ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 29, Number 1, June 2025
Available online at https://ojs.utlib.ee/index.php/ACUTM

Invariant metrics on nilpotent Lie algebras

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Abstract. In this work we state criteria for a nilpotent Lie algebra $\mathfrak{g}$ to admit an invariant metric. We use that $\mathfrak{g}$ possesses two canonical abelian ideals $\mathfrak{i}(\mathfrak{g}) \subset \mathfrak{J}(\mathfrak{g})$ to decompose the underlying vector space of $\mathfrak{g}$. By using this decomposition we state sufficiency conditions for $\mathfrak{g}$ to admit an invariant metric. The properties of the ideal $\mathfrak{J}(\mathfrak{g})$ allow to prove that if a current Lie algebra $\mathfrak{g} \otimes \mathcal{S}$ admits an invariant metric, then there must be an invariant and non-degenerate bilinear map from $\mathcal{S} \times \mathcal{S}$ into the space of centroids of $\mathfrak{g}/\mathfrak{J}(\mathfrak{g})$. We also prove that in any nilpotent Lie algebra $\mathfrak{g}$ there exists a non-zero, symmetric and invariant bilinear form. This bilinear form allows to reconstruct $\mathfrak{g}$ by means of an algebra with unit. We prove that this algebra is simple if and only if the bilinear form is an invariant metric on $\mathfrak{g}$.

1. Introduction

Let $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,])$ be a Lie algebra over a field $\mathbb{F}$ with bracket $[\cdot,\,\cdot]$. The Lie algebra $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,])$ is said to be *quadratic* if it is equipped with a non-degenerate, symmetric and bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$, which is *invariant* under the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, that is: $B([x, y], z) = -B(y, [x, z])$, for all x, y, z in $\mathfrak{g}$. A quadratic Lie algebra is denoted by $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,], B)$, and B is called an *invariant metric*.

A semisimple Lie algebra is quadratic, as the Killing form is an invariant metric defined on it. A reductive Lie algebra is an example of a non-semisimple quadratic Lie algebra; another more elaborated example is the following: Let $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,])$ be a Lie algebra. Define the skew-symmetric and bilinear map $[\,\cdot\,,\,\cdot\,]'$ in the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$, by $[x + \alpha, y + \beta]' = [x, y] + \mathrm{ad}(x)^*(\beta) - \mathrm{ad}(y)^*(\alpha)$, and consider the symmetric bilinear form B defined

Received October 25, 2024.
2020 *Mathematics Subject Classification.* 17B30, 17B05, 15A63, 17B56, 17B20.
Key words and phrases. Associative and commutative algebra, invariant bilinear form, nilpotent Lie algebra, simple algebra, tensor product.
https://doi.org/10.12697/ACUTM.2025.29.03
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by $B(x+\alpha, y+\beta) = \alpha(y) + \beta(x)$, for all x, y in $\mathfrak{g}$ and α, β in $\mathfrak{g}^*$. The triple $(\mathfrak{g} \oplus \mathfrak{g}^*, [\cdot\,,\,\cdot]', B)$ is a non-semisimple quadratic Lie algebra, as $\mathfrak{g}^*$ is an abelian ideal.

The algebraic study of quadratic non-semisimple Lie algebras can be traced back to the work of Kac [7], presenting an inductive way to construct solvable quadratic Lie algebras. It is also shown there that any solvable quadratic Lie algebra can be identified with this construction. In [3], the conditions are stated for which these type of quadratic Lie algebras are isomorphic. In [8], Medina and Revoy generalize the results of Kac to any indecomposable, non-semisimple and quadratic Lie algebra; the proof of these results depends on the choice of a minimal ideal. In [6], Kath and Olbrich propose an alternative approach by using canonical ideals to obtain classification results.

If the Killing form is non-degenerate, the Lie algebra is semisimple and, therefore, is quadratic. For nilpotent Lie algebras, the Killing form is zero, so this bilinear form is of no help to determine whether a nilpotent Lie algebra is quadratic. One of the aims of this work is to provide criteria under which a nilpotent Lie algebra is quadratic. Although much has been said about nilpotent quadratic Lie algebras, we hope that these results, which are based on classical tools and well-known results, can take a step forward in obtaining a criterion that determines conditions for which an arbitrary Lie algebra is quadratic.

If $\mathfrak{g}$ is a nilpotent Lie algebra then there are canonical abelian ideals $\mathfrak{i}(\mathfrak{g}) \subset \mathfrak{J}(\mathfrak{g})$ such that $[\mathfrak{g}, \mathfrak{J}(\mathfrak{g})] \subset \mathfrak{i}(\mathfrak{g})$; further, if $\mathfrak{g}$ admits an invariant metric then $\mathfrak{i}(\mathfrak{g})^{\perp} = \mathfrak{J}(\mathfrak{g})$ (see Lemma 4.2 in [6] and Lemma 1 below). If $\mathfrak{h}$ is a subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{J}(\mathfrak{g})$, we prove that a necessary condition for a *current Lie algebra* $\mathfrak{g} \otimes \mathcal{S}$ to admit an invariant metric, where $\mathcal{S}$ is a commutative and associtive algebra with unit, is that there must be a bilinear map $\epsilon : \mathcal{S} \times \mathcal{S} \to \mathrm{Cent}(\mathfrak{h})$ with the same properties as that of an invariant metric; that is, ϵ is symmetric, invariant and non-degenerate (see Theorem 1). In [9] it is proved that if $\mathfrak{g}$ is simple then $\mathcal{S}$ admits an invariant metric. In Corollary 1.3 of [4] this result is generalized to the case when $\mathfrak{g}$ is indecomposable. The difference of the result obtained here is that the image of the bilinear map ϵ is contained in $\mathrm{Cent}(\mathfrak{h})$, although $\mathfrak{h}$ does not admit an invariant metric and $\mathcal{S}$ is not necessarily finite dimensional. In addition, in [4] it is assumed that the ground field is algebraically closed.

In Section 2 we start by giving a review of the theory of abelian extensions of Lie algebras, as well as some standard results that we need in the sequel. Given an arbitrary Lie algebra $\mathfrak{h}$ and a vector space $\mathfrak{a}$, in Proposition 1 we give a method to construct a Lie algebra in the space $\mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{h}^*$. In Proposition 3 we prove that any non-abelian nilpotent quadratic Lie algebra can be constructed in this way. In Propositions 5 and 6 we give sufficiency conditions for such a Lie algebra to admit an invariant metric. In Section 3 we prove that if a current Lie algebra $\mathfrak{g}\otimes\mathcal{S}$ admits an invariant metric, where $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{a}\oplus\mathfrak{h}^*$ is nilpotent and $\mathfrak{h}$ is non-abelian, then there must be a non-degenerate bilinear map $\epsilon:\mathcal{S}\times\mathcal{S}\to\operatorname{Cent}(\mathfrak{h})$ such that $\epsilon(s,t)=\epsilon(st,1)$, for all s,t in $\mathcal{S}$ (see Theorem 1). In §4 we prove that any nilpotent Lie algebra has a non-zero, symmetric and invariant bilinear form. Using this bilinear form, we construct an algebra with unit from which we can recover the original Lie algebra structure, and we prove that this algebra is simple if and only if the bilinear form is an invariant metric (see Theorem 2). All the algebras and vector spaces considered in this work are finite dimensional over a unique field $\mathbb{F}$ of zero characteristic.

2. Abelian extensions of Lie algebras

Let $\rho:\mathfrak{g}\to\mathfrak{gl}(V)$ be a representation of a Lie algebra $(\mathfrak{g},[\,\cdot\,,\,\cdot\,])$ on a vector space V. We call V a $\mathfrak{g}$-*module*. Let $C(\mathfrak{g};V)$ be the standard cochain complex with differential map d_ρ given by:

$$\begin{aligned}\mathrm{d}_\rho(\lambda)(x_1,\dots,x_{p+1})=&\sum_{j=1}^{p+1}(-1)^{j-1}\rho(x_j)(\lambda(x_1,\dots,x_{p+1}))\\&+\sum_{i<j}(-1)^{i+j}\lambda([x_i,x_j],x_1,\dots,x_{\hat{i}},\dots,x_{\hat{j}},\dots,x_{p+1}),\end{aligned}$$

where p is a non-negative integer, λ belongs to $C^p(\mathfrak{g};V)$, and x_j belongs to $\mathfrak{g}$, for all $1\le j\le p+1$. If $\mathrm{d}_\rho(\lambda)=0$ then λ is called a p-cocycle. If the representation ρ is zero, we say that V is a *trivial* $\mathfrak{g}$-*module* and we denote the differential map in $C(\mathfrak{h};V)$ by d_V.

Let $\mathfrak{J}$ be an abelian ideal of $(\mathfrak{g},[\,\cdot\,,\,\cdot\,])$ and $\mathfrak{h}$ a vector subspace complementary to $\mathfrak{J}$, that is $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{J}$. For each pair x,y in $\mathfrak{h}$, let $[x,y]_{\mathfrak{h}}$ be in $\mathfrak{h}$ and $\Lambda(x,y)$ be in $\mathfrak{J}$ such that

$$[x,y]=[x,y]_{\mathfrak{h}}+\Lambda(x,y). \tag{1}$$

Let $[\,\cdot\,,\,\cdot\,]_{\mathfrak{h}}$ be the skew-symmetric and bilinear map on $\mathfrak{h}$ defined by $(x,y)\mapsto[x,y]_{\mathfrak{h}}$. Since the bracket $[\,\cdot\,,\,\cdot\,]$ satisfies the Jacobi identity, $[\,\cdot\,,\,\cdot\,]_{\mathfrak{h}}$ is a Lie bracket in $\mathfrak{h}$. By $\Lambda:\mathfrak{h}\times\mathfrak{h}\to\mathfrak{J}$, we denote the skew-symmetric and bilinear map $(x,y)\mapsto\Lambda(x,y)$.

Let $R:\mathfrak{h}\to\mathfrak{gl}(\mathfrak{J})$ be the linear map defined by $R(x)(U)=[x,U]$, for all x in $\mathfrak{h}$ and U in $\mathfrak{J}$. Let x,y be in $\mathfrak{h}$ and U be in $\mathfrak{J}$; the Jacobi identity, i.e., the vanishing on the cyclic sum of on $[x,[y,U]]$ and the fact that $\mathfrak{J}$ is abelian imply that R is a representation of $(\mathfrak{h},[\,\cdot\,,\,\cdot\,]_{\mathfrak{h}})$. In addition, the Jacobi identity on $[x,[y,z]]$, where x,y and z are in $\mathfrak{h}$, implies

$$\begin{aligned}&R(x)(\Lambda(y,z)+R(y)(\Lambda(z,x))+R(z)(\Lambda(x,y))\\&+\Lambda(x,[y,z]_{\mathfrak{h}})+\Lambda(y,[z,x]_{\mathfrak{h}})+\Lambda(z,[x,y]_{\mathfrak{h}})=0.\end{aligned}$$

Then Λ is a 2-cocycle in the complex $C(\mathfrak{h};\mathfrak{J})$.

Conversely, let $R:\mathfrak{h}\to\mathfrak{gl}(\mathfrak{J})$ be a representation of a Lie algebra $(\mathfrak{h},[\,\cdot\,,\,\cdot\,]_{\mathfrak{h}})$ in a vector space $\mathfrak{J}$ and let Λ be a 2-cocycle in $C(\mathfrak{h};\mathfrak{J})$. The skew-symmetric and bilinear map $[\,\cdot\,,\,\cdot\,]$ defined in $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{J}$ by

$$\begin{aligned}[x,y]=[x,y]_{\mathfrak{h}}+\Lambda(x,y),\quad [x,U]=R(x)(U),\\ [U,V]=0,\quad \text{for all } x,y\in\mathfrak{g},\ \text{ and } U,V\in\mathfrak{J},\end{aligned}\tag{2}$$

is a Lie bracket. The pair $(\mathfrak{g},[\,\cdot\,,\,\cdot\,])$ is the *abelian extension of* $(\mathfrak{h},[\,\cdot\,,\,\cdot\,]_{\mathfrak{h}})$ *by* $\mathfrak{J}$ *associated to* R *and* Λ. We denote this Lie algebra by $\mathfrak{h}(\Lambda,R)$.

For our purposes, in the following we assume that $(\mathfrak{g},[\,\cdot\,,\,\cdot\,])$ has abelian ideals $\mathfrak{i}$ and $\mathfrak{J}$ such that

$$\text{(a)}\quad \mathfrak{J}\ \text{ is abelian;}\qquad \text{(b)}\quad \mathfrak{i}\subset\mathfrak{J};\qquad \text{(c)}\quad [\mathfrak{g},\mathfrak{J}]\subset\mathfrak{i}.\tag{3}$$

For nilpotent non-abelian Lie algebras these ideals always exist (see Lemma 1 below).

Let $\mathfrak{h}$ be a subspace of $\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{J}$, and let $\mathfrak{a}$ be a subspace of $\mathfrak{g}$ satisfying $\mathfrak{J}=\mathfrak{a}\oplus\mathfrak{i}$; thus $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{a}\oplus\mathfrak{i}$. For x,y in $\mathfrak{h}$, we write $[x,y]=[x,y]_{\mathfrak{h}}+\Lambda(x,y)$, where $[x,y]_{\mathfrak{h}}$ belongs to $\mathfrak{h}$ and $\Lambda(x,y)$ belongs to $\mathfrak{J}$ (see (1)). For skew-symmetric and bilinear maps $\lambda:\mathfrak{h}\times\mathfrak{h}\to\mathfrak{a}$ and $\mu:\mathfrak{h}\times\mathfrak{h}\to\mathfrak{i}$, we descompose Λ as follows: $\Lambda(x,y)=\lambda(x,y)+\mu(x,y)$.

Since $\mathfrak{J}$ is equal to $\mathfrak{a}\oplus\mathfrak{i}$ and $[\mathfrak{g},\mathfrak{J}]$ is contained in $\mathfrak{i}$, the representation $R:\mathfrak{h}\to\mathfrak{gl}(\mathfrak{J})$ has the following decomposition:

$$\varphi(x)=R(x)|_{\mathfrak{a}}:\mathfrak{a}\to\mathfrak{i},\qquad \text{and}\qquad \rho(x)=R(x)|_{\mathfrak{i}}:\mathfrak{i}\to\mathfrak{i},\qquad \text{where } x\in\mathfrak{h}.$$

This yields linear maps $\varphi:\mathfrak{h}\to\operatorname{Hom}(\mathfrak{a};\mathfrak{i})$ and $\rho:\mathfrak{h}\to\mathfrak{gl}(\mathfrak{i})$ such that $[x,u+\alpha]=R(x)(u+\alpha)=\varphi(x)(u)+\rho(x)(\alpha)$, for all u in $\mathfrak{a}$ and α in $\mathfrak{i}$, and we write $R=(\varphi,\rho)$. Thus, the bracket $[\,\cdot\,,\,\cdot\,]$ in $\mathfrak{g}$ takes the form

$$\begin{aligned}[x,y]&=[x,y]_{\mathfrak{h}}+\lambda(x,y)+\mu(x,y),\\ [x,u+\alpha]&=\varphi(x)(u)+\rho(x)(\alpha),\end{aligned}$$

for all x,y in $\mathfrak{h}$, u in $\mathfrak{a}$ and α in $\mathfrak{i}$. Due to $R=(\varphi,\rho)$ being a representation, we have the identities

$$\varphi([x,y]_{\mathfrak{h}})=\rho(x)\circ\varphi(y)-\rho(y)\circ\varphi(x),\tag{4}$$

$$\rho([x,y]_{\mathfrak{h}})=\rho(x)\circ\rho(y)-\rho(y)\circ\rho(x),\quad \text{for all } x,y\in\mathfrak{h}.\tag{5}$$

From (5) it follows that $\rho:\mathfrak{h}\to\mathfrak{gl}(\mathfrak{i})$ is a representation of $(\mathfrak{h},[\,\cdot\,,\,\cdot\,]_{\mathfrak{h}})$ in $\mathfrak{i}$. The condition (4) states that φ is a 1-cocycle in the complex $C(\mathfrak{h};\operatorname{Hom}(\mathfrak{a},\mathfrak{i}))$ with representation $\bar\rho:\mathfrak{h}\to\mathfrak{gl}(\operatorname{Hom}(\mathfrak{a};\mathfrak{i}))$ defined by

$$\bar\rho(x)(\tau)=\rho(x)\circ\tau,\quad \text{for all } \tau\in\operatorname{Hom}(\mathfrak{a};\mathfrak{i}).\tag{6}$$

Since $[\mathfrak{h},\mathfrak{a}]$ is contained in $\mathfrak{i}$ and has no component in $\mathfrak{a}$, we assume that $\mathfrak{a}$ is a trivial $\mathfrak{h}$-module. Due to $\mathfrak{J}=\mathfrak{a}\oplus\mathfrak{i}$, we write the differential map d_R of $C(\mathfrak{h};\mathfrak{J})$ as follows.

Let $x_1, \dots, x_p$ be in $\mathfrak{g}$,

$$\mathbf{X}_p = (x_1, \dots, x_p), \quad \text{and} \quad \mathbf{X}_p^j = (x_1, \dots, x_{\hat{j}} \dots, x_p), \quad 1 \le j \le p.$$

For a given Λ in $C^p(\mathfrak{h}; \mathfrak{J})$, let $\lambda(\mathbf{X}_p)$ be in $\mathfrak{a}$ and $\mu(\mathbf{X}_p)$ be in $\mathfrak{i}$ such that $\Lambda(\mathbf{X}_p) = \lambda(\mathbf{X}_p) + \mu(\mathbf{X}_p)$. The maps $\mathbf{X}_p \mapsto \lambda(\mathbf{X}_p)$ and $\mathbf{X}_p \mapsto \mu(\mathbf{X}_p)$, belong to $C^p(\mathfrak{h}; \mathfrak{a})$ and $C^p(\mathfrak{h}; \mathfrak{i})$, respectively; we denote these maps by λ and μ, respectively. Let $\mathrm{d}_{\mathfrak{a}}$ be the differential in $C(\mathfrak{h}; \mathfrak{a})$ and let d_ρ be the differential in $C(\mathfrak{h}; \mathfrak{i})$. Due to $R = (\varphi, \rho)$, we have

$$\begin{aligned} &\mathrm{d}_R(\Lambda)(\mathbf{X}_{p+1}) = \\ &\mathrm{d}_{\mathfrak{a}}(\lambda)(\mathbf{X}_{p+1}) + \sum_{j=1}^{p+1} (-1)^{j-1} \varphi(x_j) \left(\lambda(\mathbf{X}_{p+1}^j) \right) + \mathrm{d}_\rho(\mu)(\mathbf{X}_{p+1}). \end{aligned} \tag{7}$$

This suggests to consider the map $e_\varphi : C(\mathfrak{h}; \mathfrak{a}) \to C(\mathfrak{h}; \mathfrak{i})$, defined by

$$e_\varphi(\lambda)(\mathbf{X}_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j-1} \varphi(x_j) \left(\lambda(\mathbf{X}_{p+1}^j) \right), \quad \text{for all } \lambda \in C^p(\mathfrak{h}; \mathfrak{a}). \tag{8}$$

Thus, we can rewrite (7) as

$$\mathrm{d}_R(\Lambda)(\mathbf{X}_{p+1}) = \mathrm{d}_{\mathfrak{a}}(\lambda)(\mathbf{X}_{p+1}) + e_\varphi(\lambda)(\mathbf{X}_{p+1}) + \mathrm{d}_\rho(\mu)(\mathbf{X}_{p+1}), \tag{9}$$

where $\mathrm{d}_{\mathfrak{a}}(\lambda)(\mathbf{X}_{p+1})$ belongs to $\mathfrak{a}$ and $e_\varphi(\lambda)(\mathbf{X}_{p+1}) + \mathrm{d}_\rho(\mu)(\mathbf{X}_{p+1})$ belongs to $\mathfrak{i}$. We make the identification $\Lambda = (\lambda, \mu)$, where λ is in $C^p(\mathfrak{h}; \mathfrak{a})$ and μ is in $C^p(\mathfrak{h}; \mathfrak{i})$. By (9), the complex $C(\mathfrak{h}; \mathfrak{J}) = C(\mathfrak{h}; \mathfrak{a}) \oplus C(\mathfrak{h}; \mathfrak{i})$ has the differential map

$$\mathrm{d}_R(\lambda, \mu) = (\mathrm{d}_{\mathfrak{a}}(\lambda), e_\varphi(\lambda) + \mathrm{d}_\rho(\mu)), \quad \text{for all } (\lambda, \mu) \in C(\mathfrak{h}; \mathfrak{a}) \oplus C(\mathfrak{h}; \mathfrak{i}). \tag{10}$$

Since $\mathrm{d}_R^2 = \mathrm{d}_{\mathfrak{a}}^2 = \mathrm{d}_\rho^2 = 0$, one has $e_\varphi \circ \mathrm{d}_{\mathfrak{a}} = -\mathrm{d}_\rho \circ e_\varphi$. We summarize what we obtained in the following statement.

Proposition 1. *Let $(\mathfrak{h}, [\,\cdot\,,\,\cdot\,]_{\mathfrak{h}})$ be a Lie algebra, $\mathfrak{a}$ be a trivial $\mathfrak{h}$-module, and $\rho : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{i})$ be a representation of $(\mathfrak{h}, [\,\cdot\,,\,\cdot\,]_{\mathfrak{h}})$ in a vector space $\mathfrak{i}$. Let $\varphi : \mathfrak{h} \to \mathrm{Hom}(\mathfrak{a}; \mathfrak{i})$ be a 1-cocycle in the complex $C(\mathfrak{h}; \mathrm{Hom}(\mathfrak{a}; \mathfrak{i}))$ associated to the representation $\bar{\rho} : \mathfrak{h} \to \mathfrak{gl}(\mathrm{Hom}(\mathfrak{a}, \mathfrak{i}))$ defined by* (6). *Let $C(\mathfrak{h}; \mathfrak{a}) \oplus C(\mathfrak{h}; \mathfrak{i})$ be the complex with differential map d_R given in* (10). *For each (λ, μ) in $C^2(\mathfrak{h}; \mathfrak{a}) \oplus C^2(\mathfrak{h}; \mathfrak{i})$, let $[\,\cdot\,,\,\cdot\,]$ be the skew-symmetric and bilinear map in $\mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{i}$ defined by*

$$\begin{aligned} &[x, y] = [x, y]_{\mathfrak{h}} + \lambda(x, y) + \mu(x, y), \qquad \text{for all } x, y \in \mathfrak{h}, \\ &[x, u + \alpha] = \varphi(x)(u) + \rho(x)(\alpha), \quad \text{for all } x \in \mathfrak{h},\ u \in \mathfrak{a}, \text{ and } \alpha \in \mathfrak{i}. \end{aligned}$$

Then $[\,\cdot\,,\,\cdot\,]$ is a Lie bracket in $\mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{i}$ if and only if $\mathrm{d}_R(\lambda, \mu) = 0$.

Proof. It is clear that if $[\,\cdot\,,\,\cdot\,]$ is a Lie bracket, then $\mathrm{d}_R(\lambda, \mu) = 0$. Now suppose that $\mathrm{d}_R(\lambda, \mu) = 0$, where $R = (\varphi, \rho) : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{a} \oplus \mathfrak{i})$. By hypothesis, φ and ρ satisfy (4) and (5), respectively, then R is a representation of $(\mathfrak{h}, [\,\cdot\,,\,\cdot\,]_{\mathfrak{h}})$

on $\mathfrak{J} = \mathfrak{a} \oplus \mathfrak{i}$, for which $\Lambda = (\lambda, \mu)$ is a 2-cocycle in $C(\mathfrak{h}; \mathfrak{a} \oplus \mathfrak{i})$. Then, by (2), $[\,\cdot\,,\,\cdot\,]$ is a Lie bracket in $\mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{i}$. □

The Lie algebra in $\mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{i}$ of Proposition 1, is an abelian extension of $(\mathfrak{h}, [\,\cdot\,,\,\cdot\,]_{\mathfrak{h}})$ by $\mathfrak{J} = \mathfrak{a} \oplus \mathfrak{i}$, associated to the representation $R = (\varphi, \rho) : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{J})$ and the cocycle $\Lambda = (\lambda, \mu)$. We denote the Lie algebra $\mathfrak{h}(\Lambda, R)$ of Proposition 1, by $\mathfrak{h}(\lambda, \mu, \varphi, \rho)$. In addition we have the following short exact sequence of complexes:

$$0 \longrightarrow C(\mathfrak{h}; \mathfrak{i}) \longrightarrow C(\mathfrak{h}; \mathfrak{a}) \oplus C(\mathfrak{h}; \mathfrak{i}) \longrightarrow C(\mathfrak{h}; \mathfrak{a}) \longrightarrow 0 \tag{11}$$

where $e_{\varphi} : C(\mathfrak{h}; \mathfrak{a}) \to C(\mathfrak{h}; \mathfrak{i})$ is the connecting homomorphism. We denote the cohomology group of $C(\mathfrak{h}; \mathfrak{a}) \oplus C(\mathfrak{h}; \mathfrak{i})$ by $H(\mathfrak{h}; \mathfrak{a} \oplus \mathfrak{i})$. For abelian extensions, the elements in $H^2(\mathfrak{h}, \mathfrak{J})$ are in one-to-one correspondence with isomorphism classes of extensions of $(\mathfrak{h}, [\,\cdot\,,\,\cdot\,]_{\mathfrak{h}})$ by $R = (\varphi, \rho)$ (see [1], Theorem 26.2). In accordance with the decomposition $\mathfrak{J} = \mathfrak{a} \oplus \mathfrak{i}$, we have the following.

Proposition 2. *Let $\mathfrak{h}(\lambda, \mu, \varphi, \rho)$ and $\mathfrak{h}(\lambda', \mu', \varphi, \rho)$ be Lie algebras constructed as in Proposition 1, associated to the same representation $R = (\varphi, \rho)$. If (λ, μ) and (λ', μ') are in the same cohomology class in $H^2(\mathfrak{h}; \mathfrak{a} \oplus \mathfrak{i})$, then there exists an isomorphism Ψ of Lie algebras between $\mathfrak{h}(\lambda, \mu, \varphi, \rho)$ and $\mathfrak{h}(\lambda', \mu', \varphi, \rho)$ such that $\Psi|_{\mathfrak{a}} = \mathrm{Id}_{\mathfrak{a}}$ and $\Psi|_{\mathfrak{i}} = \mathrm{Id}_{\mathfrak{i}}$.*

Proof. If (λ, μ) and (λ', μ') are in the same cohomology class in $H(\mathfrak{h}; \mathfrak{a} \oplus \mathfrak{i})$, then there are linear maps $L : \mathfrak{h} \to \mathfrak{a}$ and $M : \mathfrak{h} \to \mathfrak{i}$ such that $(\lambda', \mu') = (\lambda, \mu) + \mathrm{d}_R(L, M)$. Let Ψ be the map between $\mathfrak{h}(\lambda, \mu, \varphi, \rho)$ and $\mathfrak{h}(\lambda', \mu', \varphi, \rho)$, defined by $\Psi(x) = x - L(x) - M(x)$, for all x in $\mathfrak{h}$, and $\Psi(u + \alpha) = u + \alpha$, for all u in $\mathfrak{a}$ and α in $\mathfrak{i}$. Then Ψ is an isomorphism of Lie algebras. □

We denote by $Z(\mathfrak{g})$ the center of $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,])$. The *derived central series* $Z_1(\mathfrak{g}) \subset Z_2(\mathfrak{g}) \subset \cdots \subset Z_\ell(\mathfrak{g}) \subset \cdots$ is defined by $Z_1(\mathfrak{g}) = Z(\mathfrak{g})$ and $Z_\ell(\mathfrak{g}) = \pi_{\ell-1}^{-1}\left(Z\left(\mathfrak{g}/Z_{\ell-1}(\mathfrak{g})\right)\right)$, where $\ell > 1$ and $\pi_{\ell-1} : \mathfrak{g} \to \mathfrak{g}/Z_{\ell-1}(\mathfrak{g})$ is the canonical projection. The *descending central series* $\mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \cdots \supset \mathfrak{g}^\ell \supset \cdots$ is defined by $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^\ell = [\mathfrak{g}, \mathfrak{g}^{\ell-1}]$, where $\ell > 1$.

The following result defines the ideals $\mathfrak{i}(\mathfrak{g})$ and $\mathfrak{J}(\mathfrak{g})$ for a nilpotent Lie algebra $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,])$. This result is a particular case of the one given in Lemma 4.2 of [6] and its proof can be consulted there.

Lemma 1. *Let $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,])$ be a finite dimensional nilpotent Lie algebra over a field $\mathbb{F}$. Let*

$$\mathfrak{i}(\mathfrak{g}) = \sum_{k \in \mathbb{N}} Z_k(\mathfrak{g}) \cap \mathfrak{g}^k \quad \text{and} \quad \mathfrak{J}(\mathfrak{g}) = \bigcap_{k \in \mathbb{N}} (Z_k(\mathfrak{g}) + \mathfrak{g}^k).$$

Then the following hold.

(*i*) $\mathfrak{i}(\mathfrak{g}) \subset \mathfrak{J}(\mathfrak{g})$.

(***ii***) *There exists* $m \geq 1$ *such that* $\mathfrak{J}(\mathfrak{g}) = Z(\mathfrak{g}) + \sum_{k=1}^{m-1} Z_{k+1}(\mathfrak{g}) \cap \mathfrak{g}^k$.

(***iii***) $\mathfrak{J}(\mathfrak{g})$ *is abelian and* $[\mathfrak{g}, \mathfrak{J}(\mathfrak{g})] \subset \mathfrak{i}(\mathfrak{g})$.

(***iv***) *If* $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,])$ *admits an invariant metric, then* $\mathfrak{i}(\mathfrak{g})^{\perp} = \mathfrak{J}(\mathfrak{g})$.

3. Quadratic Lie algebras with abelian ideals $\mathfrak{i}$ and $\mathfrak{J}$

For a bilinear form B on a vector space $\mathfrak{g}$, let $B^{\flat} : \mathfrak{g} \to \mathfrak{g}^*$ be the map defined by $B^{\flat}(x)(y) = B(x, y)$, for all x, y in $\mathfrak{g}$. If B is non-degenerate, then there exists the inverse map $B^{\sharp} : \mathfrak{g}^* \to \mathfrak{g}$, of $B^{\flat}$.

Suppose that a quadratic Lie algebra $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,], B)$ has abelian ideals $\mathfrak{i}$ and $\mathfrak{J}$ such that $\mathfrak{i} \subset \mathfrak{J}$, $\mathfrak{J} = \mathfrak{i}^{\perp}$ and $[\mathfrak{g}, \mathfrak{J}] \subset \mathfrak{i}$. For example, if $\mathfrak{g}$ is nilpotent and non-abelian, then $Z(\mathfrak{g}) \neq \{0\}$, for $\mathfrak{g}$ has a one-dimensional ideal which is contained in $Z(\mathfrak{g})$. By Lemma 3.3 of [2], $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}] \neq \{0\}$, and by Lemma 1, $Z(\mathfrak{g}) \subset \mathfrak{J} = \mathfrak{J}(\mathfrak{g})$, and $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{i} = \mathfrak{i}(\mathfrak{g})$. In addition, $\mathfrak{i}$ and $\mathfrak{J}$ are abelian.

By Proposition 1 the vector space $\mathfrak{g}$ can be written as: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{J} = \mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{i}$, where $\mathfrak{J} = \mathfrak{a} \oplus \mathfrak{i}$. Since $\mathfrak{i}$ is isotropic, by the Witt-decomposition we consider $\mathfrak{h}$ as an isotropic subspace of $\mathfrak{g}$ such that $\mathfrak{a}^{\perp} = \mathfrak{h} \oplus \mathfrak{i}$. In addition, B restricted to $\mathfrak{a} \times \mathfrak{a}$ is non-degenerate; we denote by $B_{\mathfrak{a}}$ the restriction $B\mid_{\mathfrak{a} \times \mathfrak{a}}$.

By Proposition 1, there are a Lie bracket $[\,\cdot\,,\,\cdot\,]_{\mathfrak{h}}$ in $\mathfrak{h}$, a representation $\rho : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{i})$, a 1-cocycle φ in $C(\mathfrak{h}; \mathrm{Hom}(\mathfrak{a}; \mathfrak{i}))$ associated to $\bar{\rho} : \mathfrak{h} \to \mathfrak{gl}(\mathrm{Hom}(\mathfrak{a}; \mathfrak{i}))$ and a 2-cocycle (λ, μ) in $C(\mathfrak{h}; \mathfrak{a}) \oplus C(\mathfrak{h}; \mathfrak{i})$ associated to the representation $R = (\varphi, \rho) : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{J})$, such that

$$[x, y] = [x, y]_{\mathfrak{h}} + \lambda(x, y) + \mu(x, y), \quad [x, u + \alpha] = \varphi(x)(u) + \rho(x)(\alpha),$$

for all x, y in $\mathfrak{h}$, u in $\mathfrak{a}$ and α in $\mathfrak{i}(\mathfrak{g})$; then $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,]) = \mathfrak{h}(\lambda, \mu, \varphi, \rho)$. In addition, due to the fact that B is non-degenerate and invariant, the map

$$\phi : \mathfrak{i} \to \mathfrak{h}^*, \quad \alpha \mapsto B^{\flat}(\alpha)\mid_{\mathfrak{h}} \tag{12}$$

is an isomorphism of $\mathfrak{h}$-modules, that is, $\mathrm{ad}_{\mathfrak{h}}^*(x) \circ \phi = \phi \circ \rho(x)$, for all x in $\mathfrak{h}$. Since $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{i}$, the invariant metric B takes the form

$$B(x + u + \alpha, y + v + \beta) = \phi(\alpha)(y) + \phi(\beta)(x) + B_{\mathfrak{a}}(u, v), \tag{13}$$

for all x, y in $\mathfrak{h}$, u, v in $\mathfrak{a}$ and α, β in $\mathfrak{i}$. Using that B is invariant under $[\,\cdot\,,\,\cdot\,]$ we obtain: $B(\varphi(x)(u), y) = B([x, u], y) = -B(u, [x, y]) = -B_{\mathfrak{a}}(\lambda(x, y), u)$. Then

$$\phi(\varphi(x)(u))(y) = -B_{\mathfrak{a}}^{\flat}(\lambda(x, y))(u), \text{ for all } x, y \in \mathfrak{h}, \text{ and } u \in \mathfrak{a}. \tag{14}$$

Similarly, using that B is invariant under $[\,\cdot\,,\,\cdot\,]$, for x, y, z in $\mathfrak{h}$ we get

$$\begin{aligned} \phi(\mu(x, y))(z) &= B(\mu(x, y), z) = B([x, y], z) \\ &= B(x, [y, z]) = B(x, \mu(y, z)) = \phi(\mu(y, z))(x). \end{aligned} \tag{15}$$

Let $\varphi' : \mathfrak{h} \to \mathrm{Hom}(\mathfrak{a};\mathfrak{h}^*)$ be the map defined by $\varphi'(x) = \phi \circ \varphi(x)$, for all x in $\mathfrak{h}$. Thus, the map $x + u + \alpha \mapsto x + u + \phi(\alpha)$ is an isomorphism between $\mathfrak{h}(\lambda, \mu, \varphi, \rho)$ and $\mathfrak{h}(\lambda, \phi \circ \mu, \varphi', \mathrm{ad}_{\mathfrak{h}}^*)$. Further, let B' be the symmetric bilinear form on $\mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{h}^*$ defined by

$$B'(x + u + \alpha', y + v + \beta') = \alpha'(y) + \beta'(x) + B_{\mathfrak{a}}(u, v), \tag{16}$$

for all x, y in $\mathfrak{h}$, u, v in $\mathfrak{a}$ and α', β' in $\mathfrak{h}^*$. Then B' is an invariant metric on $\mathfrak{h}(\lambda, \phi \circ \mu, \varphi', \mathrm{ad}_{\mathfrak{h}}^*)$, making it into a quadratic Lie algebra isometric to $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,], B)$ (see (13) and (16)). If we make the assumptions $\varphi' = \varphi$, $\phi \circ \mu$, and $\rho = \mathrm{ad}_{\mathfrak{h}}^*$, induced by the isomorphism $\phi : \mathfrak{i} \to \mathfrak{h}^*$ (see (12)), then by (14) and (15), φ and μ satisfy

$$\varphi : \mathfrak{h} \to \mathrm{Hom}(\mathfrak{a};\mathfrak{h}^*), \quad \varphi(x)(u)(y) = -B_{\mathfrak{a}}(\lambda(x, y), u), \text{ and} \tag{17}$$

$$\mu : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}^*, \quad \mu(x, y)(z) = \mu(y, z)(x), \tag{18}$$

for all x, y, z in $\mathfrak{h}$ and u in $\mathfrak{a}$.

Proposition 3. *Let $\mathfrak{h}(\lambda, \mu, \varphi, \mathrm{ad}_{\mathfrak{h}}^*)$ be a Lie algebra in the sense of Proposition 1. Let $B_{\mathfrak{a}}$ be a symmetric and non-degenerate bilinear form on $\mathfrak{a}$. The symmetric and non-degenerate bilinear form B defined on $\mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{h}^*$ by*

$$B(x + u + \alpha, y + v + \beta) = \alpha(x) + \beta(x) + B_{\mathfrak{a}}(u, v) \tag{19}$$

is invariant if and only if the conditions in (17) *and* (18) *hold. In addition, any nilpotent non-abelian quadratic Lie algebra can be identified with this construction.*

One of the aims of this work is to provide conditions for a nilpotent Lie algebra to admit an invariant metric. By Proposition 3, the structure of a nilpotent non-abelian quadratic Lie algebra is of the type $\mathfrak{h}(\lambda, \mu, \varphi, \mathrm{ad}_{\mathfrak{h}}^*)$, with underlying vector space $\mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{h}^*$. Thus, from now on we consider Lie algebras of the type $\mathfrak{h}(\lambda, \mu, \varphi, \mathrm{ad}_{\mathfrak{h}}^*)$, in the sense of Proposition 1, with underlying vector space $\mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{h}^*$, and bracket

$$\begin{aligned} x, y \in \mathfrak{h}, &\quad [x, y] = [x, y]_{\mathfrak{h}} + \lambda(x, y) + \mu(x, y), \\ x \in \mathfrak{h},\ u \in \mathfrak{a}, &\quad [x, u] = \varphi(x)(u), \\ x \in \mathfrak{h},\ \alpha \in \mathfrak{h}^* &\quad [x, \alpha] = \mathrm{ad}_{\mathfrak{h}}^*(x)(\alpha). \end{aligned}$$

where $\lambda : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{a}$ and $\mu : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}^*$ are skew-symmetric bilinear maps, and $\varphi : \mathfrak{h} \to \mathrm{Hom}(\mathfrak{a};\mathfrak{h}^*)$ is a linear map.

As we saw in (18), if the bilinear form B defined in (19) is an invariant metric on $\mathfrak{h}(\lambda, \mu, \varphi, \mathrm{ad}_{\mathfrak{h}}^*)$, then μ satisfies the *cyclic condition* $\mu(x, y)(z) = \mu(y, z)(x)$, and λ and φ are related by $\varphi(x)(u)(y) = -B_{\mathfrak{a}}(\lambda(x, y), u)$.

In the next result, we will show that we can associate to φ a bilinear map $\lambda_{\varphi} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{a}$ such that $\varphi(x)(u)(y) = -B_{\mathfrak{a}}(\lambda_{\varphi}(x, y), u)$, regardless of whether $\mathfrak{h}(\lambda, \mu, \varphi, \mathrm{ad}_{\mathfrak{h}}^*)$ admits an invariant metric or not.

Proposition 4. *Let $\mathfrak{h}(\lambda,\mu,\varphi,\mathrm{ad}^*_{\mathfrak{h}})$ be a Lie algebra, in the sense of Proposition 1. Let $B_{\mathfrak{a}}$ be a non-degenerate, symmetric and bilinear form on $\mathfrak{a}$. Then, there exists a bilinear map $\lambda_\varphi : \mathfrak{h}\times\mathfrak{h}\to\mathfrak{a}$ such that $\varphi(x)(u)(y) = -B_{\mathfrak{a}}(\lambda_\varphi(x,y),u)$, for all x,y in $\mathfrak{h}$ and u in $\mathfrak{a}$.*

Proof. Since φ is a linear map between $\mathfrak{h}$ and $\mathrm{Hom}(\mathfrak{a};\mathfrak{h}^*)$, for each pair x,y in $\mathfrak{h}$, let us consider the map $T(x,y):\mathfrak{a}\to\mathbb{F}$, defined by:

$$T(x,y)(u) = -\varphi(x)(u)(y), \qquad \text{for all } u\in\mathfrak{a}.$$

As φ is linear, $T(x,y)$ is a linear map and belongs to $\mathfrak{a}^*$. Since $B_{\mathfrak{a}}$ is non-degenerate, let us consider the element $\lambda_\varphi(x,y)$ in $\mathfrak{a}$, defined by $\lambda_\varphi(x,y) = B^\sharp_{\mathfrak{a}}(T(x,y))$, that is, $B^\flat_{\mathfrak{a}}(\lambda_\varphi(x,y)) = T(x,y)$ and

$$\varphi(x)(u)(y) = -T(x,y)(u) = -B_{\mathfrak{a}}(\lambda_\varphi(x,y),u), \qquad \text{for all } u\in\mathfrak{a}.$$

Let $\lambda_\varphi:\mathfrak{h}\times\mathfrak{h}\to\mathfrak{a}$ be defined by $(x,y)\mapsto\lambda_\varphi(x,y)$, which is bilinear because φ is linear and $B_{\mathfrak{a}}$ is non-degenerate. □

If $\lambda_\varphi=\lambda$ and μ satisfies (18), by Proposition 3, $\mathfrak{h}(\lambda,\mu,\varphi,\mathrm{ad}^*_{\mathfrak{h}})$ admits an invariant metric. If $\lambda_\varphi\neq\lambda$ but $\lambda_\varphi-\lambda$ is a coboundary, we can use Proposition 2 to obtain the following criteria that determine sufficient conditions on a Lie algebra $\mathfrak{h}(\lambda,\mu,\varphi,\mathrm{ad}^*_{\mathfrak{h}})$ to admit an invariant metric.

Proposition 5. *Let $\mathfrak{h}(\lambda,\mu,\varphi,\mathrm{ad}^*_{\mathfrak{h}})$ be a Lie algebra in the sense of Proposition 1 such that $\mu(x,y)(z)=\mu(y,z)(x)$ for all x,y,z in $\mathfrak{h}$. Let $\lambda_\varphi:\mathfrak{h}\times\mathfrak{h}\to\mathfrak{a}$ be the map of Proposition 4 and $e_\varphi: C(\mathfrak{h};\mathfrak{a})\to C(\mathfrak{h};\mathfrak{h}^*)$ be the map in (8). If there exists a linear map $L:\mathfrak{h}\to\mathfrak{a}$ such that*

$$(\boldsymbol{i})\ e_\varphi(L)=0, \qquad \textit{and} \qquad (\boldsymbol{ii})\ \lambda_\varphi=\lambda+\mathrm{d}_{\mathfrak{a}}(L),$$

then $\mathfrak{h}(\lambda,\mu,\varphi,\mathrm{ad}^*_{\mathfrak{h}})$ admits an invariant metric.

Proof. Since $\mathrm{d}_R(\lambda,\mu)=0$, $e_\varphi\circ\mathrm{d}_{\mathfrak{a}} = -\mathrm{d}_\rho\circ e_\varphi$ and $e_\varphi(L)=0$, we have $\mathrm{d}_R(\lambda+\mathrm{d}_{\mathfrak{a}}(L),\mu)=\mathrm{d}_R(\lambda_\varphi,\mu)=0$; so, by Proposition 1, $\mathfrak{h}(\lambda_\varphi,\mu,\varphi,\mathrm{ad}^*_{\mathfrak{h}})$ is a Lie algebra. Due to $(\lambda+\mathrm{d}(L),\mu)=(\lambda,\mu)+\mathrm{d}_R(L,0)$, by Proposition 2 the Lie algebra $\mathfrak{h}(\lambda,\mu,\varphi,\mathrm{ad}^*_{\mathfrak{h}})$ is isomorphic to $\mathfrak{h}(\lambda_\varphi,\mu,\varphi,\mathrm{ad}^*_{\mathfrak{h}})$. Let B be the symmetric and non-degenerate bilinear form on $\mathfrak{h}\oplus\mathfrak{a}\oplus\mathfrak{h}^*$ defined by (19). For $\varphi(x)(u)(y)=-B_{\mathfrak{a}}(\lambda_\varphi(x,y),u)$, (17) holds true. As μ satisfies (18), by Proposition 3, the bilinear form B is an invariant metric on $\mathfrak{h}(\lambda+\mathrm{d}_{\mathfrak{a}}(L),\mu,\varphi,\mathrm{ad}^*_{\mathfrak{h}})$, and therefore $\mathfrak{h}(\lambda,\mu,\varphi,\mathrm{ad}^*_{\mathfrak{h}})$ admits an invariant metric. □

3.1. Example. Let $\mathfrak{h}=\mathrm{Span}_{\mathbb{F}}\{x_1,x_2,x_3\}$ be the 3-dimensional Heisenberg Lie algebra, with $[x_1,x_2]_{\mathfrak{h}}=x_3$. Let $\mathfrak{h}^*=\mathrm{Span}_{\mathbb{F}}\{\alpha_1,\alpha_2,\alpha_3\}$, where $\alpha_i(x_j)=\delta_{ij}$. Consider a 3-dimensional vector space $\mathfrak{a}=\mathrm{Span}_{\mathbb{F}}\{u_1,u_2,u_3\}$ with non-degenerate, symmetric and bilinear form $B_{\mathfrak{a}}(u_i,u_j)=\delta_{ij}$. We shall use the criterion in Proposition 5 in a Lie algebra of the type $\mathfrak{h}(\lambda,\mu,\varphi,\mathrm{ad}^*_{\mathfrak{h}})$, with the following data. First we assume $\mu=0$. The map $\varphi:\mathfrak{h}\to\mathrm{Hom}(\mathfrak{a};\mathfrak{h}^*)$ is given by $\varphi(x_1)(u_1)=\alpha_2$, $\varphi(x_2)(u_1)=-\alpha_1$ and $\varphi(x_3)=0$. Then $\varphi([x,y]_{\mathfrak{h}})=$

$\mathrm{ad}^*_{\mathfrak{h}}(x) \circ \varphi(y) - \mathrm{ad}^*_{\mathfrak{h}}(y) \circ \varphi(y)$, for all x, y in $\mathfrak{h}$, which means that φ is a 1-cocycle in the complex $C(\mathfrak{h}; \mathrm{Hom}(\mathfrak{a}; \mathfrak{h}^*))$, associated to the representation $\overline{\mathrm{ad}^*_{\mathfrak{h}}} : \mathfrak{h} \to \mathfrak{gl}(\mathrm{Hom}(\mathfrak{a}; \mathfrak{h}^*))$. Observe that $\mathrm{d}_{\mathfrak{a}}(\lambda) = 0$ for any λ in $C^2(\mathfrak{h}; \mathfrak{a})$, that is:

$$\mathrm{d}_a(\lambda)(x_1, x_2, x_3) = -\lambda([x_1, x_2]_{\mathfrak{h}}, x_3) + \lambda([x_1, x_3]_{\mathfrak{h}}, x_2) - \lambda([x_2, x_3]_{\mathfrak{h}}, x_1) = 0.$$

Let $\lambda : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{a}$ be the skew-symmetric and bilinear map defined by $\lambda(x_1, x_2) = -u_1 + u_3$ and $\lambda(x_2, x_3) = \lambda(x_3, x_1) = 0$, then

$$\begin{aligned} &e_\varphi(\lambda)(x_1, x_2, x_3) \\ &= \varphi(x_1)(\lambda(x_2, x_3)) + \varphi(x_2)(\lambda(x_3, x_1)) + \varphi(x_3)(\lambda(x_1, x_2)) = 0. \end{aligned}$$

Hence $\mathrm{d}_R(\lambda, 0) = (\mathrm{d}_{\mathfrak{a}}(\lambda), e_\varphi(\lambda)) = 0$ and, by Proposition 1, $\mathfrak{h}(\lambda, 0, \varphi, \mathrm{ad}^*_{\mathfrak{h}})$ is a Lie algebra.

Since $\varphi(x)(u)(y) = -B_{\mathfrak{a}}(\lambda_\varphi(x, y), u)$, we shall determine the bilinear map $\lambda_\varphi : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{a}$ of Proposition 4, by

$$\lambda_\varphi(x, y) = -\varphi(x)(u_1)(y)u_1 - \varphi(x)(u_2)(y)u_2 - \varphi(x)(u_3)(y)u_3,$$

for all x, y in $\mathfrak{h}$. Hence, $\lambda_\varphi(x_1, x_2) = -u_1 = -\lambda_\varphi(x_2, x_1)$ and $\lambda_\varphi(x_k, x_3) = \lambda_\varphi(x_3, x_k) = 0$, for all $1 \leq k \leq 3$.

Let $L : \mathfrak{h} \to \mathfrak{a}$ be the linear map defined by $L(x_3) = u_3$; then

$$e_\varphi(L)(x_1, x_2) = \varphi(x_1)(L(x_2)) - \varphi(x_2)(L(x_1)) = 0.$$

Analogously we can verify $e_\varphi(L)(x_2, x_3) = e_\varphi(L)(x_3, x_1) = 0$; thus $e_\varphi(L) = 0$. In addition $\mathrm{d}_{\mathfrak{a}}(L)(x_1, x_2) = -L([x_1, x_2]_{\mathfrak{h}}) = -L(x_3) = -u_3$, then $\lambda(x_1, x_2) = \lambda_\varphi(x_1, x_2) - \mathrm{d}_{\mathfrak{a}}(L)(x_1, x_2)$, and $\lambda(x_j, x_3) = \lambda_\varphi(x_3, x_j) = 0$ for all j; thus $\lambda_\varphi = \lambda + \mathrm{d}_{\mathfrak{a}}(L)$. By Proposition 5, the Lie algebra $\mathfrak{h}(\lambda, 0, \varphi, \mathrm{ad}^*_{\mathfrak{h}})$ admits an invariant metric. The bracket of $\mathfrak{h}(\lambda, 0, \varphi, \mathrm{ad}^*_{\mathfrak{h}})$ is given by

$$\begin{aligned} &[x_1, x_2] = [x_1, x_3]_{\mathfrak{h}} + \lambda(x_1, x_2) = x_3 - u_1 + u_3, \\ &[x_1, u_1] = \varphi(x_1)(u_1) = \alpha_2, \quad [x_2, u_1] = \varphi(x_2)(u_1) = -\alpha_1 \\ &[x_1, \alpha_3] = \mathrm{ad}^*_{\mathfrak{h}}(x_1)(\alpha_3) = -\alpha_2, \quad [x_2, \alpha_3] = \mathrm{ad}^*_{\mathfrak{h}}(x_2)(\alpha_3) = \alpha_1. \end{aligned}$$

Let B be the symmetric and bilinear form on $\mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{h}^*$ defined by $B(x_3, u_3) = -1$ and $B(u_i, u_j) = B(x_i, \alpha_j) = \delta_{ij}$. Then B is an invariant metric on $\mathfrak{h}(\lambda, 0, \varphi, \mathrm{ad}^*_{\mathfrak{h}})$.

Proposition 6. *Let $\mathfrak{h}(\lambda, \mu, \varphi, \mathrm{ad}^*_{\mathfrak{h}})$ be a Lie algebra in the sense of Proposition 1 such that $\mu(x, y)(z) = \mu(y, z)(x)$ for all x, y, z in $\mathfrak{h}$. Let $\lambda_\varphi : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{a}$ be the map of Proposition 4 and $B_{\mathfrak{a}}$ be a symmetric and non-degenerate bilinear form on $\mathfrak{a}$. If there exists a linear map $L : \mathfrak{h} \to \mathfrak{a}$ such that*

$$(\boldsymbol{i})\ \lambda_\varphi = \lambda + \mathrm{d}_{\mathfrak{a}}(L), \quad \text{and} \quad (\boldsymbol{ii})\ B_{\mathfrak{a}}(\lambda_\varphi(y, z), L(x)) = B_{\mathfrak{a}}(\lambda_\varphi(x, y), L(z)),$$

for all x, y, z in $\mathfrak{h}$, then $\mathfrak{h}(\lambda, \mu, \varphi, \mathrm{ad}^*_{\mathfrak{h}})$ admits an invariant metric.

Proof. Let $\mu' = \mu + e_\varphi(L)$. We claim that $\mu'(x,y)(z) = \mu'(y,z)(x)$, for all x, y, z in $\mathfrak{h}$. Since μ satisfies the cyclic condition (18), we have $\mu'(x,y)(z) = \mu'(y,z)(x)$ if and only if $e_\varphi(L)(x,y)(z) = e_\varphi(L)(y,z)(x)$. By the definition of e_φ (see (8)), we have

$$\begin{aligned} e_\varphi(L)(x,y)(z) &= \varphi(x)(L(y))(z) - \varphi(y)(L(x))(z), \text{ and} \\ e_\varphi(L)(y,z)(x) &= \varphi(y)(L(z))(x) - \varphi(z)(L(y))(x). \end{aligned} \tag{20}$$

By Proposition 4, the equations in (20) can be written as

$$\begin{aligned} e_\varphi(L)(x,y)(z) &= -B_{\mathfrak{a}}(\lambda_\varphi(x,z), L(y)) + B_{\mathfrak{a}}(\lambda_\varphi(y,z), L(x)), \text{ and} \\ e_\varphi(L)(y,z)(x) &= -B_{\mathfrak{a}}(\lambda_\varphi(y,x), L(z)) + B_{\mathfrak{a}}(\lambda_\varphi(z,x), L(y)). \end{aligned} \tag{21}$$

Since $\lambda_\varphi = \lambda + \mathrm{d}_{\mathfrak{a}}(L)$, then $\lambda_\varphi : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{a}$ is skew-symmetric. From (21) we deduce that $e_\varphi(L)(x,y)(z) = e_\varphi(L)(y,z)(x)$ follows from $B_{\mathfrak{a}}(\lambda_\varphi(y,z), L(x)) = B_{\mathfrak{a}}(\lambda_\varphi(x,y), L(z))$, which is the hypothesis in (**ii**). Then, for all x, y, z in $\mathfrak{h}$, $\mu'(x,y)(z) = \mu'(y,z)(x)$. Observe that

$$(\lambda_\varphi, \mu') = (\lambda + \mathrm{d}_{\mathfrak{a}}(L), \mu + e_\varphi(L)) = (\lambda, \mu) + \mathrm{d}_R(L, 0), \tag{22}$$

then $\mathrm{d}_R(\lambda_\varphi, \mu') = 0$. Thus, the Lie algebra $\mathfrak{h}(\lambda_\varphi, \mu', \varphi, \mathrm{ad}_{\mathfrak{h}}^*)$ satisfies

$$\varphi(x)(u)(y) = -B_{\mathfrak{a}}(\lambda_\varphi(x,y), u), \qquad \text{and} \qquad \mu'(x,y)(z) = \mu'(y,z)(x).$$

Hence, by Proposition 3, the symmetric and non-degenerate bilinear form B defined by (19), is an invariant metric on $\mathfrak{h}(\lambda_\varphi, \mu', \varphi, \mathrm{ad}_{\mathfrak{h}}^*)$. By Proposition 2 and (22), $\mathfrak{h}(\lambda_\varphi, \mu', \varphi, \mathrm{ad}_{\mathfrak{h}}^*)$ and $\mathfrak{h}(\lambda, \mu, \varphi, \mathrm{ad}_{\mathfrak{h}}^*)$ are isomorphic, thus $\mathfrak{h}(\lambda, \mu, \varphi, \mathrm{ad}_{\mathfrak{h}}^*)$ admits an invariant metric. $\square$

3.2. Example. Let $\mathfrak{h} = \mathrm{Span}_{\mathbb{F}}\{x_1, x_2, x_3\}$ be the 3-dimensional Heisenberg Lie algebra with $[x_1, x_2]_{\mathfrak{h}} = x_3$. Let $\mathfrak{h}^* = \mathrm{Span}_{\mathbb{F}}\{\alpha_1, \alpha_2, \alpha_3\}$, where $\alpha_i(x_j) = \delta_{ij}$. Let $\mathfrak{a} = \mathrm{Span}_{\mathbb{F}}\{u_1, u_2, u_3\}$ with the non-degenerate, symmetric and bilinear form $B_{\mathfrak{a}}(u_i, u_j) = \delta_{ij}$. We shall use the criterion in Proposition 6 in a Lie algebra $\mathfrak{h}(\lambda, \mu, \varphi, \mathrm{ad}_{\mathfrak{h}}^*)$, with the following data. The map $\varphi : \mathfrak{h} \to \mathrm{Hom}(\mathfrak{a}; \mathfrak{h}^*)$ is given by

$$\begin{aligned} &\varphi(x_1)(u_2) = -\varphi(x_2)(u_1) = \alpha_3, \quad \varphi(x_2)(u_3) = -\varphi(x_3)(u_2) = \alpha_1, \\ &\varphi(x_3)(u_1) = -\varphi(x_1)(u_3) = \alpha_2. \end{aligned} \tag{23}$$

Then $\varphi([x,y]_{\mathfrak{h}}) = \mathrm{ad}_{\mathfrak{h}}^*(x) \circ \varphi(y) - \mathrm{ad}_{\mathfrak{h}}^*(y) \circ \varphi(x)$, for all x, y in $\mathfrak{h}$, which amounts to say that φ is a 1-cocycle in the complex $C(\mathfrak{h}; \mathrm{Hom}(\mathfrak{a}; \mathfrak{h}^*))$, associated to the representation $\overline{\mathrm{ad}_{\mathfrak{h}}^*} : \mathfrak{h} \to \mathfrak{gl}(\mathrm{Hom}(\mathfrak{a}; \mathfrak{h}^*))$. The skew-symmetric bilinear map $\lambda : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{a}$ is given by

$$\lambda(x_1, x_2) = (1 + \xi) u_3, \quad \lambda(x_2, x_3) = u_1, \quad \lambda(x_3, x_1) = u_2,$$

where ξ is in $\mathbb{F}$. As we pointed out in the previous example, $\mathrm{d}_{\mathfrak{a}}(\lambda) = 0$, for any λ in $C^2(\mathfrak{h}; \mathfrak{a})$. The skew-symmetric bilinear map $\mu : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}^*$ is given by

$$\mu(x_1, x_2) = -2\xi\, \alpha_3, \quad \mu(x_2, x_3) = -2\xi\, \alpha_1, \quad \mu(x_3, x_1) = -2\xi\, \alpha_2.$$

Since $\varphi(x_j)(u_j) = 0$ for all $1 \leq j \leq 3$,

$$\begin{aligned}&e_\varphi(\lambda)(x_1, x_2, x_3) = \\ &\varphi(x_1)(\lambda(x_2, x_3)) + \varphi(x_2)(\lambda(x_3, x_1)) + \varphi(x_3)(\lambda(x_1, x_2)) = 0.\end{aligned}$$

Due to $\operatorname{ad}_{\mathfrak{h}}^*(x_j)(\alpha_j) = 0$ for all j, we have

$$\begin{aligned}&\mathrm{d}_{\operatorname{ad}_{\mathfrak{h}}^*}(\mu)(x_1, x_2, x_3) = \operatorname{ad}_{\mathfrak{h}}^*(x_1)(\mu(x_2, x_3)) \\ &+ \operatorname{ad}_{\mathfrak{h}}^*(x_2)(\mu(x_3, x_1)) + \operatorname{ad}_{\mathfrak{h}}^*(x_3)(\mu(x_1, x_2)) + \mu(x_1, [x_2, x_3]_{\mathfrak{h}}) \\ &+ \mu(x_2, [x_3, x_1]_{\mathfrak{h}}) + \mu(x_3, [x_1, x_2]_{\mathfrak{h}}) = 0.\end{aligned}$$

Then $\mathrm{d}_R(\lambda, \mu) = (\mathrm{d}_{\mathfrak{a}}(\lambda), e_\varphi(\lambda) + \mathrm{d}_{\operatorname{ad}_{\mathfrak{h}}^*}(\mu)) = 0$ and, by Proposition 1, $\mathfrak{h}(\lambda, \mu, \varphi, \operatorname{ad}_{\mathfrak{h}}^*)$ is a Lie algebra. By (23), the map $\lambda_\varphi : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{a}$, defined in Proposition 4, is skew-symmetric and is given by

$$\lambda_\varphi(x_1, x_2) = u_3, \quad \lambda_\varphi(x_2, x_3) = u_1, \quad \lambda_\varphi(x_3, x_1) = u_2.$$

Let $L : \mathfrak{h} \to \mathfrak{a}$ be the linear map defined by $L(x_j) = \xi\, u_j$ for all j. It is straightforward to verify that

$$\begin{aligned}&B_{\mathfrak{a}}(\lambda_\varphi(x_1, x_2), L(x_3)) = B_{\mathfrak{a}}(\lambda_\varphi(x_2, x_3), L(x_1)) \\ &= B_{\mathfrak{a}}(\lambda_\varphi(x_3, x_1), L(x_2)) = \xi.\end{aligned}$$

In addition,

$$\begin{aligned}&\mathrm{d}_{\mathfrak{a}}(L)(x_1, x_2) = -L([x_1, x_2]_{\mathfrak{h}}) = -L(x_3) = -\xi u_3, \quad \text{and} \\ &\mathrm{d}_{\mathfrak{a}}(L)(x_2, x_3) = -L([x_2, x_3]_{\mathfrak{h}}) = 0 = -L([x_3, x_1]_{\mathfrak{h}}) = \mathrm{d}_{\mathfrak{a}}(L)(x_3, x_1).\end{aligned}$$

Then, $\lambda_\varphi = \lambda + \mathrm{d}_{\mathfrak{a}}(L)$. By Proposition 6, $\mathfrak{h}(\lambda, \mu, \varphi, \operatorname{ad}_{\mathfrak{h}}^*)$ admits an invariant metric. The bracket on $\mathfrak{h}(\lambda, \mu, \varphi, \operatorname{ad}_{\mathfrak{h}}^*)$ is given by

$$\begin{aligned}&[x_1, x_2] = x_3 + (1 + \xi)u_3 - 2\xi\alpha_3, \quad [x_2, x_3] = u_1 - 2\xi\alpha_1, \\ &[x_3, x_1] = u_2 - 2\xi\alpha_2, \quad [x_1, u_2] = -[x_2, u_1] = \alpha_3, \quad [x_2, u_3] = -[x_3, u_2] = \alpha_1, \\ &[x_3, u_1] = -[x_1, u_3] = \alpha_2, \quad [x_1, \alpha_3] = -\alpha_2, \quad [x_2, \alpha_3] = \alpha_1.\end{aligned}$$

The invariant metric B on $\mathfrak{h}(\lambda, \mu, \varphi, \operatorname{ad}_{\mathfrak{h}}^*)$ is $B(x_j, u_k) = -\xi\delta_{jk}$, and $B(u_j, u_k) = B(x_j, \alpha_k) = \delta_{jk}$, for all j, k.

It is a straightforward to verify that

$$\begin{aligned}&Z(\mathfrak{g}) = \mathrm{Span}_{\mathbb{F}}\{\alpha_1, \alpha_2\}, \quad Z_2(\mathfrak{g}) = \mathfrak{h}^*, \quad Z_3(\mathfrak{g}) = \mathfrak{a} \oplus \mathfrak{h}^*, \\ &Z_4(\mathfrak{g}) = \mathbb{F}x_3 \oplus Z_3(\mathfrak{g}), \quad Z_5(\mathfrak{g}) = \mathfrak{g}, \quad \mathfrak{g}^1 = Z_4(\mathfrak{g}), \\ &\mathfrak{g}^2 = \mathfrak{a} \oplus \mathfrak{h}^*, \quad \mathfrak{g}^3 = \mathfrak{h}^*, \quad \mathfrak{g}^4 = Z(\mathfrak{g}), \quad \mathfrak{g}^5 = \{0\}.\end{aligned}$$

Thus, $\mathfrak{i}(\mathfrak{g}) = \mathfrak{h}^*$ and $\mathfrak{j}(\mathfrak{g}) = \mathfrak{a} \oplus \mathfrak{h}^*$.

4. Current nilpotent Lie algebras

Let $\mathfrak{g} = \mathfrak{h}(\lambda, \mu, \varphi, \mathrm{ad}^*_{\mathfrak{h}})$ be a non-abelian nilpotent quadratic Lie algebra with invariant metric B; by Lemma 1, $\mathfrak{J}(\mathfrak{g}) = \mathfrak{a} \oplus \mathfrak{h}^*$. Let $\mathcal{S}$ be an associative and commutative algebra with unit 1 over $\mathbb{F}$. In the vector space $\mathfrak{g} \otimes \mathcal{S}$, we consider the bracket $[X \otimes s, Y \otimes t]_{\mathfrak{g}\otimes\mathcal{S}} = [X, Y] \otimes st$, for all X, Y in $\mathfrak{g}$ and s, t in $\mathcal{S}$. Then $(\mathfrak{g} \otimes \mathcal{S}, [\,\cdot\,,\,\cdot\,]_{\mathfrak{g}\otimes\mathcal{S}})$ is a Lie algebra which is called the *current Lie algebra of* $\mathfrak{g}$ *by* $\mathcal{S}$.

Since $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{J}(\mathfrak{g})$, $\mathfrak{g} \otimes \mathcal{S} = (\mathfrak{h} \otimes \mathcal{S}) \oplus (\mathfrak{J}(\mathfrak{g}) \otimes \mathcal{S})$. Thus, by (2), the bracket $[\,\cdot\,,\,\cdot\,]_{\mathfrak{g}\otimes\mathcal{S}}$ in $\mathfrak{g} \otimes \mathcal{S}$ takes the form

$$\begin{aligned} &[x \otimes s, y \otimes t]_{\mathfrak{g}\otimes\mathcal{S}} = [x, y]_{\mathfrak{h}} \otimes st + \Lambda(x, y) \otimes st, \\ &[x \otimes s, U \otimes t]_{\mathfrak{g}\otimes\mathcal{S}} = R(x)(U) \otimes st. \end{aligned} \tag{24}$$

where x, y are in $\mathfrak{h}$, U is in $\mathfrak{J}(\mathfrak{g})$, and s, t are in $\mathcal{S}$. From now on we write $[\,\cdot\,,\,\cdot\,]$ to denote the bracket $[\,\cdot\,,\,\cdot\,]_{\mathfrak{g}\otimes\mathcal{S}}$ in $\mathfrak{g} \otimes \mathcal{S}$.

Let $\mathcal{W} = \{f : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F} \mid f \text{ is bilinear and invariant } \}$. Suppose that $\mathfrak{g} \otimes \mathcal{S}$ admits an invariant metric $\bar{B}$. For each pair s, t in $\mathcal{S}$, consider the map $\mathcal{L}(s, t) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ defined by

$$\mathcal{L}(s, t)(X, Y) = \bar{B}(X \otimes s, Y \otimes t), \qquad \text{for all } X, Y \in \mathfrak{g}.$$

We assert that $\mathcal{L}(s, t)$ belongs to $\mathcal{W}$. Indeed, let X, Y, Z be in $\mathfrak{g}$ and using that $\bar{B}$ is invariant, we get

$$\begin{aligned} &\mathcal{L}(s, t)([X, Y], Z) = \bar{B}([X, Y] \otimes s, Z \otimes t) \\ &= \bar{B}([X \otimes s, Y \otimes 1], Z \otimes t) = \bar{B}(X \otimes s, [Y \otimes 1, Z \otimes t]) \\ &= \bar{B}(X \otimes s, [Y, Z] \otimes t) = \mathcal{L}(s, t)(X, [Y, Z]). \end{aligned} \tag{25}$$

Then $\mathcal{L}(s, t)$ belongs to $\mathcal{W}$. Following the arguments in (25), we obtain

$$\begin{aligned} &\mathcal{L}(s, t)([X, Y], Z) = \bar{B}([X \otimes 1, Y \otimes s], Z \otimes t) \\ &= \bar{B}(X \otimes 1, [Y, Z] \otimes st) = \mathcal{L}(1, st)(X, [Y, Z]) \\ &= \mathcal{L}(1, st)([X, Y], Z). \end{aligned} \tag{26}$$

Using the same arguments as in (26) and that $\bar{B}$ is symmetric, we get

$$\begin{aligned} &\mathcal{L}(s, t)([X, Y], Z) = \bar{B}(Z \otimes t, [X \otimes s, Y \otimes 1]) \\ &= \bar{B}([Z, X] \otimes st, Y \otimes 1) = -\bar{B}([Z \otimes s, X \otimes t], Y \otimes 1) \\ &= \bar{B}(Z \otimes s, [X, Y] \otimes t) = \mathcal{L}(s, t)(Z, [X, Y]). \end{aligned} \tag{27}$$

Since $\mathcal{S}$ is commutative, from (26) and (27) we deduce that $\mathcal{L}(s, t)|_{[\mathfrak{g},\mathfrak{g}]\times\mathfrak{g}}$ is symmetric and invariant, that is,

$$\mathcal{L}(s, t)(X, Y) = \mathcal{L}(t, s)(X, Y) = \mathcal{L}(st, 1)(X, Y) = \mathcal{L}(s, t)(Y, X), \tag{28}$$

for all X in $[\mathfrak{g}, \mathfrak{g}]$ and Y in $\mathfrak{g}$.

Let $\operatorname{Cent}(\mathfrak{g}) = \{T \in \mathfrak{gl}(\mathfrak{g}) \mid T([X,Y]) = [X, T(Y)] \text{ for all } X, Y \in \mathfrak{g}\}$. Since B is a non-degenerate bilinear form, for each pair s, t in $\mathcal{S}$, let us consider the map $\Gamma(s,t) : \mathfrak{g} \to \mathfrak{g}$ defined by $\Gamma(s,t) = B^\sharp \circ \mathcal{L}(s,t)^\flat$, that is, $B^\flat \circ \Gamma(s,t) = \mathcal{L}(s,t)^\flat$ and

$$B(\Gamma(s,t)(X), Y) = \mathcal{L}(s,t)(X,Y) = \bar{B}(X \otimes s, Y \otimes t), \tag{29}$$

where X, Y are in $\mathfrak{g}$. As $\mathcal{L}(s,t)$ is bilinear and invariant, $\Gamma(s,t)$ belongs to $\operatorname{Cent}(\mathfrak{g})$. In addition, as $\mathcal{L}(s,t)|_{[\mathfrak{g},\mathfrak{g}]\times g}$ is symmetric and invariant,

$$\begin{aligned}\mathcal{L}(s,t)([X,Y],Z) = B(\Gamma(s,t)([X,Y]),Z) &= \mathcal{L}(st,1)([X,Y],Z)\\ &= B(\Gamma(st,1)([X,Y],Z)), \quad \text{for all } X,Y,Z \in \mathfrak{g}.\end{aligned}$$

Thus $\Gamma(s,t)([X,Y]) = \Gamma(st,1)([X,Y])$ and $\Gamma(s,t)|_{[\mathfrak{g},\mathfrak{g}]}$ is symmetric and invariant, that is,

$$\Gamma(s,t)|_{[\mathfrak{g},\mathfrak{g}]} = \Gamma(t,s)|_{[\mathfrak{g},\mathfrak{g}])} = \Gamma(st,1)|_{[\mathfrak{g},\mathfrak{g}]}, \quad \text{for all } s,t \in \mathcal{S}. \tag{30}$$

It is not difficult to verify that the canonical ideal $\mathfrak{J}(\mathfrak{g})$ is invariant under $\Gamma(s,t)$. Thus, for every x in $\mathfrak{h}$ and U in $\mathfrak{J}(\mathfrak{g})$, there are $\epsilon(s,t,x)$ in $\mathfrak{h}$, $\Theta(s,t,x)$ in $\mathfrak{J}(\mathfrak{g})$ and $\vartheta(s,t,U)$ in $\mathfrak{J}(\mathfrak{g})$ such that

$$\Gamma(s,t)(x) = \epsilon(s,t,x) + \Theta(s,t,x), \qquad \text{and} \qquad \Gamma(s,t)(U) = \vartheta(s,t,U).$$

Let $\epsilon : \mathcal{S} \times \mathcal{S} \to \mathfrak{gl}(\mathfrak{h})$ be the map defined by $\epsilon(s,t)(x) = \epsilon(s,t,x)$. Similarly, we define $\Theta : \mathcal{S} \times \mathcal{S} \to \operatorname{Hom}(\mathfrak{h}, \mathfrak{J}(\mathfrak{g}))$ by $\Theta(s,t)(x) = \Theta(s,t,x)$ and $\vartheta : \mathcal{S} \times \mathcal{S} \to \mathfrak{gl}(\mathfrak{J}(\mathfrak{g}))$ by $\vartheta(s,t)(U) = \vartheta(s,t,U)$.

Since $\Gamma(s,t)([x,y]) = [x, \Gamma(s,t)(y)]$ for all x, y in $\mathfrak{h}$, by (2) we deduce that $\epsilon(s,t)([x,y]_{\mathfrak{h}}) = [x, \epsilon(s,t)(y)]_{\mathfrak{h}}$; that is, $\epsilon(s,t)$ belongs to $\operatorname{Cent}(\mathfrak{h})$.

From (30) we get

$$[\Gamma(s,t)(X), Y] = \Gamma(s,t)([X,Y]) = \Gamma(st,1)([X,Y]) = [\Gamma(st,1)(X), Y]$$

for all X, Y in $\mathfrak{g}$, then $\Gamma(s,t)(\mathfrak{g}) - \Gamma(st,1)(\mathfrak{g}) \subset Z(\mathfrak{g})$. This implies that $\Gamma(s,t)(x) - \Gamma(st,1)(x)$ belongs to $Z(\mathfrak{g}) \subset \mathfrak{J}(\mathfrak{g})$ for all x in $\mathfrak{h}$, that is,

$$(\epsilon(s,t)(x) - \epsilon(st,1)(x)) + (\Theta(s,t)(x) - \Theta(st,1)(x)) \in Z(\mathfrak{g}) \subset \mathfrak{J}(\mathfrak{g}).$$

As $\Theta(s,t)(x) - \Theta(st,1)(x)$ belongs to $\mathfrak{J}(\mathfrak{g})$, then $\epsilon(s,t)(x) - \epsilon(st,1)(x)$ belongs to $\mathfrak{h} \cap \mathfrak{J}(\mathfrak{g}) = \{0\}$. Hence, $\epsilon(s,t) = \epsilon(st,1)$.

As $[x,U] = R(x)(U)$ and $\Gamma(s,t)([x,U]) = [\Gamma(s,t)(x), U] = [x, \Gamma(s,t)(U)]$ for all x in $\mathfrak{h}$, and U in $\mathfrak{J}(\mathfrak{g})$, by (2) we obtain that

$$\vartheta(s,t)(R(x)(U)) = R(\epsilon(s,t)(x))(U) = R(x)(\vartheta(s,t)(U)). \tag{31}$$

Suppose there exists s' in $\mathcal{S}$ such that $\epsilon(s',t) = 0$ for all t in $\mathcal{S}$. By (31) it follows that $\Gamma(s',t)(R(x)(U)) = \vartheta(s',t)(R(x)(U)) = 0$ for all x in $\mathfrak{h}$, U in $\mathfrak{J}(\mathfrak{g})$ and t in $\mathcal{S}$. By (29) this amounts to saying that

$$\bar{B}(R(x)(u) \otimes s', Y \otimes t) = B(\Gamma(s',t)(R(x)(u)), Y) = 0, \tag{32}$$

where Y is in $\mathfrak{g}$. As $\bar{B}$ is non-degenerate and Y and t are arbitrary, from (32) we deduce that $R(x)(U)\otimes s' = 0$. If s' is non-zero, then $R(x)(U) = [x,U] = 0$. Since x and U are arbitrary, it follows that $[\mathfrak{h},\mathfrak{J}] = \{0\}$. As $\mathfrak{h}^* = \mathfrak{i}(\mathfrak{g}) \subset \mathfrak{J}(\mathfrak{g})$, then $[\mathfrak{h},\mathfrak{h}^*] = \mathrm{ad}^*_{\mathfrak{h}}(\mathfrak{h})(\mathfrak{h}^*) = \{0\}$, which implies that $\mathfrak{h}$ is abelian. Therefore, if $\mathfrak{h}$ is non-abelian, $s' = 0$.

Theorem 1. *Let $\mathfrak{g} = \mathfrak{h}(\lambda,\mu,\varphi,\mathrm{ad}^*_{\mathfrak{h}})$ be a non-abelian nilpotent quadratic Lie algebra. Let $\mathcal{S}$ be a commutative and associative algebra with unit. If the current Lie algebra $\mathfrak{g}\otimes\mathcal{S}$ admits an invariant metric and $\mathfrak{h}$ is non-abelian, then there exists a bilinear map $\epsilon : \mathcal{S}\times\mathcal{S} \to \mathrm{Cent}(\mathfrak{h})$ such that*

(***i***) *$\epsilon(s,t) = \epsilon(s\,t,1)$, for all s,t in $\mathcal{S}$;*

(***ii***) *if there exists s such that $\epsilon(s,t) = 0$, for all t in $\mathcal{S}$, then $s = 0$.*

Remark 1. Observe that ϵ has the same properties as an invariant metric. What Theorem 1 says is that if $\mathfrak{g}\otimes\mathcal{S}$ admits an invariant metric, then $\mathcal{S}$ possesses a bilinear map ϵ into the centroids of $\mathfrak{g}/\mathfrak{J}(\mathfrak{g})$, which has the same behaviour as an invariant metric. In Corollary 1.3 of [4], it is shown that $\mathfrak{g}\otimes\mathcal{S}$ admits an invariant metric if and only if $\mathcal{S}$ admits an invariant metric; under the hypothesis that $\mathfrak{g}$ is indecomposable, $\mathcal{S}$ is finite dimensional and $\mathbb{F}$ is algebraically closed.

5. Invariant forms on nilpotent Lie algebras

Let $(\mathfrak{g},[\,\cdot\,,\,\cdot\,])$ be a finite dimensional nilpotent Lie algebra over a field $\mathbb{F}$ of zero characteristic. Let $\mathcal{V}$ be the vector space generated by all the symmetric and bilinear forms $f : \mathfrak{g}\times\mathfrak{g} \to \mathbb{F}$. Let $\varrho : \mathfrak{g} \to \mathfrak{gl}(\mathcal{V})$ be the map defined by

$$\varrho(X)(f)(Y,Z) = -f([X,Y],Z) - f(Y,[X,Z]), \qquad \text{for all } X,Y,Z \in \mathfrak{g}. \tag{33}$$

Then ϱ is a representation of $(\mathfrak{g},[\,\cdot\,,\,\cdot\,])$ in $\mathcal{V}$. As $(\mathfrak{g},[\,\cdot\,,\,\cdot\,])$ is nilpotent, by Theorem 3.3 in [2], there exists a non-zero f in $\mathcal{V}$ such that $\varrho(X)(f) = 0$ for all X in $\mathfrak{g}$. From (33), this amounts to saying that f is an invariant form in $(\mathfrak{g},[\,\cdot\,,\,\cdot\,])$.

For finite dimensional nilpotent Lie algebras, the above shows that there exists an invariant, symmetric and non-zero bilinear form. To find out whether it is non-degenerate, to such a bilinear form, we can associate an algebra with unit from which we can recover the Lie algebra structure in $\mathfrak{g}$, and we can prove that this algebra is simple if and only if the bilinear form is non-degenerate.

Theorem 2. *Let $(\mathfrak{g},[\,\cdot\,,\,\cdot\,])$ be a finite dimensional nilpotent Lie algebra over a field $\mathbb{F}$ of zero characteristic. Let f be an invariant, symmetric and non-zero bilinear form on $(\mathfrak{g},[\,\cdot\,,\,\cdot\,])$. Let $\mathcal{A}_f = \mathbb{F}\times\mathfrak{g}$ be the algebra with product defined by*

$$(\xi,X)(\eta,Y) = (\,\xi\,\eta + f(X,Y)\,,\,\xi\,Y + \eta\,X + \frac{1}{2}[X,Y]\,), \tag{34}$$

for all ξ, η *in* $\mathbb{F}$ *and* X, Y *in* $\mathfrak{g}$. *Suppose that* $\dim \mathfrak{g} > 1$.

(***i***) *The invariant form* f *is non-degenerate if and only if* $\mathcal{A}_f$ *is a simple algebra.*

(***ii***) *Let* $[\,\cdot\,,\,\cdot\,]_f$ *be the commutator of* (34). *Then* $\mathcal{A}_f/\mathbb{F}(1,0)$ *with bracket induced by* $[\,\cdot\,,\,\cdot\,]_f$, *is a Lie algebra isomorphic to* $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,])$.

Proof.

(**i**) We define equality, addition and multiplication by scalars in $\mathcal{A}_f$ in the obvious manner. Suppose f is non-degenerate and let I be a non-zero right ideal of $\mathcal{A}_f$. We shall prove that $(1,0)$ belongs to I. We assert that there exists an element (ξ, X) in I with $\xi \neq 0$. Let (ξ, X) be a non-zero element in I. If $\xi = 0$, then X is non-zero. As f is non-degenerate, there exists Y in $\mathfrak{g}$ such that $f(X,Y)$ is non-zero. By (34), we have that $(0,X)(\eta,Y) = \big(f(X,Y), \eta X + \frac{1}{2}[X,Y]\big)$ belongs to I, which proves our assertion.

Let (ξ, x) be in I, then $(\xi, X)(-\xi, X) = (-\xi^2 + f(X,X), 0)$ is in I. If $-\xi^2 + f(X,X)$ is non-zero, then $(1,0)$ belongs to I and consequently $I = \mathcal{A}_f$. Thus, from now on we assume the following:

$$\text{for all } (\xi, X) \text{ in } I, \quad \xi^2 = f(X,X). \tag{$\clubsuit$}$$

Let (ξ, X) be in I, with $\xi \neq 0$ and $X \neq 0$ (if $X = 0$ we are done), and let (η, Y) be an arbitrary element in $\mathcal{A}_f$. Then $(\xi, X)(\eta, Y) = (\xi\eta + f(X,Y), \xi y + \eta X + \frac{1}{2}[X,Y])$ belongs to I. Applying ($\clubsuit$), we obtain

$$f(X,Y)^2 = \xi^2 f(Y,Y) + \frac{1}{4} f([X,Y],[X,Y]). \tag{35}$$

As Y is arbitrary, we apply (35) to $Y + Z$, and using that f is invariant, we get

$$4f(X,Y)f(X,Z) = 4\xi^2 f(Y,Z) - f([X,[X,Y]],Z). \tag{36}$$

Using that f is non-degenerate, from (36) it follows that

$$4f(X,Y)X = 4\xi^2 Y - \operatorname{ad}(X)^2(Y). \tag{37}$$

Applying the adjoint representation $\operatorname{ad}(X)$ in (37), we have

$$\operatorname{ad}(X)^2(\operatorname{ad}(X)(Y)) = 4\xi^2 \operatorname{ad}(X)(Y). \tag{38}$$

If $\operatorname{ad}(X)^2 = 0$ then from (36) we get

$$f(X,Y)f(X,Z) = \xi^2 f(Y,Z). \tag{39}$$

As f is non-degenerate and ξ is non-zero, from (39) we obtain $Y = \xi^{-2} f(X,Y)X$. Since Y is arbitrary, we deduce that $\mathfrak{g} = \mathbb{F}X$ and $\dim \mathfrak{g} = 1$, a contradiction.

If $\operatorname{ad}(X)^2$ is non-zero, then $\operatorname{ad}(X) \neq 0$ and there exists Y in $\mathfrak{g}$ such that $\operatorname{ad}(X)(Y) \neq 0$. Then $\operatorname{ad}(X)(Y) \neq 0$ is an eigenvector of $\operatorname{ad}(X)$ with eigenvalue $4\xi^2 \neq 0$ by (38), which contradicts that $\operatorname{ad}(X)$ is nilpotent.

Now suppose that $\mathcal{A}_f$ is a simple algebra and we will prove that f is non-degenerate. Let R be the subspace of $\mathfrak{g}$ generated by those elements X such that $f(X,Y)=0$ for all Y in $\mathfrak{g}$. We claim that $\{0\}\times R$ is a two-sided ideal of $\mathcal{A}_f$. Indeed, as f is invariant it is clear that R is an ideal of $\mathfrak{g}$. Let X be in R and (η,Y) be in $\mathcal{A}_f$. Then $(0,X)(\eta,Y)=(0,\eta X+\frac{1}{2}[X,Y])=(\eta,Y)(0,X)$ belongs to $\{0\}\times R$. Since $\mathcal{A}_f$ is simple and f is non-zero, then $\{0\}\times R$ is zero, which implies that $R=\{0\}$ and f is non-degenerate.

(**ii**) Let (ξ,X) and (η,Y) be in $\mathcal{A}_f$, then $[(\xi,X),(\eta,Y)]_f=(0,[X,Y])$. The kernel of the projection map $(\xi,X)\mapsto x$, between $\mathcal{A}_f$ and $\mathfrak{g}$, is $\mathbb{F}(1,0)$. Then the map $(\xi,X)+\mathbb{F}(1,0)\mapsto X$ is an isomorphism between $\mathcal{A}_f/\mathbb{F}(1,0)$ with bracket induced by $[\,\cdot\,,\,\cdot\,]_f$, and $(\mathfrak{g},[\,\cdot\,,\,\cdot\,])$.

$\square$

Theorem 2 can be considered as a generalization of the results given in [5] from the abelian to the nilpotent case.

Acknowledgements

The author acknowledges the support provided by the post-doctoral fellowship CONAHCYT 769309. The author thanks the referee for valuable comments and suggestions on this article. The author declares he has no conflicts to disclose.

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