pretopological spaces and the operator $\mathfrak{c}$ is also termed a pseudo closure, however, we will refer to $\mathfrak{c}$ as the P-closure over the pretopological space $(\mathfrak{P}, \mathfrak{c})$ in our text for future reference. Moreover, let $(\mathfrak{P}, \mathfrak{i})$ be a pretopological space and $S \subseteq \mathfrak{P}$, then $\mathfrak{c}(S) = \{\mathfrak{p} \in \mathfrak{P} | J \cap S \neq \phi$ for all $J \in \mathfrak{N}(\mathfrak{p})\}$ ([12, p. 243]). From example 2, $\mathfrak{c}(\{\mathfrak{p}_1, \mathfrak{p}_2\}) = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\} = \mathfrak{P}$. Also, $\mathfrak{c}(\{\mathfrak{p}_2\}) = \{\mathfrak{p}_2\}$.

Definition 4. Let $(\mathfrak{P}, \mathfrak{c})$ be a pretopological space and S be subset of $\mathfrak{P}$, then S is said to be a P-*closed* set if and only if $\mathfrak{c}(S) = S$.

In Example 2, P-closed sets are $\{\mathfrak{p}_2\}, \{\mathfrak{p}_2, \mathfrak{p}_3\}, \mathfrak{P}$. Let us take $S = \{\mathfrak{p}_1\} \subset \mathfrak{P}$. Now, $\mathfrak{c}(S) = \{\mathfrak{p}_1, \mathfrak{p}_3\}$ which is not P-closed. Hence, the fact that $\mathfrak{c}(S)$ is always closed may not be true in a pretopological space. Thus, we have the following result.

Proposition 1. *Let $(\mathfrak{P}, \mathfrak{c})$ be a pretopological space and S be a subset of $\mathfrak{P}$. Then $\mathfrak{c}(S)$ is P-closed if S is P-closed.*

Proof. Let S be P-closed. Then $\mathfrak{c}(S) = S$. This implies that $\mathfrak{c}(\mathfrak{c}(S)) = \mathfrak{c}(S)$. Hence, $\mathfrak{c}(S)$ is P-closed. □

Definition 5. Suppose that $\mathfrak{P}$ is a set which is non-empty and $P(\mathfrak{P})$ is the family of all subsets of $\mathfrak{P}$. Define a mapping $\mathfrak{i} : P(\mathfrak{P}) \to P(\mathfrak{P})$ for which the following conditions are true:

(i1) $\mathfrak{i}(\mathfrak{P}) = \mathfrak{P}$;
(i2) $S \subseteq S' \Rightarrow \mathfrak{i}(S) \subseteq \mathfrak{i}(S')$;
$\mathfrak{i}(S) \cup \mathfrak{i}(S') \subseteq \mathfrak{i}(S \cup S')$;
$\mathfrak{i}(S \cap S') \subseteq \mathfrak{i}(S) \cap \mathfrak{i}(S')$;
(i3) $\mathfrak{i}(S) \subseteq S$;
(i4) $\mathfrak{i}(S) \cap \mathfrak{i}(S') \subseteq \mathfrak{i}(S \cap S')$;
(i5) $\mathfrak{i}(\mathfrak{i}(S)) = \mathfrak{i}(S)$.

If the ordered pair $(\mathfrak{P}, \mathfrak{i})$ satisfies (i1)–(i4) it is called a *pretopological space*. If in addition it also satisfies (i5) then it will be called a *topological space* ([12, pp. 241, 250], [61, p. 586]).

To distinguish the definition of topological space from that of pretopological space we will now call the operator $\mathfrak{i}$ as P-*interior* over the pretopological space $(\mathfrak{P}, \mathfrak{i})$.

Definition 6. Let $(\mathfrak{P}, \mathfrak{i})$ be a pretopological space. Consider an arbitrary subset S of $\mathfrak{P}$, we say that S is P-*open* if $\mathfrak{i}(S) = S$ or S is a P-neighborhood of each of its points.

Remark 2. Let S be any arbitrary subset of $\mathfrak{P}$. Then, S is P-open if and only if $\mathfrak{P} - S$ is P-closed. Moreover, let $(\mathfrak{P}, \mathfrak{i})$ be a pretopological space and $S, J \subseteq \mathfrak{P}$. Then $\mathfrak{i}(S) = \{\mathfrak{p} \in S | S \in \mathfrak{N}(\mathfrak{p})\}$ and $\mathfrak{N}(\mathfrak{p}) = \{J \subseteq \mathfrak{P} | \mathfrak{p} \in \mathfrak{i}(J)\}$ ([29, p. 44], [62, p. 578]). From Example 1, $\mathfrak{i}(\{2,3,4\}) = \{2,3\}$.

Remark 3. P-open sets are in the context of pretopological spaces and are not referred as pre-open sets as discussed in [3].

In Example 1, P-open sets are $\phi, \mathbb{N}$. Let us take $S = \{2, 3, 9, 10\} \subset \mathbb{N}$. Now $\mathfrak{i}(S) = \{2, 9\}$, which is not P-open. Hence, the fact that $\mathfrak{i}(S)$ is being open in a topological space is not true in a pretopological space. Thus, we have the following result.

Proposition 2. *Let* $(\mathfrak{P}, \mathfrak{i})$ *be a pretopological space and* S *a subset of* $\mathfrak{P}$. *Then* $\mathfrak{i}(S)$ *is P-open if* S *is P-open.*

Proof. Let S be P-open. Then $\mathfrak{i}(S) = S$. This implies that $\mathfrak{i}(\mathfrak{i}(S)) = \mathfrak{i}(S)$. Hence, $\mathfrak{i}(S)$ is P-open. □

Remark 4. Suppose $\mathfrak{i}, \mathfrak{c}$ and $\mathfrak{N}(\mathfrak{p})$ are the P-interior function, the P-closure function and the P-neighborhood system on a pretopological space $\mathfrak{P}$, respectively, and $S, J \subseteq \mathfrak{P}$. Then ([12, pp. 240, 241], [29, p. 44], [62, p. 578])

(1) $\mathfrak{i}(S) = \mathfrak{P} - \mathfrak{c}(\mathfrak{P} - S)$;
(2) $J \in \mathfrak{N}(\mathfrak{p}) \iff \mathfrak{p} \in \mathfrak{i}(J)$;
(3) $\mathfrak{p} \in \mathfrak{c}(S) \iff (\mathfrak{P} - S) \notin \mathfrak{N}(\mathfrak{p})$;
(4) $\mathfrak{c}(S) = \mathfrak{P} - \mathfrak{i}(\mathfrak{P} - S)$.

Using these properties (1)–(4), one can show that the axioms given in the definition 1, 3 and 5 are equivalent to each other (see [61]).

Definition 7 ([53] p.75)**.** Let $\mathfrak{T}$ be a topological space (pretopological space), then $\mathfrak{T}$ is said to be a *clopen* (a P-clopen) topological (pretopological) space if every open (P-open) set in $\mathfrak{T}$ is closed (P-closed).

Example 3. Let $\mathfrak{T} = \{\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3\}$ and let the neighbourhood system on $\mathfrak{T}$ be defined as follows:
$\mathfrak{N}(\mathfrak{t}_1) = \{\{\mathfrak{t}_1, \mathfrak{t}_2\}, \mathfrak{T}\}$,
$\mathfrak{N}(\mathfrak{t}_2) = \{\{\mathfrak{t}_1, \mathfrak{t}_2\}, \mathfrak{T}\}$,
$\mathfrak{N}(\mathfrak{t}_3) = \{\{\mathfrak{t}_3\}, \{\mathfrak{t}_1, \mathfrak{t}_3\}, \{\mathfrak{t}_2, \mathfrak{t}_3\}, \mathfrak{T}\}$.
Since every topological space is a pretopological space, from Definition 6, open sets of $\mathfrak{T}$ are $\phi, \{\mathfrak{t}_1, \mathfrak{t}_2\}, \{\mathfrak{t}_3\}, \mathfrak{T}$ which are also closed in $\mathfrak{T}$. Hence, $\mathfrak{T}$ is a clopen topological space.

Example 4. Let $\mathfrak{P} = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4\}$ and let the P-neighborhood system on $\mathfrak{P}$ be defined as follows:

$$\mathfrak{N}(\mathfrak{p}_1) = \{\{\mathfrak{p}_1, \mathfrak{p}_2\}, \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}, \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_4\}, \mathfrak{P}\},$$
$$\mathfrak{N}(\mathfrak{p}_2) = \{\{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}, \mathfrak{P}\},$$

$$\begin{aligned}\mathfrak{N}(\mathfrak{p}_3) =& \{\{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3\},\mathfrak{P}\},\\ \mathfrak{N}(\mathfrak{p}_4) =& \{\{\mathfrak{p}_4\},\{\mathfrak{p}_1,\mathfrak{p}_4\},\{\mathfrak{p}_2,\mathfrak{p}_4\},\{\mathfrak{p}_3,\mathfrak{p}_4\},\{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_4\},\{\mathfrak{p}_2,\mathfrak{p}_3,\mathfrak{p}_4\},\\ & \{\mathfrak{p}_1,\mathfrak{p}_3,\mathfrak{p}_4\},\mathfrak{P}\}.\end{aligned}$$

From Definition 6, P-open sets of $\mathfrak{P}$ are $\phi, \{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3\}, \{\mathfrak{p}_4\}, \mathfrak{P}$ which are also P-closed in $\mathfrak{P}$. Hence, $\mathfrak{P}$ is a P-clopen pretopological space.

The generalization of open sets in a topological space is available in the literature. Mashhour [44] has given the notion of pre-open sets and Levine [37] has introduced the idea of semi-open sets. Below we discuss few extensions of these open sets in pretopological space.

Definition 8. Suppose that $\mathfrak{P}$ is a pretopological space. We will call $S \subseteq \mathfrak{P}$ a *pre-P-open set* if $S \subseteq \mathfrak{i}(\mathfrak{c}(S))$ and S as *pre-P-closed set* if $S \supseteq \mathfrak{c}(\mathfrak{i}(S))$.

Remark 5. It is easy to see that if $S \subseteq \mathfrak{P}$ is P-open then it is always pre-P-open.

Definition 9. Let $(\mathfrak{P}, \mathfrak{c})$ be a pretopological space. Then we will call $S \subseteq \mathfrak{P}$ a *semi-P-open set* if there exists a P-open set G such that $G \subseteq S \subseteq \mathfrak{c}(G)$.

Proposition 3. *Let $\mathfrak{P}$ be a pretopological space. If $S \subseteq \mathfrak{P}$ is semi-P-open, then $S \subseteq \mathfrak{c}(\mathfrak{i}(S))$.*

Proof. Let $S \subseteq \mathfrak{P}$ be semi-P-open. Then there exists a P-open set G such that $G \subseteq S \subseteq \mathfrak{c}(G)$. Now, $G = \mathfrak{i}(G) \subseteq \mathfrak{i}(S) \subseteq \mathfrak{i}(\mathfrak{c}(G))$ implies $G \subseteq \mathfrak{i}(S)$, hence $\mathfrak{c}(G) \subseteq \mathfrak{c}(\mathfrak{i}(S))$. Since $G \subseteq S \subseteq c(G)$, we can write $S \subseteq \mathfrak{i}(\mathfrak{c}(S))$. Hence, the claim is proved. ☐

Remark 6. It is easy to show that, in a clopen topological space $\mathfrak{T}$, $S \subseteq \mathfrak{T}$ is semi-open if and only if S is open.

Remark 7 ([37], p. 36). In a topological space $\mathfrak{T}$, $S \subseteq \mathfrak{T}$ is semi-open if and only if $\subseteq \mathfrak{c}(\mathfrak{i}(S))$.

We will give an example that shows that the converse of Proposition 3 is not true.

Example 5. Let $\mathfrak{P} = \{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3,\mathfrak{p}_4\}$ be a pretopological space with the neighborhood system defined as follows:
$\mathfrak{N}(\mathfrak{p}_1) = \{\{\mathfrak{p}_1,\mathfrak{p}_2\},\{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3\},\{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_4\},\mathfrak{P}\}$,
$\mathfrak{N}(\mathfrak{p}_2) = \{\{\mathfrak{p}_2,\mathfrak{p}_3\},\{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3\},\{\mathfrak{p}_2,\mathfrak{p}_3,\mathfrak{p}_4\},\mathfrak{P}\}$,
$\mathfrak{N}(\mathfrak{p}_3) = \{\{\mathfrak{p}_2,\mathfrak{p}_3,\mathfrak{p}_4\},\mathfrak{P}\}$,
$\mathfrak{N}(\mathfrak{p}_4) = \{\{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_4\},\mathfrak{P}\}$.
Now, consider a subset $S = \{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3\} \subset \mathfrak{P}$. Then $\mathfrak{i}(S) = \{\mathfrak{p}_1,\mathfrak{p}_2\}, \mathfrak{c}(\mathfrak{i}(S)) = \{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3,\mathfrak{p}_4\} = \mathfrak{P}$. Therefore, $S \subseteq \mathfrak{c}(\mathfrak{i}(S))$. Here P-open sets are $\phi, \mathfrak{P}$. Thus, there exist no P-open set G such that $G \subseteq S \subseteq \mathfrak{c}(G)$. Hence, S is not a semi-P-open set.

Proposition 4. *Let $\mathfrak{P}$ be a P-clopen pretopological space. Then $S \subseteq \mathfrak{P}$ is P-open if and only if S is semi-P-open.*

Proof. Let $S \subseteq \mathfrak{P}$ be P-open. Therefore, we can write $S \subseteq S \subseteq \mathfrak{c}(S)$. Hence, S is semi-P-open. Conversely, let $S \subseteq \mathfrak{P}$ be semi-P-open. Then there exists a P-open set G, such that $G \subseteq S \subseteq \mathfrak{c}(G)$. Since $\mathfrak{P}$ is P-clopen, $\mathfrak{c}(G) = G$. Therefore, $G \subseteq S \subseteq G$ implies that $S = G$. Hence, S is P-open. □

Below we discuss a special type of topological space which demonstrates an interesting property.

Definition 10 ([29], p.76)**.** If every subset of a topological space $\mathfrak{T}$ is either open or closed (or both) then such a space is called a *door space.*

Remark 8. In a door space $\mathfrak{T}$, every pre-open set is always open ([55, p. 89]).

3. Compactness in pretopological space

Compactness in a topological space has been discussed by many authors ([6], [45], [54], [59]). Here, we extend the property of compactness to a pretopological space. Since, in a pretopological space, P-interior of a set is not always P-open, compactness in a pretopological space cannot be the same as that in a toplogical space. We will now introduce a new type of compactness which we will call 'Full compactness'.

We begin with the definitions of various types of covers in pretopological spaces which are simply the extensions of covers in topological spaces. A cover for a pretopological space $\mathfrak{P}$ is, simply, a collection of subsets of $\mathfrak{P}$ whose union is $\mathfrak{P}$ itself.

Definition 11 ([12], p. 285, [18], p. 16)**.** Let $\mathfrak{P}$ be a pretopological space and $\mathfrak{i}$ be the P-interior operator on $\mathfrak{P}$. Then a family $\mathbb{I}$ of subsets of $\mathfrak{P}$ is said to be a *P-interior cover* of $\mathfrak{P}$ if

$$\mathfrak{P} = \bigcup_{S \in \mathbb{I}} \mathfrak{i}(S).$$

Definition 12 ([18], p. 16)**.** Let $\mathfrak{P}$ be a pretopological space and $\mathfrak{N}(\mathfrak{p})$ be a P-neighborhood system on $\mathfrak{P}$ for $\mathfrak{p} \in \mathfrak{P}$. A cover $\mathbb{U}$ of $\mathfrak{P}$ is said to be a *P-neighborhood cover* of $\mathfrak{P}$ if for each $\mathfrak{p} \in \mathfrak{P}$, there exists $J \in \mathbb{U}$ such that $J \in \mathfrak{N}(\mathfrak{p})$.

It should be noted that every P-neighborhood cover is a P-interior cover for a pretopological space and vice versa (see [18, p. 16]).

Definition 13. Let $\mathfrak{P}$ be a pretopological space. We will call a cover $\mathbb{P}$ of $\mathfrak{P}$ a *pre-P-open cover* if it consists of pre-P-open sets of $\mathfrak{P}$ (i.e. $\mathbb{P} = \{S | S \subseteq \mathfrak{i}(\mathfrak{c}(S))\}$).

Definition 14. Let $\mathfrak{P}$ be a pretopological space. We will call a cover $\mathbb{O}$ a *P-open cover* if it consists of P-open sets of $\mathfrak{P}$ (i.e. $\mathbb{O} = \{S|\mathfrak{i}(S) = S\}$).

Definition 15. Let $\mathfrak{P}$ be a pretopological space and $\mathbb{P}$ be a pre-P-open cover of $\mathfrak{P}$. Let $\mathbb{G}$ be a cover of $\mathfrak{P}$ such that for each pre-P-open set $P \in \mathbb{P}$ there exists a P-open set $G \in \mathbb{G}$ such that $P \subseteq G \subseteq \mathfrak{c}(P)$, that is, $\mathbb{G} = \{G|P \in \mathbb{P}, P \subseteq G \subseteq \mathfrak{c}(P)\}$. We will call the cover $\mathbb{G}$ a *P-open super cover* of $\mathfrak{P}$ with respect to $\mathbb{P}$.

Definition 16. Let $\mathfrak{P}$ be a pretopological space. We will call a cover $\mathbb{S}$ an *S-cover* of $\mathfrak{P}$ if $\mathbb{S}$ consists of semi-P-open sets of $\mathfrak{P}$.

Let $\mathcal{C}$ be any cover of $\mathfrak{P}$. Recall that if a subcollection $\mathcal{S}$ of $\mathcal{C}$ also covers $\mathfrak{P}$, then $\mathcal{S}$ is called a subcover of $\mathcal{C}$. When this subcover consists of finite number of elements, then $\mathcal{S}$ is said to be a finite subcover of $\mathfrak{P}$. Based on the different covers defined, there are different types of compactness in a pretopological space that can be defined.

Definition 17 ([12], p. 783, [19], p.77)**.** Let $\mathfrak{P}$ be a pretopological space and $\mathbb{I}$ be any P-interior cover of $\mathfrak{P}$. Then $\mathfrak{P}$ is *compact* if

$$\mathfrak{P} = \bigcup_{j=1}^{m} \mathfrak{i}(S_{k_j}) \text{ where } S_{k_j} \in \mathbb{I}.$$

That is, for every P-interior cover $\mathbb{I} = \{S_k|k \in I\}$ of $\mathfrak{P}$, there exists a finite collection $\{S_{k_j} \in \mathbb{I}|j = 1, 2, .., m,\ m \text{ is finite}\}$ which also covers $\mathfrak{P}$.

Since every interior of a set is an open set in a topological space, compactness in a topological space is defined in terms of open covers, that is, a topological space is compact if every open cover has a finite subcover.

Proposition 5. *Let $\mathbb{U}$ be any P-neighborhood cover of a pretopological space $\mathfrak{P}$. Then $\mathfrak{P}$ is compact if and only if there exists a finite subcover $\{J_k|1 \le k \le n, n \in \mathbb{N}\}$ of $\mathbb{U}$ such that, for each $\mathfrak{p} \in \mathfrak{P}$, we have a k with $J_k \in \mathfrak{N}(\mathfrak{p})$.*

Proof. Let $\mathfrak{P}$ be a pretopological space which is compact and $\mathbb{U}$ be any P-neighborhood cover of $\mathfrak{P}$. Then $\mathbb{U}$ is a P-interior cover of $\mathfrak{P}$. Since $\mathfrak{P}$ is compact, $\mathfrak{P} = \bigcup_{k=1}^{n} \mathfrak{i}(J_k)$ where $J_k \in \mathbb{U}$. This implies that, for each $\mathfrak{p} \in \mathfrak{P}$, we have atleast one k with $\mathfrak{p} \in \mathfrak{i}(J_k)$ where $1 \le k \le n$. Consequently, for each $\mathfrak{p} \in \mathfrak{P}$, we have a k with $J_k \in \mathfrak{N}(\mathfrak{p})$ where $1 \le k \le n$. Hence, there exists a finite subcover $\{J_k|1 \le k \le n, n \in \mathbb{N}\}$ of $\mathbb{U}$ such that, for each $\mathfrak{p} \in \mathfrak{P}$, we have a k with $J_k \in \mathfrak{N}(\mathfrak{p})$.

Conversely, suppose that any P-neighborhood cover $\mathbb{U}$ of $\mathfrak{P}$ has a finite subcover $\{J_k \in \mathbb{U}|1 \le k \le n, n \in \mathbb{N}\}$ such that for each $\mathfrak{p} \in \mathfrak{P}$, we have a k with $J_k \in \mathfrak{N}(\mathfrak{p})$. We will show that $\mathfrak{P}$ is compact. By Definition 17, $\mathfrak{P}$ is compact if and only if every P-interior cover of $\mathfrak{P}$ has a finite subcover.

Consider any P-interior cover $\mathbb{I}$ of $\mathfrak{P}$. Then $\mathfrak{P} = \bigcup_{S\in\mathbb{I}} \mathfrak{i}(S)$ and so for each $\mathfrak{p} \in \mathfrak{P}$, there exists $S \in \mathbb{I}$ such that $\mathfrak{p} \in \mathfrak{i}(S)$. By Remark 4, this implies that, for each $\mathfrak{p} \in \mathfrak{P}$, there exists $S \in \mathbb{I}$ such that $S \in \mathfrak{N}(\mathfrak{p})$. Therefore, $\mathbb{I} = \{S|S \in \mathfrak{N}(\mathfrak{p})\}$ is also a P-neighborhood cover of $\mathfrak{P}$. By assumption, $\mathbb{I}$ has a finite subcover $\{S_k \in \mathbb{I}|1 \leq k \leq n, n \in \mathbb{N}\}$ such that for each $\mathfrak{p} \in \mathfrak{P}$, we have a k with $S_k \in \mathfrak{N}(\mathfrak{p})$. That is to say, the P-interior cover $\mathbb{I}$ of $\mathfrak{P}$ has a finite subcover. Since $\mathbb{I}$ is an arbitrary P-interior cover, it follows that $\mathfrak{P}$ is compact. □

Definition 18. A pretopological space $\mathfrak{P}$ is called a *nearly compact space* if, for each P-open cover $\mathbb{O} = \{O \subseteq \mathfrak{P}|\mathfrak{i}(O) = O\}$ of $\mathfrak{P}$, there exists a finite subcollection $\{O_k|O_k \in \mathbb{O}, 1 \leq k \leq n\}$ of $\mathbb{O}$ such that

$$\mathfrak{P} = \bigcup_{k=1}^{n} \{\mathfrak{i}(\mathfrak{c}(O_k))|O_k \in \mathbb{O}\}.$$

Definition 19. A pretopological space $\mathfrak{P}$ is said to be *po-compact space* if each pre-P-open cover $\mathbb{P}$ of $\mathfrak{P}$ has a finite P-open super cover that covers $\mathfrak{P}$.

Definition 20. A pretopological space is *strongly compact* if each pre-P-open cover of $\mathfrak{P}$ has a finite subcover.

Definition 21. A pretopological space $\mathfrak{P}$ is said to be an *S-compact* space if, for each S-cover of $\mathfrak{P}$, there exists a finite subcover.

Below we provide a new definition of a type of compactness which we find the strongest version among all types of compactness.

Definition 22. Let $\mathfrak{P}$ be a pretopological space. We will call $\mathfrak{P}$ a *fully compact space* if, for every P-neighbourhood cover $\mathbb{U}$ of $\mathfrak{P}$, there exists a finite subcollection $\mathbb{G} = \{J_k \in \mathbb{U}|\mathfrak{i}(J_k) = J_k, 1 \leq k \leq n\}$ of $\mathbb{U}$, such that

$$\mathfrak{P} = \bigcup_{J_k\in\mathbb{G}} J_k.$$

Full compactness in a topological space can be defined in a similar way, that is, a topological space $\mathfrak{P}$ is fully compact if for every neighbourhood cover $\mathbb{U}$ there exist a finite subcollection $\mathbb{G} = \{J_k \in \mathbb{U}|\mathfrak{i}(J_k) = J_k, 1 \leq k \leq n\}$ of $\mathbb{U}$ such that

$$\mathfrak{P} = \bigcup_{J_k\in\mathbb{G}} J_k.$$

Below we present some relations between different compactnesses in a pretopological space and in a topological space.

Proposition 6. *If $\mathfrak{P}$ is a fully compact pretopological space, then it is compact.*

Proof. Let $\mathfrak{P}$ be the fully compact pretopological space. Now, consider a P-neighborhood cover $\mathbb{U}$ of $\mathfrak{P}$. Since $\mathfrak{P}$ is fully compact, there exists a finite subcollection $\mathbb{G} = \{J_k \in \mathbb{U} | \mathfrak{i}(J_k) = J_k; 1 \leq k \leq n\}$ such that

$$\mathfrak{P} = \bigcup_{J_k \in \mathbb{G}} J_k.$$

Thus, there exists a finite subcover $\mathbb{G}$ of $\mathbb{U}$ such that, for each $\mathfrak{p} \in \mathfrak{P}$, we have k with $J_k \in \mathbb{G}$ and $\mathfrak{p} \in J_k = \mathfrak{i}(J_k)$. That means that there exists a finite subcover $\mathbb{G}$ of $\mathbb{U}$ such that for each $\mathfrak{p} \in \mathfrak{P}$, we have k with $J_k \in \mathbb{G}$ and $J_k \in \mathfrak{N}(\mathfrak{p})$. Thus, $\mathfrak{P}$ is compact. □

Proposition 7. *If $\mathfrak{P}$ is a strongly compact pretopological space, then it is a nearly compact space.*

Proof. Let $\mathfrak{P}$ be a strongly compact pretopological space. Consider a P-open cover $\mathbb{O}$ of $\mathfrak{P}$. Since every P-open set is a pre-P-open set, $\mathbb{O}$ is a pre-P-open cover of $\mathfrak{P}$. Because $\mathfrak{P}$ is strongly compact, $\mathfrak{P} = \bigcup_{k=1}^{n} O_k \subseteq \bigcup_{k=1}^{n} \mathfrak{i}(c(O_k))$ where $O_k \in \mathbb{O}$. Therefore, $\mathfrak{P} = \bigcup_{k=1}^{n} \mathfrak{i}(\mathfrak{c}(O_k))$ and so $\mathfrak{P}$ is nearly compact. □

Proposition 8. *Let $\mathfrak{P}$ be an S-compact pretopological space. Then it is nearly compact.*

Proof. Let $\mathfrak{P}$ be an S-compact pretopological space. Consider a P-open cover $\mathbb{O}$ of $\mathfrak{P}$. For $O \in \mathbb{O}$, we have $O \subseteq O \subseteq \mathfrak{c}(O)$. Hence $O \in \mathbb{O}$ is a semi-P-open set. Therefore, $\mathbb{O}$ is an S-cover of $\mathfrak{P}$. Since $\mathfrak{P}$ is S-compact, $\mathfrak{P} = \bigcup_{k=1}^{n} O_k \subseteq \bigcup_{k=1}^{n} \mathfrak{i}(\mathfrak{c}(O_k))$ where $O_k \in \mathbb{O}$ being P-open is pre-P-open. Therefore, $\mathfrak{P} = \bigcup_{k=1}^{n} \mathfrak{i}(\mathfrak{c}(O_k))$ where $O_k \in \mathbb{O}$ and hence, $\mathfrak{P}$ is nearly compact. □

Proposition 9. *If $\mathfrak{P}$ is a compact pretopological space, then it is a nearly compact space.*

Proof. Let $\mathfrak{P}$ be a compact pretopological space. Consider a P-open cover $\mathbb{O}$ of $\mathfrak{P}$. Since every P-open set is a P-neighborhood of all its points, the P-open cover $\mathbb{O}$ is also a P-neighborhood cover of $\mathfrak{P}$. Now, there exists a finite subcover $\{O_k \in \mathbb{O} | 1 \leq k \leq n\}$ of $\mathbb{O}$ as $\mathfrak{P}$ is compact. Thus, for each $\mathfrak{p} \in \mathfrak{P}$, we have $k, 1 \leq k \leq n$, with $\mathfrak{p} \in O_k$ and $O_k \in \mathfrak{N}(\mathfrak{p})$. Consequently, $\mathfrak{P} = \bigcup_{k=1}^{n} O_k$ where $O_k \in \mathbb{O}$. Now, $O_k \subseteq \mathfrak{i}(\mathfrak{c}(O_k))$ as O_k is P-open. Therefore, $\mathfrak{P} = \bigcup_{k=1}^{n} \mathfrak{i}(\mathfrak{c}(O_k))$ and so $\mathfrak{P}$ is nearly compact. □

Proposition 10. *If a pretopological space $\mathfrak{P}$ is po-compact, then it is always nearly compact.*

Proof. Let $\mathfrak{P}$ be a po-compact space. Let $\mathbb{O}$ be a P-open cover of $\mathfrak{P}$. Since every P-open set is pre-P-open, $\mathbb{O}$ is a pre-P-open cover of $\mathfrak{P}$. For each $O \in \mathbb{O} \Rightarrow O \subseteq O \subseteq \mathfrak{c}(O)$. So, $\mathbb{O}$ is a P-open super cover of itself. Because $\mathfrak{P}$ is po-compact, there exists finite P-open super cover $\mathbb{G}$ of $\mathbb{O}$ that covers $\mathfrak{P}$. Therefore, $\mathfrak{P} = \bigcup_{k=1}^{n} \{O_k \in \mathbb{G}\}$. Clearly, O_k being P-open implies that $O_k \subseteq \mathfrak{i}(\mathfrak{c}(O_k))$. This gives $\mathfrak{P} = \bigcup_{k=1}^{n} \{\mathfrak{i}(\mathfrak{c}(O_k))|O_k \in \mathbb{G}\}$. Hence, $\mathfrak{P}$ is nearly compact. □

Remark 9. In a topological space, the converse of Proposition 10 is also true [6, p. 140], but in a pretopological space, the converse is not always true because (c5) may not be satisfied.

Proposition 11. *Let $\mathfrak{T}$ be topological space. If $\mathfrak{T}$ is fully compact, then it is compact.*

Proof. Since $\mathfrak{T}$ is a topological space, it is a pretopological space. The proof follows as in Proposition 6. □

Door spaces are special type of topological spaces. Below we present an interesting observation regarding a door space.

Theorem 1. *Let $\mathfrak{T}$ be the door space. Then $\mathfrak{T}$ is compact if and only if it is strongly compact.*

Proof. Suppose that $\mathfrak{T}$ is a compact space. Consider any pre-open cover $\mathbb{P}$ of $\mathfrak{T}$. Since $\mathfrak{T}$ is a door space, every pre-open set is open. Since $\mathbb{P}$ is an open cover of $\mathfrak{T}$, it is also a neighborhood cover of $\mathfrak{T}$. Since $\mathfrak{T}$ is compact, there is a finite subcover of the neighborhood cover $\mathbb{P}$. Therefore, $\mathfrak{T}$ is strongly compact.

Conversely, assume that $\mathfrak{T}$ is strongly compact. Consider a neighborhood cover $\mathbb{U}$ of $\mathfrak{T}$. Thus, for every $\mathfrak{t} \in \mathfrak{T}$, there exists $J \in \mathbb{U}$ such that $J \in \mathfrak{N}(\mathfrak{t})$. We know that, for every $J \in \mathfrak{N}(\mathfrak{t})$, there exists an open set $O \in \mathfrak{N}(\mathfrak{t})$, such that $O \subseteq J$. Therefore, for each $\mathfrak{t} \in \mathfrak{T}$, there exists an open set $O \subseteq J$ such that $J \in \mathbb{U}$. Therefore, $\mathbb{O} = \{O \subseteq J | O \in \mathfrak{N}(\mathfrak{t}), J \in \mathbb{U} \text{ and } \mathfrak{t} \in \mathfrak{T}\}$ is an open cover of $\mathfrak{T}$. Since every open set is pre-open, $\mathbb{O}$ is a pre-open cover of $\mathfrak{T}$. Thus, there exists a finite subcover $\{O_k | O_k \in \mathbb{O}, 1 \leq k \leq n\}$ of $\mathbb{O}$. This implies that, for every $\mathfrak{t} \in \mathfrak{T}$, there exists k such that $1 \leq k \leq n$ and $O_k \in \mathfrak{N}(\mathfrak{t})$. Since $O_k \subseteq J_k \in \mathbb{U}$, we can say that, for every $\mathfrak{t} \in \mathfrak{T}$, there exists $J_k \in \mathbb{U}$ such that $J_k \in \mathfrak{N}(\mathfrak{t})$, $1 \leq k \leq n$. Thus, $\mathfrak{T}$ is compact. □

Now, we will prove some results related to compactness in clopen spaces.

Theorem 2. *Let $\mathfrak{T}$ be a clopen topological space. Then $\mathfrak{T}$ is S-compact if and only if $\mathfrak{T}$ is compact.*

Proof. If $\mathfrak{T}$ is an S-compact clopen topological space then it is easy to show that $\mathfrak{T}$ is compact. We prove the converse part. Let $\mathfrak{T}$ be a compact clopen topological space. Then every open cover of $\mathfrak{T}$ has a finite subcover. Let $\mathbb{S}$ be any semi-open cover of $\mathfrak{T}$. Here, $\mathfrak{T}$ is clopen. Thus, every semi-open set of $\mathfrak{T}$ will be open. This implies that $\mathbb{S}$ is an open cover of $\mathfrak{T}$. Since $\mathfrak{T}$ is compact, we can write $\mathfrak{T} = \bigcup_{k=1}^{n} S_k$, where $S_k \in \mathbb{S}$. Hence, $\mathfrak{T}$ is S-compact. □

Remark 10. If $\mathfrak{P}$ is clopen pretopological space, then $\mathfrak{P}$ is S-compact if $\mathfrak{P}$ is compact.

Theorem 3. *Let $\mathfrak{T}$ be a clopen topological space. Then $\mathfrak{T}$ is compact if and only if $\mathfrak{T}$ is nearly compact.*

Proof. If $\mathfrak{T}$ is compact, then it is easily seen that $\mathfrak{T}$ is nearly compact. We prove the converse part. Let $\mathfrak{T}$ be nearly compact. We will show that $\mathfrak{T}$ is compact. Consider any open cover $\mathbb{O}$ of $\mathfrak{T}$. Since $\mathfrak{T}$ is nearly compact, we can write $\mathfrak{T} = \bigcup_{k=1}^{n} \mathfrak{i}(\mathfrak{c}(O_k))$, where $O_k \in \mathbb{O}$. Here, $\mathfrak{T}$ is clopen space, and therefore, we can write $\mathfrak{i}(\mathfrak{c}(O_k)) = \mathfrak{i}(O_k) = O_k$ for all $O_k \in \mathbb{O}$. Therefore, $\mathfrak{T} = \bigcup_{k=1}^{n} \mathfrak{i}(\mathfrak{c}(O_k)) = \bigcup_{k=1}^{n} O_k$, where $O_k \in \mathbb{O}$. Hence, $\mathfrak{T}$ is compact. □

Theorem 4. *Let $\mathfrak{T}$ be a clopen topological space. Then $\mathfrak{T}$ is S-compact if and only if $\mathfrak{T}$ is nearly compact.*

Proof. If $\mathfrak{T}$ is S-compact, then it is easily seen that $\mathfrak{T}$ is nearly compact. We will prove the converse part. Let $\mathfrak{T}$ be nearly compact. We will show that $\mathfrak{T}$ is S-compact. Consider an arbitrary S-cover $\mathbb{S}$ of $\mathfrak{T}$. Since $\mathfrak{T}$ is a clopen topological space, every semi-open set will be an open set. Therefore, $\mathbb{S}$ will be an open cover. Moreover, $\mathfrak{T}$ is nearly compact. Thus, $\mathfrak{T} = \bigcup_{k=1}^{n} \mathfrak{i}(\mathfrak{c}(S_k))$, where $S_k \in \mathbb{S}$. Here, S_k is an open set and so $\mathfrak{i}(\mathfrak{c}(S_k)) = S_k$. Thus, $\mathfrak{T} = \bigcup_{k=1}^{n} (S_k)$ with $S_k \in \mathbb{S}$. Hence, $\mathfrak{T}$ is S-compact. □

Remark 11. Just like in the case of clopen topological spaces, we can prove that a P-clopen pretopological space is S-compact if and only if it is nearly compact.

We summarize the above relationship between the types of compactness in both pretopological and topological spaces by Figures 2, 3, 4, 5 and 6.

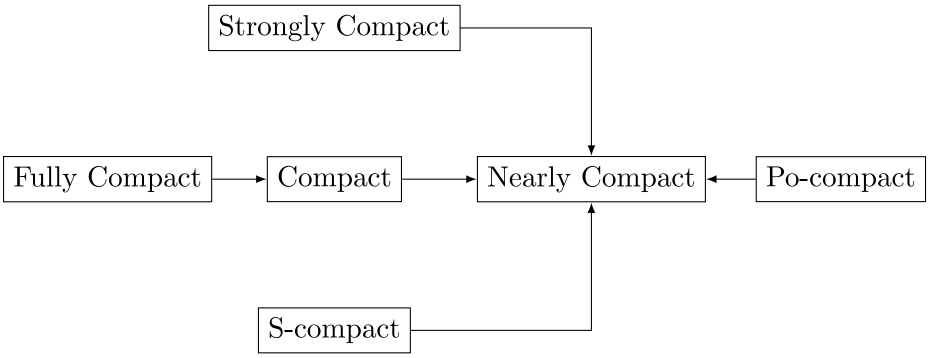

FIGURE 2. Compactness in a pretopological space.

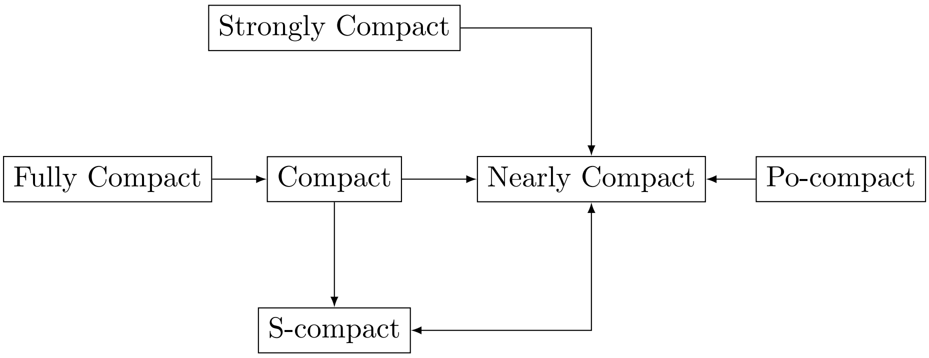

FIGURE 3. Compactness in a P-clopen pretopological space.

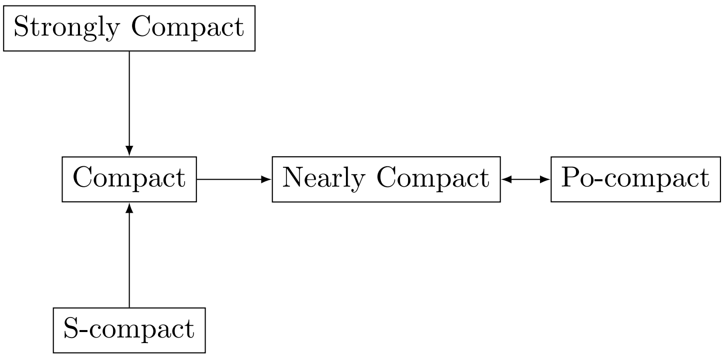

FIGURE 4. Compactness in a topological space [6].

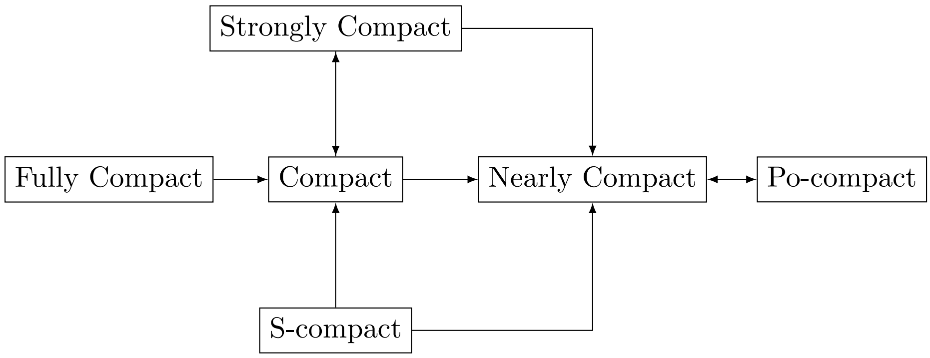

Figure 5. Compactness in a door space.

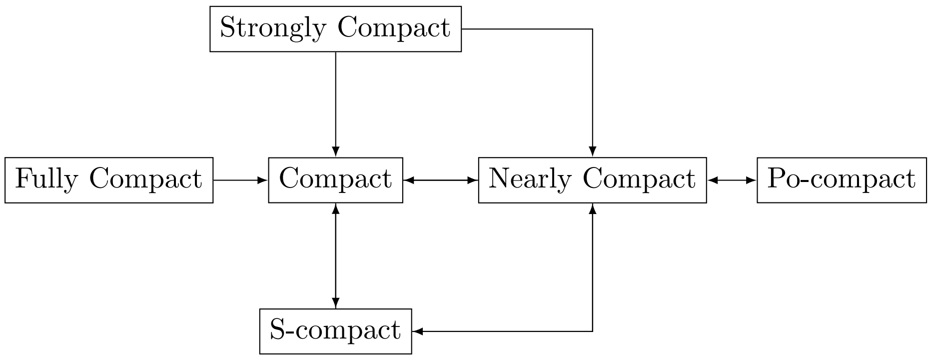

Figure 6. Compactness in a clopen topological space.

4. Conclusion

In this paper, we have explored various open sets in a pretopological space and subsequently investigated the concept of compactness. Our examination has revealed that "full compactness" is the strongest version and "near compactness" is the weakest version of compactness, both within the realms of topological and pretopological spaces. We have also discussed the compactness of few special types of topological spaces to establish our results. It becomes evident that many conclusions in pretopological spaces align with those in topological spaces, yet certain results do exhibit disparities.

Acknowledgements

The first author is thankful to Marwadi University, Rajkot, India for providing research support under the full-time PhD scholarship program. All the authors are thankful to the anonymous reviewers for their invaluable feedback and suggestions, which have significantly contributed to the improvement of this manuscript.

References

[1] M. Ahat, S. B. Amor, M. Bui, S. Jhean-Larose, and G. Denhiere, *Document classification with LSA and pretopology*, Stud. Inform. Univ. **8** (2010), 125–144. URL

[2] M. Ahat, S. B. Amor, M. Bui, M. Lamure, and M.-F. Courel, *Pollution modeling and simulation with multi-agent and pretopology.* In: J. Zhou (eds) Complex Sciences, Lecture Notes of the Institute for Computer Sciences, Social Informatics and Telecommunications Engineering, 225–231, Springer, 2009. DOI

[3] R. A. H. Al-Abdulla, *Pre-open sets in topological spaces*, Al-Qadisiyah J. Pure Science **12** (2007), 203–213.

[4] J. P. Auray, G. Duru, and M. Mougeot, *A pre-topological analysis of the input-output model*, Econom. Lett. **2** (1979), 343–347. DOI

[5] A. A. Azzam, A. M. Nasr, H. ElGhawalby, and R. Mareay, *Application on similarity relation and pretopology*, Fractal Fract. **7**, (2023), 168, 16 pp. DOI

[6] K. B. Bagchi, A. Mukharjee, and M. Paul, *A new approach to nearly compact spaces*, Acta Comment. Univ. Tartu. Math. **22** (2018), 137–147. DOI

[7] S. Bonnevay, *Pretopological operators for gray-level image analysis*, Stud. Inform. Univ. **7** (2009), 27–44. URL

[8] Q. V. Bui, S. Ben Amor, M. Bui, *Stochastic pretopology as a tool for complex networks analysis*, J. Information and Telecommunication **3** (2019), 135–155. DOI

[9] Q. V. Bui, K. Sayadi, and M. Bui, *A multi-criteria document clustering method based on topic modeling and pseudoclosure function*, Informatica **40** (2016), 169–180. DOI

[10] G. Caillaut and G. Cleuziou, *Multi-instance learning of pretopological spaces to model complex propagation phenomena: Application to lexical taxonomy learning*, Artificial Intelligence **301** (2021), 103556, 74 pp. DOI

[11] G. Caillaut, G. Cleuziou, and N. Dugué, *Learning pretopological spaces to extract ego-centered communities.* In: Q. Yang et al. (eds) Advances in Knowledge Discovery and Data Mining: 23rd Pacific-Asia Conference, Macau, China, April 14-17, 488–500, 2019. DOI

[12] E. Čech, Z. Frolík, and M. Katětov, *Topological Spaces*, Academia, Czechoslovak Academy of Sciences, Prague, 1966. URL

[13] G. Choquet, *Convergences*, Annales de l'université de Grenoble **23** (1948), 57–11. URL

[14] G. Cleuziou, D. Buscaldi, V. Levorato, and G. Dias, *A pretopological framework for the automatic construction of lexical-semantic structures from texts.* In: Proceedings of the 20th ACM International Conference on Information and Knowledge Management, 2453–2456, Association for Computing Machinery, New York, 2011. DOI

[15] G. Cleuziou and G. Dias, *Learning pretopological spaces for lexical taxonomy acquisition.* In: A. Appice et al. (eds) Machine Learning and Knowledge Discovery in Databases: European Conference, September 7-11, **9285**, 493–508, Porto, Portugal, 2015. DOI

[16] M. Dalud-Vincent, M. Brissaud, and M. Lamure, *Connectivities and partitions in a pretopological space*, International Mathematical Forum **6** (2011), 2201–2215. URL

[17] M. M. Day, *Convergence, closure and neighborhoods*, Duke Math. J. **11** (1944), 181–199. DOI

[18] S. Dolecki, *An initiation into convergence theory*, Beyond Topology, Contemp. Math. **486** (2009), 115–161. DOI

[19] S. Dolecki, G. H. Greco, and A. Lechicki, *Compactoid and compact filters*, Pacific J. Math. **117** (1985), 69–98. DOI

[20] L. Frank and E. Hubert, *Pretopological approach for supervised learning.* In: Proceedings of 13th International Conference on Pattern Recognition, August 25-29,**4**, 256–260, Vienna, Austria, 1996, DOI

[21] M. M. Fréchet, *Sur quelques points du calcul fonctionnel*, Rend. Circ. Mat. Palermo **22** (1906), 1–72. DOI

[22] C. Frélicot and H. Emptoz, *A pretopological approach for pattern classification with reject options.* In: A. Amin et al. (eds) Advances in Pattern Recognition, August 11-13, Lecture Notes in Computer Science **1451**, 707–715, Sydney, Australia, 1998. DOI

[23] C. Frelicot and F. Lebourgeois, *A pretopology-based supervised pattern classifier.* In: Proceedings, Fourteenth International Conference on Pattern Recognition (Cat. No. 98EX170) **1**, 106–109, Brisbane, Australia, 1998. DOI

[24] D. Friboulet, M. P. Lamure, I. E. Magnin, J.-J. Milan, and N. Nicoloyannis, *Topological model based method for 3D reconstruction of surfaces. Application: Cardiac Imaging.* In: S. J. Dwyer et al. (eds) Medical Imaging III: Image Processing, Society of Photo-Optical Instrumentation Engineers Conference Series, May, **1092**, 502–510, 1989. DOI

[25] S. Gaal, *Point Set Topology*, Courier Corporation, 2009.

[26] F. Hausdorff, *Gestufte Räume*, Fund. Math. **25** (1935), 486–502. DOI

[27] F. Hausdorff, *Grundzüge der Mengenlehre*, Verlag von Veit, Leipzig, 1914.

[28] T. K. T. Ho, Q. V. Bui, and M. Bui, *Dynamic social network analysis: A novel approach using agent-based model, author-topic model, and pretopology*, Concurrency and Computation: Practice and Experience **32** (2020), e5321, 16 pp. DOI

[29] J. L. Kelley, *General Topology*, Springer, New York, 1975.

[30] D. C. Kent and W. K. Min, *Neighborhood spaces*, Int. J. Math. Math. Sci. **32** (2002), 387–399. DOI

[31] C. Kuratowski, *Sur l'opération A de l'analysis situs*, Fund. Math. **3** (1922), 182–199. URL

[32] M. Lamure, S. Bonnevay, M. Bui, and S. Ben-Amor, *A stochastic and pretopological modeling aerial pollution of an urban area*, Stud. Inform. Univ. **7** (2009), 410–426.

[33] M. Lamure and J. J. Milan, *A new concept for binary images: the kernel.* In: V. Cantoni et al. (eds) Image Analysis and Processing II, 191–196, Springer, Boston, 1988. DOI

[34] C. Largeron and S. Bonnevay, *A pretopological approach for structural analysis*, Information Sciences **144** (2002), 169–185. DOI

[35] T. V. Le, N. Kabachi, and M. Lamure, *A clustering method associated pretopological concepts and k-means algorithm.* In: Recent Advances in Stochastic Modeling and Data Analysis, 529–536, World Scientific, 2007. DOI

[36] F. Lebourgeois, M. Bouayad, and H. Emptoz, *Structure relation between classes for supervised learning using pretopology.* In: Proceedings of the Fifth International Conference on Document Analysis and Recognition, ICDAR '99 (Cat. No.PR00318), 33–36, Bangalore, India, 1999. DOI

[37] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41. DOI

[38] V. Levorato, *Group measures and modeling for social networks*, J. Complex Systems **2014** (2014), 354385, 10 pp. DOI

[39] V. Levorato, *Modeling groups in social networks*, ECMS, Poland, 129–134, 2011. DOI

[40] L. N. Lévy, J. Bosom, G. Guerard, S. B. Amor, M. Bui, and H. Tran, *Application of pretopological hierarchical clustering for buildings portfolio*. In: 10th International Conference on Smart Cities and Green ICT Systems (SMARTGREENS 2021), 228–235, 2021. DOI

[41] J. B. Listing, *Vorstudien zur Topologie*, Gottingen Studien, 1847.

[42] D. Mammass, S. Djeziri, and F. Nouboud, *A pretopological approach for image segmentation and edge detection*, J. Math. Imaging Vision **15** (2001), 169–179. DOI

[43] D. Mammas, M. El Yassa, F. Nouboud, and A. Chalifour, *A multicriterion pretopological approach for image segmentation*, In: 3^{rd} International Conference: Sciences of Electronic Technologies of Information and Telecommunications (SETIT 2005), IEEE, March 27-31, 2005.

[44] A. S. Mashhour, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt. **53** (1982), 47–53.

[45] A. S. Mashhour, *Strongly compact spaces*, Delta J. Sci. **8** (1984), 30–46.

[46] A. Meziane, T. Iftene, and N. Selmaoui, *Satellite image segmentation by mathematical pretopology and automatic classification*, Image Processing, Signal Processing, and Synthetic Aperture Radar for Remote Sensing **3217** (1997), 232–236. DOI

[47] J. R. Munkres, *Topology*, 2nd ed., Prentice Hall, 2000.

[48] N. K. K. Nguyen and M. Bui, *Modeling cascading failures in stock markets by a pretopological framework*, Vietnam J. Computer Science **8** (2021), 23–38. DOI

[49] N. K. K. Nguyen, Q. Nguyen, and M. Bui, *Mining stock market time series and modeling stock price crash using a pretopological framework*, In: N. Nguyen et al. (eds) Computational Collective Intelligence: 11th International Conference (ICCCI 2019), September 4-6, Lecture Notes in Computer Science **11683**, 638–649, Hendaye, France, 2019. DOI

[50] E. Piegay, N. Selmaoui, and C. Leschi *Crest lines detection by valleys spreading*. In: V. Hlavac and R. Sara (eds) Computer Analysis of Images and Patterns (CAIP 1995), September 6-8, Lecture Notes in Computer Science **970**, 270–277, Springer, Berlin, 1995. DOI

[51] P. Pina, N. Selmaoui, and M. A. Fortes, *Geometrical and topological characterization of cork cells by digital image analysis*, In: P. Maragos et al. (eds) Mathematical Morphology and Its Applications to Image and Signal Processing, Computational Imaging and Vision **5**, 459–66, Springer, Boston, 1996. DOI

[52] H. Poincare, *Analysis Situs*, Gauthier-Villars Paris, France, 1895.

[53] L. Polkowski,*Topology of Concepts*, In: Approximate Reasoning by Parts Intelligent Systems Reference Library **20**, Springer, Berlin 2011. DOI

[54] R. Prasad and R. S. Yadav, *On s-compact spaces*, Indian J. Math, **24** (1982), 209–214.

[55] I. L. Reilly and M. K. Vamanamurthy, *On some questions concerning pre-open sets*, Kyungpook Math. J. **30** (1990), 87–93.

[56] J. Reynolds, *H-closed spaces and H-sets in the convergence setting*, arXiv:1510.08044 [math.GN] (2015), 14 pp. DOI

[57] N. Selmaoui-Folcher, J. Tokotoko, S. Gorohouna, L. Roi, C. Leschi, and C. Ris, *Concept of temporal pretopology for the analysis for structural changes: application to econometrics*, International J. Data Warehousing and Mining (IJDWM), **18** (2022), 1–17. DOI

[58] J. N. Sharma, *Topology*, Krishna Prakashan Mandir, 1975.

[59] M. K. Singal and A. Mathur, *On nearly compact spaces*, Boll. Unione Mat. Ital. **4** (1969), 702–710.

[60] V. Solis, O. Cordova, and J. M. Marquez, *A pretopology analysis for the dissemination of pulses in the economy: a case of study for Mexico and Brazil*, In: 23^{rd} International

Input-Output Conference & 5th Edition of the International School of I-O analysis, June 22-26, Mexico, 2015.

[61] B. M. R. Stadler and P. F. Stadler, *Basic properties of closure spaces*, J. Chem, Inf. Comput. Sci. **42** (2002), 577–585.

[62] B. M. R. Stadler and P. F. Stadler, *Generalized topological spaces in evolutionary theory and combinatorial chemistry*, J. Chemical Information and Computer Sciences **42** (2002), 577–585. DOI

[63] B. M. R. Stadler, P. F. Stadler, M. Shpak, and G. P. Wagner, *Recombination spaces, metrics, and pretopologies*, Zeitschrift für Physikalische Chemie **216** (2002), 17 pp. DOI

[64] B. M. R. Stadler, P. F. Stadler, G. P. Wagner, and W. Fontana, *The topology of the possible: Formal spaces underlying patterns of evolutionary change*, J. Theoret. Biol. **213** (2001), 241–274. DOI

[65] T. Van Le, T. N. Truong, H. N. Nguyen, and T. V. Pham, *An efficient pretopological approach for document clustering.* In: 5th International Conference on Intelligent Networking and Collaborative Systems, September 9-11, 114–120, Xi'an city, Shaanxi province, China, 2013. DOI

[66] S. Willard, *General Topology*, Courier Corporation, 2012.

Department of Mathematics, Marwadi University, Morbi road, Rajkot, 360003, Gujarat, India

E-mail address: devangi.sojitra.211@gmail.com

E-mail address: biswas.minakshi@gmail.com

E-mail address: gautamhathiwala1717@gmail.com