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Cohomology and deformations of crossed homomorphisms between Lie–Yamaguti superalgebras

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Abstract. In this study, we propose the idea of crossed homomorphisms between Lie–Yamaguti superalgebras and develop the Yamaguti cohomology theory of crossed homomorphisms. In light of this, we characterize linear deformations of crossing homomorphisms between Lie–Yamaguti superalgebras using this cohomology. We demonstrate that if two linear or formal deformations of a crossing homomorphism are similar, then their infinitesimals are in the same cohomology class in the first cohomology group. In addition, we show that an order n deformation of a crossing homomorphism can be extended to an order $n+1$ deformation if and only if the obstruction class in the second cohomology group is trivial.

1. Introduction

In mathematical deformation theory, one studies how an object in a certain category of spaces can be varied in dependence of the points of a parameter space. In other words, deformation theory thus deals with the structure of families of objects like varieties, singularities, vector bundles, coherent sheaves, algebras or differentiable maps. Deformation problems appear in various areas of mathematics, in particular in algebra, algebraic and analytic geometry, and mathematical physics. The mathematical theory of deformations has been proved to be a powerful tool in modeling physical reality. For example, (algebras associated with) classical quantum mechanics (and field theory) on a Poisson phase space can be deformed to (algebras associated with) quantum mechanics (and quantum field theory). That is a frontier domain in mathematics and theoretical physics called deformation quantization, with multiple ramifications, avatars and connections in both mathematics and physics. These include representation theory, quantum groups

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(when considering Hopf algebras instead of associative or Lie algebras), noncommutative geometry and manifolds, algebraic geometry, number theory. The notion of deformation can be applied to a variety of categories that are used to express mathematically the physical reality. The deformation of algebraic systems has been one of the problems that many mathematical researchers are interested in, since Gerstenhaber studied the deformation theory of algebras in a series of papers ([2], [3]). For example, it has been extended to covariant functors from a small category to algebras (see [4]) and to algebraic systems, bialgebras, Hopf algebras (see [5]) by Gerstenhaber and Schack, also to Leibniz pairs and Poisson algebras (see [1]) by Flato et al. to Lie triple systems (see [8]) by Kubo and Taniguchi.

Lie−Yamaguti algebras (first called generalized Lie triple systems) were introduced in [22] as an algebraic treatment of characteristic properties of the torsion and curvature of reductive homogeneous spaces. Later on, these algebras were called Lie triple algebras in [9] and the recent terminology was introduced in [10]. Cohomology groups of Lie−Yamaguti algebras were defined in [21] while their deformations and extensions were considered in [23, 12, 13].

Lie superalgebras as a $\mathbb{Z}_2$-graded generalization of Lie algebras are considered in [7, 17] while a $\mathbb{Z}_2$-graded generalization of Lie triple systems (called Lie supertriple systems) was first considered in [20]. For an application of Lie supertriple systems in physics, one may refer to [14]. Next, Lie−Yamaguti superalgebras as a $\mathbb{Z}_2$-graded generalization of Lie−Yamaguti algebras were first considered in [15]. In [18] the authors analyzed product structures and complex structures on Lie−Yamaguti algebras by means of Nijenhuis operators. Takahashi studied modules over quandles using representations of Lie−Yamaguti algebras in [19].

The notion of crossed homomorphism on Lie algebras was introduced when non abelian extension of Lie algebras was studied [11]. An example of crossed homomorphisms is a differential operator of weight 1, and a flat connection 1-form of a trivial principle bundle is also a crossed homomorphism. In [16], authors showed that the category of weak representations (resp. admissible representations) of Lie–Rinehart algebras (resp. Leibniz pairs) is a left module category over the monoidal category of representations of Lie algebras using crossed homomorphisms. Later, cohomology and deformations of crossed homomorphisms on 3-Lie algebras were also studied in [6]. Thus it is natural to consider cohomology and deformations of crossed homomorphisms between Lie−Yamaguti superalgebras. More precisely, for a crossed homomorphism $H : \mathfrak{g} \longrightarrow \mathfrak{h}$ from a Lie−Yamaguti superalgebra $\mathfrak{g}$ to another Lie−Yamaguti superalgebra $\mathfrak{h}$ with respect to an action (ρ, μ), the most important step is to establish the cohomology theory of H. Our strategy is as follows. First we introduce linear maps $\rho_H : \mathfrak{g} \longrightarrow \mathfrak{h}$ and $\mu_H : \otimes^2\mathfrak{g} \longrightarrow \mathfrak{h}$ via H, and prove that $(\mathfrak{h};\rho_H,\mu_H)$ is a representation of $\mathfrak{g}$ on the vector space $\mathfrak{h}$. Consequently, we obtain a corresponding Yamaguti cohomology of Lie–Yamaguti superalgebra $\mathfrak{g}$ with coefficients in the representation $(\mathfrak{h};\rho_H,\mu_H)$. Note that Yamaguti cohomology starts from 1-cochains. Thus the second step is to construct 0-cochains and the corresponding coboundary maps, which is a difficulty to overcome. Once the cohomology theory is established, we are able to explore the relationship between cohomology and deformations of crossed homomorphisms. For this purpose, we intend to investigate three kinds of deformations: linear, formal, and higher order deformations. Note that a Lie–Yamaguti superalgebra can be reduced to a Lie supertriple system when the ternary bracket is trivial, thus the notion of crossed homomorphisms and the cohomology and deformation theory of those between Lie supertriple systems can be obtained directly from the present paper.

The paper is structured as follows. In Section 2, we review several fundamental concepts like representations, cohomology, and Lie–Yamaguti superalgebras. We define crossed homomorphisms between Lie–Yamaguti superalgebras in Section 3 and demonstrate that there is a one-to-one correspondence between crossed homomorphisms and Lie–Yamaguti superalgebra homomorphisms. The cohomology of crossed homomorphisms on Lie–Yamaguti superalgebras is established in Section 4, and a functorial property of the cohomology theory is examined. Finally, in Section 5, we examine three different types of deformations and demonstrate how cohomology can control the infinitesimal linear and formal deformations, as well as how a higher deformations extension is described by a certain cohomology class.

In this paper, all vector spaces are assumed to be over a field $\mathbb{K}$ of characteristic 0 and finite-dimensional.

2. Preliminaries

Let $\mathfrak{g}$ be a $\mathbb{Z}_2$-graded linear space with direct sum $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$. The elements of $\mathfrak{g}_j$, are said to be homogeneous of degree (parity) $j\in\mathbb{Z}_2$. The set of all homogeneous elements of $\mathfrak{g}$ is $\mathcal{H}(\mathfrak{g})=\mathfrak{g}_{\bar{0}}\cup\mathfrak{g}_{\bar{1}}$. Usually $\mid x\mid$ denotes parity of a homogeneous element $x\in\mathcal{H}(\mathfrak{g})$.

Now we recall some basic notions such as Lie–Yamaguti superalgebras, representations and their cohomology theories. The notion of Lie–Yamaguti superalgebras was introduced by Yamaguti in [22].

Definition 1 ([22, 24, 25])**.** A *Lie–Yamaguti superalgebra* is a triple $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$ where $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ is $\mathbb{K}$-vector superspace, i.e, $\mathbb{Z}_2$-graded vector space, $[\cdot,\cdot]$ a bilinear map (the binary superoperation on $\mathfrak{g}$) and $[\![\cdot,\cdot,\cdot]\!]$ a trilinear map (the ternary superoperation on $\mathfrak{g}$) such that

$$[x,y]=(-1)^{|x||y|+1}[y,x],$$
$$[\![x,y,z]\!]=(-1)^{|x||y|+1}[\![y,x,z]\!],$$

$$[\![x,y,z]\!]+(-1)^{|x|(|y|+|z|)}[\![y,z,x]\!]+(-1)^{|z|(|x|+|y|)}[\![z,x,y]\!]+J(x,y,z)=0, \quad (1)$$

$$[\![[x,y],z,u]\!]+(-1)^{|x|(|y|+|z|)}[\![[y,z],x,u]\!]+(-1)^{|z|(|x|+|y|)}[\![[z,x],y,u]\!]=0, \quad (2)$$

$$[\![x,y,[u,v]]\!]=[[\![x,y,u]\!],v]+(-1)^{|u|(|x|+|y|)}[u,[\![x,y,v]\!]], \quad (3)$$

$$\begin{aligned}[\![x,y,[\![u,v,w]\!]]\!] &= [\![[\![x,y,u]\!],v,w]\!]+(-1)^{|u|(|x|+|y|)}[\![u,[\![x,y,v]\!],w]\!]\\ &\quad +(-1)^{(|x|+|y|)(|u|+|v|)}[\![u,v,[\![x,y,w]\!]]\!],\end{aligned}$$

for all x,y,z,u,v,w in $\mathfrak{g}$ and

$$J(x,y,z)=[[x,y],z]+(-1)^{|x|(|y|+|z|)}[[y,z],x]+(-1)^{|z|(|x|+|y|)}[[z,x],y]. \quad (4)$$

The relation (4) is called *super-Jacobiator*.

Observe that if $[x,y]=0$, for all x,y in $\mathfrak{g}$, then a Lie−Yamaguti superalgebra $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$ reduces to a Lie supertriple system $(\mathfrak{g},[\![\cdot,\cdot,\cdot]\!])$ as defined in [14] and if $[\![x,y,z]\!]=0$ for all x,y,z in $\mathfrak{g}$, then $(\mathfrak{g},[\cdot,\cdot],[\![\cdot,\cdot,\cdot]\!])$ is a Lie superalgebra $(\mathfrak{g},[\cdot,\cdot])$.

Recall that a *Lie supertriple system* is a pair $(T,[\![\cdot,\cdot,\cdot]\!])$ where $T=T_0\oplus T_1$ is a $\mathbb{K}$-vector superspace and $[\![\cdot,\cdot,\cdot]\!]$ a trilinear map such that

$$[\![x,y,z]\!]=(-1)^{|x||y|}[\![y,x,z]\!],$$

$$[\![x,y,z]\!]+(-1)^{|x|(|y|+|z|)}[\![y,z,x]\!]+(-1)^{|z|(|x|+|y|)}[\![z,x,y]\!]=0,$$

$$\begin{aligned}[\![x,y,[\![u,v,w]\!]]\!] &= [\![x,y,u]\!],v,v]\!]+(-1)^{(|x|+|y|)|u|}[\![u,[\![x,y,v]\!],w]\!]\\ &\quad +(-1)^{(|x|+|y|)(|u|+|v|)}[\![u,v,\{x,y,w]\!]]\!],\end{aligned}$$

for all x,y,z in T, and a Lie superalgebra is a pair $(A,[\cdot,\cdot])$ where $A=A_0\oplus A_1$ is a $\mathbb{K}$-vector superspace with " $[\cdot,\cdot]$ " a bilinear map (the binary superoperation on A) such that

$$x*y=(-1)^{|x||y|}y*x,$$

$$J(x,y,z)=0,$$

for all x,y,z in A.

Example 1. Let $(\mathfrak{g},[\cdot,\cdot])$ be a Lie superalgebra. We define a bracket $\{\cdot,\cdot,\cdot\}:\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ by

$$\{x,y,z\}=[[x,y],z] \quad \forall x,y,z\in\mathfrak{g}.$$

Then $(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\})$ becomes naturally a Lie−Yamaguti superalgebra.

Example 2. Let $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$ a $\mathbb{Z}_2$-graded vector space where the even part $\mathfrak{g}_{\bar{0}}=\text{span}\{e_1,e_2\}$ and the odd part $\mathfrak{g}_{\bar{1}}=\text{span}\{f_1,f_2\}$. If we define a binary non-zero bracket $[\cdot,\cdot]$ and a ternary non-zero bracket $\{\cdot,\cdot,\cdot\}$ on $\mathfrak{g}$ as

$$[e_1,e_2]=e_2, \quad [e_1,f_1]=f_2, \quad [f_1,f_2]=e_1,$$

$$\{e_1,e_2,f_1\}=f_2, \quad \{e_1,f_1,f_2\}=e_1,$$

then $(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\})$ is a Lie−Yamaguti superalgebra.

Next, we recall the notion of representations of Lie–Yamaguti superalgebras.

Definition 2. Let $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ be a Lie–Yamaguti superalgebra. A representation of $\mathfrak{g}$ is a *vector superspace* V equipped with an even linear map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ and an even bilinear map $\mu : \otimes^2\mathfrak{g} \to \mathfrak{gl}(V)$, which meet the following conditions: for all $x, y, z, w \in \mathfrak{g}$,

$$\mu([x,y],z) - (-1)^{|y||z|}\mu(x,z)\rho(y) + (-1)^{|x|(|y|+|z|)}\mu(y,z)\rho(x) = 0, \tag{5}$$

$$\mu(x,[y,z]) - (-1)^{|x||y|}\rho(y)\mu(x,z) + (-1)^{|z|(|x|+|y|)}\rho(z)\mu(x,y) = 0, \tag{6}$$

$$\rho([\![x,y,z]\!]) = [D_{\rho,\mu}(x,y), \rho(z)], \tag{7}$$

$$\begin{aligned}&(-1)^{(|x|+|y|)(|z|+|w|)}\mu(z,w)\mu(x,y) - (-1)^{(|x||y|+|z||w|)}\mu(y,w)\mu(x,z)\\ &-\mu(x,[\![y,z,w]\!]) + (-1)^{|x|(|y|+|z|)}D_{\rho,\mu}(y,z)\mu(x,w) = 0,\end{aligned} \tag{8}$$

$$\mu([\![x,y,z]\!],w) + (-1)^{|z|(|x|+|y|)}\mu(z,[\![x,y,w]\!]) = [D_{\rho,\mu}(x,y), \mu(z,w)], \tag{9}$$

where the bilinear map $D_{\rho,\mu} : \otimes^2\mathfrak{g} \to \mathfrak{gl}(V)$ is given by

$$D_{\rho,\mu}(x,y) := (-1)^{|x||y|}\mu(y,x) - \mu(x,y) + [\rho(x),\rho(y)] - \rho([x,y]), \quad \forall x,y \in \mathfrak{g}. \tag{10}$$

It is obvious that $D_{\rho,\mu}$ is skew-symmetric, and we write D in the sequel if there is no ambiguities. We denote a representation of $\mathfrak{g}$ by $(V;\rho,\mu)$.

By computation, we obtain the following lemma.

Lemma 1. *Suppose that $(V;\rho,\mu)$ is a representation of a Lie–Yamaguti superalgebra $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$. Then the following equalities are satisfied:*

$$D([x,y],z) + (-1)^{|x|(|y|+|z|)}D([y,z],x) + (-1)^{|z|(|x|+|y|)}D([z,x],y) = 0$$

$$D([\![x,y,z]\!],y,w) + (-1)^{|z|(|x|+|y|)}D(z,[\![x,y,w]\!]) = [D(x,y), D_{\rho,\mu}(z,w)]$$

$$\begin{aligned}\mu([\![x,y,z]\!],w) &= (-1)^{|w|(|y|+|z|)+|y||z|}\mu(x,w)\mu(z,y)\\ &\quad -(-1)^{|w|(|x|+|z|)+|x||z|}\mu(y,w)\mu(z,x)\\ &\quad -(-1)^{(|z|+|w|)(|x|+|y|)}\mu(z,w)D(x,y).\end{aligned}$$

Example 3. Let $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ be a Lie–Yamaguti superalgebra. We define linear maps $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ and $\mathfrak{R} : \otimes^2\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ by $x \mapsto \mathrm{ad}_x$ and $(x,y) \mapsto \mathfrak{R}_{x,y}$, respectively, where $\mathrm{ad}_x z = [x,z]$ and $\mathfrak{R}_{x,y}z = [\![z,x,y]\!]$ for all $z \in \mathfrak{g}$. Then $(\mathrm{ad},\mathfrak{R})$ forms a representation of $\mathfrak{g}$ on itself, where $\mathfrak{L} := D_{\mathrm{ad},\mathfrak{R}}$ is given by

$$\mathfrak{L}_{x,y} = (-1)^{|x||y|}\mathfrak{R}_{y,x} - \mathfrak{R}_{x,y} + [\mathrm{ad}_x, \mathrm{ad}_y] - \mathrm{ad}_{[x,y]}, \quad \forall x,y \in \mathfrak{g}.$$

By (1), we have

$$\mathfrak{L}_{x,y}z = [\![x,y,z]\!], \quad \forall z \in \mathfrak{g}.$$

In this case, $(\mathfrak{g};\mathrm{ad},\mathfrak{R})$ is called the *adjoint representation* of $\mathfrak{g}$.

Let $(\mathfrak{g}, [\cdot,\cdot], [\![\cdot,\cdot,\cdot]\!])$ be a Lie–Yamaguti superalgebra and $(V;\rho,\mu)$ a representation of $\mathfrak{g}$. We denote the set of p-cochains by $C^p_{\text{LieY}}(\mathfrak{g}, V)$ $(p \geqslant 1)$, where

$$C^{n+1}_{\text{LieY}}(\mathfrak{g}, V) \triangleq \begin{cases} \text{Hom}(\underbrace{\wedge^2\mathfrak{g}\otimes\cdots\otimes\wedge^2\mathfrak{g}}_{n}, V) \times \text{Hom}(\underbrace{\wedge^2\mathfrak{g}\otimes\cdots\otimes\wedge^2\mathfrak{g}}_{n}\otimes\mathfrak{g}, V), & \\ & \forall n \geqslant 1, \\ \text{Hom}(\mathfrak{g}, V), & n = 0. \end{cases}$$

In the sequel, we recall the coboundary map of p-cochains.

- If $n \geqslant 1$, for any $(f, g) \in C^{n+1}_{\text{LieY}}(\mathfrak{g}, V)$, the coboundary map

  $$\delta = (\delta_{\text{I}}, \delta_{\text{II}}) : C^{n+1}_{\text{LieY}}(\mathfrak{g}, V) \to C^{n+2}_{\text{LieY}}(\mathfrak{g}, V),\ (f, g) \mapsto (\delta_{\text{I}}(f, g), \delta_{\text{II}}(f, g))$$

  is given as follows:

$$\begin{aligned} &(\delta_{\text{I}}(f,g))(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n+1}) \\ = \ & (-1)^{n+|x_{n+1}|(|\mathfrak{X}_1|+\cdots+|\mathfrak{X}_n|)}\rho(x_{n+1})\,g(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,y_{n+1}) \\ & -(-1)^{n+|x_{n+1}|(|\mathfrak{X}_1|+\cdots+|\mathfrak{X}_n|)}\rho(y_{n+1})\,g(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,x_{n+1}) \\ & -(-1)^n g(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,[x_{n+1},y_{n+1}]) \\ & +\textstyle\sum_{k=1}^{n}(-1)^{k+1+|\mathfrak{X}_k|(|\mathfrak{X}_1|+\cdots+|\mathfrak{X}_{k-1}|)}D(\mathfrak{X}_k)\,f\left(\mathfrak{X}_1,\cdots,\widehat{\mathfrak{X}}_k,\cdots,\mathfrak{X}_{n+1}\right) \\ & +\textstyle\sum_{1\leqslant k<l\leqslant n+1}(-1)^{k+|\mathfrak{X}_k|(|\mathfrak{X}_{k+1}|+\cdots+|\mathfrak{X}_{l-1}|)}f\left(\mathfrak{X}_1,\cdots,\widehat{\mathfrak{X}}_k,\cdots,\mathfrak{X}_k\circ\mathfrak{X}_l,\cdots,\mathfrak{X}_{n+1}\right), \end{aligned} \tag{11}$$

$$\begin{aligned} &(\delta_{\text{II}}(f,g))(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n+1},z) \\ = \ & (-1)^{n+(|\mathfrak{X}_{n+1}|+|z|)(|\mathfrak{X}_1|+\cdots+|\mathfrak{X}_n|)}\mu(y_{n+1},z)\,g(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,x_{n+1}) \\ & -(-1)^{n+(|\mathfrak{X}_{n+1}|+|z|)(|\mathfrak{X}_1|+\cdots+|\mathfrak{X}_n|)}\mu(x_{n+1},z)\,g(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,y_{n+1}) \\ & +\textstyle\sum_{k=1}^{n+1}(-1)^{k+1+|\mathfrak{X}_k|(|\mathfrak{X}_1|+\cdots+|\mathfrak{X}_{k-1}|)}D(\mathfrak{X}_k)\,g\left(\mathfrak{X}_1,\cdots,\widehat{\mathfrak{X}}_k,\cdots,\mathfrak{X}_{n+1},z\right) \\ & +\textstyle\sum_{1\leqslant k<l\leqslant n+1}(-1)^{k+|\mathfrak{X}_k|(|\mathfrak{X}_{k+1}|+\cdots+|\mathfrak{X}_{l-1}|)}g\left(\mathfrak{X}_1,\cdots,\widehat{\mathfrak{X}}_k,\cdots,\mathfrak{X}_k\circ\mathfrak{X}_l,\cdots,\mathfrak{X}_{n+1},z\right) \\ & +\textstyle\sum_{k=1}^{n+1}(-1)^{k+|\mathfrak{X}_k|(|\mathfrak{X}_{k+1}|+\cdots+|\mathfrak{X}_{n+1}|)}g\left(\mathfrak{X}_1,\cdots,\widehat{\mathfrak{X}}_k,\cdots,\mathfrak{X}_{n+1},[\![x_k,y_k,z]\!]\right), \end{aligned} \tag{12}$$

  where $\mathfrak{X}_i = x_i \wedge y_i \in \wedge^2\mathfrak{g}(i = 1,\cdots,n+1)$, $|\mathfrak{X}_i| = |x_i| + |y_i|$, $z \in \mathfrak{g}$ and $\mathfrak{X}_k \circ \mathfrak{X}_l := [\![x_k,y_k,x_l]\!] \wedge y_l + x_l \wedge [\![x_k,y_k,y_l]\!]$.
- If $n = 0$, for any $f \in C^1_{\text{LieY}}(\mathfrak{g}, V)$, the coboundary map

$$\begin{aligned} \delta : C^1_{\text{LieY}}(\mathfrak{g}, V) &\to C^2_{\text{LieY}}(\mathfrak{g}, V), \\ f &\mapsto (\delta_{\text{I}}(f), \delta_{\text{II}}(f)) \end{aligned}$$

is defined to be

$$(\delta_{\mathrm{I}}(f))\,(x,y)=\rho(x)f(y)-\rho(y)f(x)-f([x,y]), \tag{13}$$

$$(\delta_{\mathrm{II}}(f))\,(x,y,z)=D(x,y)f(z)+\mu(y,z)f(x)-\mu(x,z)f(y)-f(\llbracket x,y,z\rrbracket), \tag{14}$$

for all $x,y,z\in\mathfrak{g}$.

Yamaguti proved the following fact.

Proposition 1. *With the notations above, for any $f\in C^1_{\mathrm{LieY}}(\mathfrak{g},V)$, we have*

$$\delta_{\mathrm{I}}\left(\delta_{\mathrm{I}}(f),\delta_{\mathrm{II}}(f)\right)=0 \quad \textit{and} \quad \delta_{\mathrm{II}}\left(\delta_{\mathrm{I}}(f),\delta_{\mathrm{II}}(f)\right)=0.$$

Moreover, for all $(f,g)\in C^p_{LieY}(\mathfrak{g},V)$ where $p\geqslant 2$, we have

$$\delta_{\mathrm{I}}\left(\delta_{\mathrm{I}}(f,g),\delta_{\mathrm{II}}(f,g)\right)=0 \quad \textit{and} \quad \delta_{\mathrm{II}}\left(\delta_{\mathrm{I}}(f,g),\delta_{\mathrm{II}}(f,g)\right)=0.$$

Thus the cochain complex $\left(C^*_{\mathrm{LieY}}(\mathfrak{g},V)=\bigoplus_{p=1}^{\infty}C^p_{\mathrm{Lie}}(\mathfrak{g},V),\delta\right)$ is well defined.

For convenience, we call this cohomology the *Yamaguti cohomology* in this paper.

Definition 3 ([6])**.** With the above notations, let (f,g) in $C^p_{\mathrm{LieY}}(\mathfrak{g},V)$ (resp. $f\in C^1_{\mathrm{LieY}}(\mathfrak{g},V)$ for $p=1$) be a p-cochain. If it satisfies $\delta(f,g)=0$ (resp. $\delta(f)=0$), then it is called a p-cocycle. If there exists $(h,s)\in C^{p-1}_{\mathrm{LieY}}(\mathfrak{g},V)$ (resp. $t\in C^1(\mathfrak{g},V)$, if $p=2$) such that $(f,g)=\delta(h,s)$ (resp. $(f,g)=\delta(t)$), then it is called a p-coboundary ($p\geqslant 2$). The set of p-cocycles and that of p coboundaries are denoted by $Z^p_{\mathrm{Lie}}(\mathfrak{g},V)$ and $B^p_{\mathrm{LieY}}(\mathfrak{g},V)$ respectively. The resulting p-cohomology group is defined to be the factor super space

$$H^p_{\mathrm{LieY}}(\mathfrak{g},V)=Z^p_{\mathrm{LieY}}(\mathfrak{g},V)/B^p_{\mathrm{LieY}}(\mathfrak{g},V).$$

In particular, we have

$$H^1_{\mathrm{LieY}}(\mathfrak{g},V)=\left\{f\in C^1_{\mathrm{LieY}}(\mathfrak{g},V):\delta(f)=0\right\}.$$

3. Crossed homomorphisms between Lie–Yamaguti superalgebras

The concept of crossed homomorphisms between Lie–Yamaguti superalgebras is introduced in this section, and it is demonstrated that a crossed homomorphism can be viewed as a homomorphism of Lie–Yamaguti superalgebras. A crossed homomorphism also has a 1 relative Rota-Baxter operator associated with it. We first present the idea of center of Lie–Yamaguti superalgebras. Let $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\llbracket\cdot,\cdot\rrbracket_{\mathfrak{g}})$ be a Lie–Yamaguti superalgebra. Denote the *center* of $\mathfrak{g}$ by

$$\begin{aligned}C(\mathfrak{g}):=\;&(\{x\in\mathfrak{g}\mid \llbracket x,y,z\rrbracket=0,\forall y,z\in\mathfrak{g}\}\cup\{x\in\mathfrak{g}\mid \llbracket y,x,z\rrbracket=0,\forall y,z\in\mathfrak{g}\})\\&\cap\{x\in\mathfrak{g}\mid [x,y]=0,\forall y\in\mathfrak{g}\}.\end{aligned}$$

Definition 4. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot]\!]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, [\![\cdot,\cdot]\!]_{\mathfrak{h}})$ be two Lie–Yamaguti superalgebras. Let $(\mathfrak{h};\rho,\mu)$ be a representation of $\mathfrak{g}$ on the superspace $\mathfrak{h}$, i.e., linear maps $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{b}), \mu : \otimes^2\mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$, and $D : \wedge^2\ \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ are given by (5)–(10). If for all $x, y \in \mathfrak{g}, u, v, w \in \mathfrak{h}$, the following conditions

$$\begin{aligned} \rho(x)u, \mu(x,y)u \in C(\mathfrak{h}),\\ \rho(x)[u,v]_{\mathfrak{h}} = \mu(x,y)[u,v]_{\mathfrak{h}} = 0,\\ \rho(x)[\![u,v,w]\!]_{\mathfrak{h}} = \mu(x,y)[\![u,v,w]\!]_{\mathfrak{h}} = 0 \end{aligned}$$

are satisfied, then we say that (ρ,μ) is an *action* of $\mathfrak{g}$ on $\mathfrak{h}$.

Let (ρ,μ) be an action of $\mathfrak{g}$ on $\mathfrak{h}$. By (10), we deduce that

$$D(x,y)u \in C(\mathfrak{g}), D(x,y)[u,v]_{\mathfrak{h}} = D(x,y)[\![u,v,w]\!]_{\mathfrak{h}} = 0,\ \forall x,y \in \mathfrak{g}, u,v,w \in \mathfrak{h}.$$

The following proposition show that actions of Lie–Yamaguti superalgebras can be used to characterize semidirect product Lie–Yamaguti superalgebras.

Proposition 2. *Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot]\!]_{\mathfrak{g}})$ be two Lie–Yamaguti superalgebras. Let (ρ,μ) be an action of $\mathfrak{g}$ on $\mathfrak{h}$, then there is a Lie–Yamaguti superalgebra structure on the direct sum $\mathfrak{g} \oplus \mathfrak{h}$ defined by*

$$\begin{aligned} [x+u, y+v]_{\rho,\mu} &= [x,y]_{\mathfrak{g}} + \rho(x)v - \rho(y)u + [u,v]_{\mathfrak{h}},\\ [\![x+u, y+v, z+w]\!]_{\rho,\mu} &= [\![x,y,z]\!]_{\mathfrak{g}} + D(x,y)w + (-1)^{|u|(|y|+|z|)}\mu(y,z)u\\ &\quad -(-1)^{|z||v|}\mu(x,z)v + [\![u,v,w]\!]_{\mathfrak{h}}, \end{aligned}$$

for all $x,y,z \in \mathfrak{g}$ and $u,v,w \in \mathfrak{h}$. This Lie–Yamaguti superalgebra is called the semidirect product Lie–Yamaguti superalgebra with respect to the action (ρ,μ), and is denoted by $\mathfrak{g} \ltimes_{\rho,\mu} \mathfrak{h}$.

Proof. It is a direct computation, and we omit the details. □

The following definition is standard.

Definition 5. Suppose that $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}})$ are two Lie–Yamaguti superalgebras. A homomorphism from $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}})$ to $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}})$ is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ that preserves the Lie–Yamaguti superalgebra structures, that is, for all $x,y,z \in \mathfrak{g}$,

(1) $\phi(\mathfrak{g}_i) \subseteq \mathfrak{h}_i,\ i \in \mathbb{Z}_2$,
(2) $\phi([x,y]_{\mathfrak{g}}) = [\phi(x),\phi(y)]_{\mathfrak{h}}$,
(3) $\phi([\![x,y,z]\!]_{\mathfrak{g}}) = [\![\phi(x),\phi(y),\phi(z)]\!]_{\mathfrak{h}}$.

If, moreover, ϕ is a bijection, then it is called an *isomorphism*.

Now we are ready to introduce the notion of crossed homomorphisms between Lie–Yamaguti superalgebras.

Definition 6. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot]\!]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, [\![\cdot,\cdot]\!]_{\mathfrak{h}})$ be two Lie–Yamaguti superalgebras. Let (ρ,μ) be an action of $\mathfrak{g}$ on $\mathfrak{h}$. A linear map $H : \mathfrak{g} \longrightarrow \mathfrak{h}$

is called a *crossed homomorphism* from $\mathfrak{g}$ to $\mathfrak{h}$ with respect to (ρ,μ), if H preserves the grading, that is, $H(\mathfrak{g}_i)\subseteq\mathfrak{h}_i$, $i\in\mathbb{Z}_2$, and

$$H[x,y]_{\mathfrak{g}}=(-1)^{|x||H|}\rho(x)H(y)-(-1)^{|y|(|H|+|x|)}\rho(y)H(x)+[Hx,Hy]_{\mathfrak{h}},\qquad(15)$$

$$\begin{aligned}H[\![x,y,z]\!]_{\mathfrak{g}}=\;&(-1)^{|H|(|x|+|y|)}D(x,y)H(z)+(-1)^{(|H|+|x|)(|y|+|z|)}\mu(y,z)H(x)\\&-(-1)^{|H|(|x|+|z|)+|y||z|}\mu(x,z)H(y)+[\![Hx,Hy,Hz]\!]_{\mathfrak{h}},\quad\forall x,y,z\in\mathfrak{g}.\end{aligned}\qquad(16)$$

Example 4. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie–Yamaguti superalgebras with trivial actions ($\rho=0$, $\mu=0$). Any homomorphism $H:\mathfrak{g}\to\mathfrak{h}$ is a crossed homomorphism.

Remark 1. If the action of $\mathfrak{g}$ on $\mathfrak{h}$ is trivial, then any crossed homomorphism from $\mathfrak{g}$ to $\mathfrak{h}$ is a Lie–Yamaguti superalgebra homomorphism as in Definition 5; if $\mathfrak{h}$ is commutative, then any crossed homomorphism is a derivation from $\mathfrak{g}$ to $\mathfrak{h}$ with respect to the representation $(\mathfrak{h};\rho,\mu)$.

The following theorem shows that a crossed homomorphism can be seen as a homomorphism between Lie–Yamaguti superalgebras.

Theorem 1. *Let (ρ,μ) be an action of a Lie–Yamaguti superalgebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},[\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}})$ on another Lie–Yamaguti superalgebra $(\mathfrak{h},[\cdot,\cdot]_{\mathfrak{h}},[\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}})$. Then a linear map $H:\mathfrak{g}\longrightarrow\mathfrak{h}$ is a crossed homomorphism from $\mathfrak{g}$ to $\mathfrak{h}$ if and only if the linear map $\phi_H:\mathfrak{g}\longrightarrow\mathfrak{g}\ltimes_{\rho,\mu}\mathfrak{h}$ is a Lie–Yamaguti superalgebra homomorphism, where*

$$\phi_H(x):=(x,Hx),\quad\forall x\in\mathfrak{g}.$$

Proof. For all $x,y,z\in\mathfrak{g}$, we have

$$\phi_H([x,y]_{\mathfrak{g}})=([x,y]_{\mathfrak{g}},H[x,y]_{\mathfrak{g}}),$$

$$[\phi_H(x),\phi_H(y)]_{\rho,\mu}=[(x,Hx),(y,Hy)]_{\rho,\mu}$$

$$=\Big([x,y]_{\mathfrak{g}},(-1)^{|H||x|}\rho(x)Hy-(-1)^{|y|(|x|+|H|)}\rho(y)Hx+[Hx,Hy]_{\mathfrak{h}}\Big).$$

Similarly, we also have

$$\phi_H([\![x,y,z]\!]_{\mathfrak{g}})=([\![x,y,z]\!]_{\mathfrak{g}},H[\![x,y,z]\!]_{\mathfrak{g}}),$$

$$[\![\phi_H(x),\phi_H(y),\phi_H(z)]\!]_{\rho,\mu}=[(x,Hx),(y,Hy),(z,Hz)]_{\rho,\mu}=([\![x,y,z]\!]_{\mathfrak{g}},H[\![x,y,z]\!]_{\mathfrak{g}}),$$

where

$$\begin{aligned}H[\![x,y,z]\!]_{\mathfrak{g}}=&(-1)^{|H|(|x|+|y|)}D(x,y)H(z)+(-1)^{(|H|+|x|)(|y|+|z|)}\mu(y,z)H(x)\\&-(-1)^{|H|(|x|+|z|)+|y||z|}\mu(x,z)H(y)+[\![Hx,Hy,Hz]\!]_{\mathfrak{h}}.\end{aligned}$$

Thus, we have that the linear map $\phi_H : \mathfrak{g} \longrightarrow \mathfrak{g} \ltimes_{\rho,\mu} \mathfrak{h}$ is a Lie–Yamaguti superalgebra homomorphism if and only if the following two equalities hold:

$$\begin{aligned} H[x,y]_{\mathfrak{g}} =& (-1)^{|H||x|}\rho(x)Hy - (-1)^{|y|(|x|+|H|)}\rho(y)Hx + [Hx,Hy]_{\mathfrak{h}}, \\ H[\![x,y,z]\!]_{\mathfrak{g}} =& (-1)^{|H|(|x|+|y|)}D(x,y)H(z) + (-1)^{(|H|+|x|)(|y|+|z|)}\mu(y,z)H(x) \\ &- (-1)^{|H|(|x|+|z|)+|y||z|}\mu(x,z)H(y) + [\![Hx,Hy,Hz]\!]_{\mathfrak{h}}, \ \forall x,y,z \in \mathfrak{g}, \end{aligned}$$

which implies that the linear map $H : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a crossed homomorphism from $\mathfrak{g}$ to $\mathfrak{h}$. This completes the proof. □

Remark 2. In fact, a crossed homomorphism corresponds to a split non-abelian extension of Lie–Yamaguti superalgebras. More precisely, consider the following nonabelian extension of Lie–Yamaguti superalgebras:

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \oplus \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow 0.$$

A section $s : \mathfrak{g} \longrightarrow \mathfrak{g} \oplus \mathfrak{h}$ is given by $s(x) = (x, Hx), x \in \mathfrak{g}$. Theorem 1 says that s is a Lie–Yamaguti superalgebra homomorphism if and only if H is a crossed homomorphism. Such an extension is called a split nonabelian extension.

Definition 7. Let H and H' be two crossed homomorphisms from a Lie–Yamaguti superalgebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}})$ to another Lie–Yamaguti superalgebra $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}})$ with respect to an even action (ρ,μ). A homomorphism from H' to H is a pair $(\psi_{\mathfrak{g}}, \psi_{\mathfrak{b}})$, where $\psi_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathfrak{g}$ and $\psi_{\mathfrak{b}} : \mathfrak{h} \longrightarrow \mathfrak{h}$ are two Lie–Yamaguti superalgebra homomorphisms such that $\psi_{\mathfrak{g}}(\mathfrak{g}_i) \subseteq \mathfrak{g}_i$, $\psi_{\mathfrak{h}}(\mathfrak{h}_j) \subseteq \mathfrak{h}_j$, $i,j \in \mathbb{Z}_2$, and

$$\psi_{\mathfrak{h}} \circ H' = H \circ \psi_{\mathfrak{g}}, \tag{17}$$

$$\psi_{\mathfrak{h}}(\rho(x)u) = (-1)^{|\psi_{\mathfrak{h}}|(|\psi_{\mathfrak{g}}|+|x|)}\rho(\psi_{\mathfrak{g}}(x))\,\psi_{\mathfrak{h}}(u), \tag{18}$$

$$\psi_{\mathfrak{h}}(\mu(x,y)u) = (-1)^{|\psi_{\mathfrak{h}}|(|x|+|y|)}\mu(\psi_{\mathfrak{g}}(x),\psi_{\mathfrak{g}}(y))\,\psi_{\mathfrak{h}}(u), \quad \forall x,y \in \mathfrak{g}, u \in \mathfrak{h}. \tag{19}$$

In particular, if both $\psi_{\mathfrak{g}}$ and $\psi_{\mathfrak{h}}$ are invertible, then $(\psi_{\mathfrak{g}}, \psi_h)$ is called an isomorphism from H' to H.

By equalities (18) and (19), and a direct computation, we have the following proposition.

Proposition 3. *Let H and H' be two crossed homomorphisms from a Lie–Yamaguti superalgebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}})$ to another Lie–Yamaguti superalgebra $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}})$ with respect to an action (ρ,μ). Suppose that $(\psi_{\mathfrak{g}}, \psi_{\mathfrak{h}})$ is a homomorphism from H' to H, then we have*

$$\psi_{\mathfrak{h}}(D(x,y)u) = (-1)^{|\psi_{\mathfrak{h}}|(|D|+|x|+|y|)}D(\psi_{\mathfrak{g}}(x),\psi_{\mathfrak{g}}(y))\,\psi_{\mathfrak{h}}(u), \quad \forall x,y \in \mathfrak{g}, u \in \mathfrak{h}.$$

At the end of this section, we reveal the relationship between crossed homomorphisms between Lie–Yamaguti superalgebras and relative Rota-Baxter operators of weight 1 on Lie–Yamaguti superalgebras. We give the notion of relative Rota-Baxter operators of weight λ on Lie–Yamaguti superalgebras first.

Definition 8. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot]\!]_{\mathfrak{g}})$ be a Lie–Yamaguti superalgebra and $(V;\rho,\mu)$ its representation. A even linear map $T: V \longrightarrow \mathfrak{g}$ is called a relative Rota-Baxter operator of weight λ if the equalities

$$[Tu,Tv] = T\left(\rho(Tu)v - (-1)^{|u||v|}\rho(Tv)u + \lambda[u,v]\right),$$

$$[\![Tu,Tv,Tw]\!] = \\ T\left(D(Tu,Tv)w + (-1)^{|u|(|v|+|w|)}\mu(Tv,Tw)u - (-1)^{|v||w|}\mu(Tu,Tw)v + \lambda[\![u,v,w]\!]\right),$$

hold for all $u,v,w \in V$. Relative Rota-Baxter operators of nonzero weight on Lie superalgebras stem from the classical Yang-Baxter equation and have many applications in mathematical physics. Here, we introduce the notion of relative Rota-Baxter operators of nonzero weight on Lie–Yamaguti superalgebras and explore its relation to crossed homomorphisms.

Proposition 4. *Let (ρ,μ) be an even action of a Lie–Yamaguti superalgebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}})$ on another Lie–Yamaguti superalgebra $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}})$. An invertible linear map $H: \mathfrak{g} \longrightarrow \mathfrak{h}$ is a crossed homomorphism from $\mathfrak{g}$ to $\mathfrak{h}$ with respect to (ρ,μ) if and only if $H^{-1}: \mathfrak{h} \longrightarrow \mathfrak{g}$ is a relative Rota Baxter operator of weight 1 on $\mathfrak{g}$ with respect to the representation $(\mathfrak{h};\rho,\mu)$.*

Proof. Assume that the invertible linear map $H: \mathfrak{g} \longrightarrow \mathfrak{h}$ is a crossed homomorphism, then for all $u,v,w \in \mathfrak{h}$, by (15), we have

$$\begin{aligned}\left[H^{-1}(u),H^{-1}(v)\right]_{\mathfrak{g}} &= H^{-1}\left(H\left[H^{-1}(u),H^{-1}(v)\right]_{\mathfrak{g}}\right)\\ &= H^{-1}\left(\rho\left(H^{-1}(u)\right)v - \rho\left(H^{-1}(v)\right)w + [u,v]_{\mathfrak{h}}\right).\end{aligned}$$

Similarly, by (16), we have

$$\begin{aligned}&[\![H^{-1}(u),H^{-1}(v),H^{-1}(w)]\!]_{\mathfrak{g}}\\ =&H^{-1}\left(H[\![H^{-1}(u),H^{-1}(v),H^{-1}(w)]\!]_{\mathfrak{g}}\right)\\ =&H^{-1}\left(D\left(H^{-1}(u),H^{-1}(v)\right)w + (-1)^{|u|(|v|+|w|)}\mu\left(H^{-1}(v),H^{-1}(w)\right)u\right)\\ &- H^{-1}\left((-1)^{|v||w|}\mu\left(H^{-1}(u),H^{-1}(w)\right)v - [\![u,v,w]\!]_{\mathfrak{h}}\right).\end{aligned}$$

Thus H^{-1} is a relative Rota-Baxter operator of weight 1.

Conversely, let H^{-1} be a relative Rota-Baxter operator of weight 1. For all $x,y,z \in \mathfrak{g}$, there exist $u,v,w \in \mathfrak{h}$, such that $x = H^{-1}(u)$, $y = H^{-1}(v)$

and $z = H^{-1}(w)$. Then we have

$$\begin{aligned} H[x,y]_{\mathfrak{g}} &= H\left[H^{-1}(u), H^{-1}(v)\right]_{\mathfrak{g}} \\ &= H\circ H^{-1}\Big(\rho\left(H^{-1}(u)\right)v - (-1)^{|u||v|}\rho\left(H^{-1}(v)\right)u + [u,v]_{\mathfrak{h}}\Big) \\ &= (-1)^{|H||x|}\rho(x)H(y) - (-1)^{|y|(|H|+|x|)}\rho(y)H(x) + [Hx,Hy]_{\mathfrak{h}} \end{aligned}$$

and

$$\begin{aligned} &H[\![x,y,z]\!]_{\mathfrak{g}} = H\left[H^{-1}(u), H^{-1}(v), H^{-1}(w)\right]_{\mathfrak{g}} \\ =&H\circ H^{-1}\Big(D\left(H^{-1}(u),H^{-1}(v)\right)w + (-1)^{|u|(|v|+|w|)}\mu\left(H^{-1}(v),H^{-1}(w)\right)u\Big) \\ &- H\circ H^{-1}\Big((-1)^{|v||w|}\mu\left(H^{-1}(u),H^{-1}(w)\right)v - [\![u,v,w]\!]_{\mathfrak{h}}\Big) \\ =&(-1)^{|H|(|x|+|y|)}D(x,y)H(z) + (-1)^{(|H|+|x|)(|y|+|z|)}\mu(y,z)H(x) \\ &- (-1)^{|H|(|x|+|z|)+|y||z|}\mu(x,z)H(y) + [\![Hx,Hy,Hz]\!]_{\mathfrak{h}}, \end{aligned}$$

which implies that H is a crossed homomorphism. □

4. Cohomology of crossed homomorphisms between Lie–Yamaguti superalgebras

The cohomology of crossed homomorphisms between Lie–Yamaguti superalgebras is constructed in this section. We first create a representation of a Lie–Yamaguti superalgebra via a given action. Let $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},[\![\cdot,\cdot]\!]_{\mathfrak{g}})$ and $(\mathfrak{h},[\cdot,\cdot]_{\mathfrak{h}},[\![\cdot,\cdot]\!]_{\mathfrak{h}})$ be two Lie–Yamaguti superalgebras, and $H:\mathfrak{g}\longrightarrow\mathfrak{h}$ a crossed homomorphism with respect to an action (ρ,μ). Define $\rho_H:\mathfrak{g}\longrightarrow\mathfrak{gl}(\mathrm{h})$, $\mu_H:\otimes^2\mathfrak{g}\longrightarrow\mathfrak{gl}(\mathfrak{h})$ to be

$$\rho_H(x)u := [\![Hx,u]\!]_{\mathfrak{h}} + \rho(x)u, \tag{20}$$

$$\mu_H(x,y)u := (-1)^{|u|(|x|+|y|)}[\![u,Hx,Hy]\!]_{\mathfrak{h}} + \mu(x,y)u, \quad \forall x,y\in\mathfrak{g}, u\in\mathfrak{h}. \tag{21}$$

Lemma 2. *With the assumptions above, define $D_H:\wedge^2\mathfrak{g}\longrightarrow\mathfrak{gl}(h)$ to be*

$$D_H(x,y)u := (-1)^{|x||H|}[\![Hx,Hy,u]\!]_{\mathfrak{h}} + D(x,y)u, \quad \forall x,y\in\mathfrak{g}, u\in\mathfrak{h}. \tag{22}$$

Then $D_H = D_{\rho_H,\mu_H}$.

Proof. For all $x, y \in \mathfrak{g}, u \in \mathfrak{h}$, we have

$$
\begin{aligned}
& D_{\rho_H,\mu_H}(x,y)u \\
= \quad & (-1)^{|x||y|}\mu_H(y,x)u - \mu_H(x,y)u + [\rho_H(x), \rho_H(y)](u) - \rho_H([x,y]_\mathfrak{g})u \\
= \quad & (-1)^{|u|(|x|+|y|)}[\![u, Hy, Hx]\!]_\mathfrak{h} + (-1)^{|x||y|}\mu(y,x)u - (-1)^{|u|(|x|+|y|)}[\![u, Hx, Hy]\!]_\mathfrak{h} \\
& -\mu(x,y)u + (-1)^{|x||H|}[Hx, [Hy,u]_\mathfrak{h}]_\mathfrak{h} + (-1)^{|x||H|}\rho(x)[Hy,u]_\mathfrak{h} + [Hx, \rho(y)u]_\mathfrak{h} \\
& +\rho(x)\rho(y)u - (-1)^{|y|(|x|+|H|)}[Hy, [Hx,u]_\mathfrak{h}]_\mathfrak{h} - (-1)^{|y|(|x|+|H|)}\rho(y)[Hx,u]_\mathfrak{h} \\
& -(-1)^{|x||y|}[Hy, \rho(x)u]_\mathfrak{h} - (-1)^{|x||y|}\rho(y)\rho(x)u - [H[x,y]_\mathfrak{g}, u]_\mathfrak{h} - \rho([x,y]_\mathfrak{g})u \\
\overset{(1),(10)}{=} \quad & (-1)^{|x||H|}[\![Hx, Hy, u]\!]_\mathfrak{h} + D(x,y)u \\
= \quad & D_H(x,y)u.
\end{aligned}
$$

This completes the proof. □

Proposition 5. *With the assumptions above, $(\mathfrak{h}; \rho_H, \mu_H)$ is a representation of $\mathfrak{g}$, where ρ_H, μ_H, and D_H are given by (20) − (22), respectively.*

Proof. For all $x, y, z \in \mathfrak{g}, u \in \mathfrak{h}$, we have

$$
\begin{aligned}
& \rho_H([\![x,y,z]\!]_\mathfrak{g})u - [D_H(x,y), \rho_H(z)]u \\
= \quad & [H[\![x,y,z]\!]_\mathfrak{g}, u]_\mathfrak{h} + \rho([\![x,y,z]\!]_\mathfrak{g})u - (-1)^{|H||y|}[\![Hx, Hy, [Hz,u]_\mathfrak{h}]\!]_\mathfrak{h} \\
& -(-1)^{|H|(|x|+|y|}D(x,y)[Hz,u]_\mathfrak{h} - (-1)^{|H||x|}[\![Hx, Hy, \rho(z)u]\!]_\mathfrak{h} - D(x,y)\rho(z)u \\
& +(-1)^{|z|(|x|+|y|)+|H||x|}\Big([Hz, [\![Hx,Hy,u]\!]_\mathfrak{h}]_\mathfrak{h} + \rho(z)[\![Hx,Hy,u]\!]_\mathfrak{h}\Big) \\
& +(-1)^{|z|(|D|+|x|+|y|)}\big((-1)^{|H||D|}[Hz, D(x,y)u]_\mathfrak{h} + \rho(z)D(x,y)u\big) \\
\overset{(7)}{=} \quad & (-1)^{|H||y|}\Big([[\![Hx,Hy,Hz]\!]_\mathfrak{h}, u]_\mathfrak{h} - [\![Hx, Hy, [Hz,u]_\mathfrak{h}]\!]_\mathfrak{h}\Big) \\
& +(-1)^{|z|(|x|+|y|)+|H||x|}[Hz, [\![Hx,Hy,u]\!]_\mathfrak{h}]_\mathfrak{h} \\
\overset{(3)}{=} \quad & 0
\end{aligned}
$$

and

$$
\begin{aligned}
& \mu_H([x,y]_\mathfrak{g}, z)u - (-1)^{|y||z|}\mu_H(x,z)\rho_H(y)u + (-1)^{|x|(|y|+|z|)}\mu_H(y,z)\rho_H(x)u \\
= \quad & (-1)^{|u|(|x|+|y|+|z|)}[\![u, H[x,y]_\mathfrak{g}, Hz]\!]_\mathfrak{h} + \mu([x,y],z)u \\
& -(-1)^{|x|(|y|+|u|)+|z||u|}[\![[Hy,u]_\mathfrak{h}, Hx, Hz]\!]_\mathfrak{h} - (-1)^{|H|(|x|+|z|)+|y||z|}\mu(x,z)[Hy,u]_\mathfrak{h} \\
& -(-1)^{|u|(|x|+|z|)+|x||y|}[\![\rho(y)u, Hx, Hz]\!]_\mathfrak{h} - (-1)^{|y||z|}\mu(x,z)\rho(y)u \\
& +(-1)^{|u|(|y|+|z|)}[\![[Hx,u]_\mathfrak{h}, Hy, Hz]\!]_\mathfrak{h} + (-1)^{(|H|+|x|)(|y|+|z|)}\mu(y,z)[Hx,u]_\mathfrak{h} \\
& +(-1)^{|u|(|y|+|z|)+|H||y|}[\![\rho(x)u, Hy, Hz]\!]_\mathfrak{h} + (-1)^{|x|(|y|+|z|)}\mu(y,z)\rho(x)u
\end{aligned}
$$

$$\begin{aligned}
&\overset{(6)}{=} (-1)^{|u|(|H|+|x|+|y|+|z|)+|H||y|}[\![u,[Hx,Hy]_{\mathfrak{h}},Hz]\!]_{\mathfrak{h}}\\
&\quad -(-1)^{|x|(|H|+|y|+|u|)+|z||u|}[\![[Hy,u]_{\mathfrak{h}},Hx,Hz]\!]_{\mathfrak{h}}\\
&\quad +(-1)^{|u|(|y|+|z|)+|H||y|}[\![[Hx,u]_{\mathfrak{h}},Hy,Hz]\!]_{\mathfrak{h}}\\
&\overset{(2)}{=} 0.
\end{aligned}$$

The other equalities can be obtained similarly. We omit the details. □

Let $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},[\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}})$ and $(\mathfrak{h},[\cdot,\cdot]_{\mathfrak{h}},[\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}})$ be two Lie−Yamaguti superalgebras. Let (ρ,μ) be an action of $\mathfrak{g}$ on $\mathfrak{b}$. Define $\delta:\wedge^2\mathfrak{g}\longrightarrow \mathrm{Hom}(\mathfrak{g},\mathfrak{h})$ to be

$$\begin{aligned}
(\delta(x,y))z := \;& (-1)^{(|H|+|x|)(|y|+|z|)}\mu(y,z)(Hx)-(-1)^{(|H|)(|x|+|z|)+|y||z|}\mu(x,z)(Hy)\\
&+(-1)^{|H||y|}[\![Hx,Hy,Hz]\!]_{\mathfrak{h}}, \quad \forall x,y,z\in\mathfrak{g}.
\end{aligned} \tag{23}$$

Proposition 6. *With the notations above, $\delta(x,y)$ defined by (23) is a 1-cocycle of the Lie Yamaguti superalgebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},[\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}})$ with coefficients in the representation $(\mathfrak{h};\rho_H,\mu_H)$.*

Proof. For all $x_1,x_2,x_3\in\mathfrak{g}$, we have

$$\begin{aligned}
&\delta_{\mathrm{I}}(\delta(x,y))\,(x_1,x_2)\\
=\;& (-1)^{|x_1|(|x|+|y|)}\rho_H\,(x_1)\,\delta(x,y)x_2-(-1)^{|x_2|(|x|+|y|+|x_1|)}\rho_H\,(x_2)\,\delta(x,y)x_1\\
&-\delta(x,y)\left([x_1,x_2]_{\mathfrak{g}}\right)\\
=\;& (-1)^{(|H|+|x|)(|y|+|x_1|+|x_2|)+|y||x_1|}\rho_H\,(x_1)\,\mu\,(y,x_2)\,(Hx)\\
&+(-1)^{(|H|+|x|)(|y|+|x_1|+|x_2|)+|x_2|(|y|+|x_1|)}\rho_H\,(x_2)\,\mu\,(y,x_1)\,(Hx)\\
&-(-1)^{|H|(|x|+|x_1|+|x_2|)+|y|(|x_1|+|x_2|)+|x||x_1|}\rho_H\,(x_1)\,\mu\,(x,x_2)\,(Hy)\\
&+(-1)^{|H|(|x|+|x_1|+|x_2|)+|y|(|x_1|+|x_2|)+|x_2|(|x|+|x_1|)}\rho_H\,(x_2)\,\mu\,(x,x_1)\,(Hy)\\
&+(-1)^{|x_1|(|H|+|x|+|y|)+|H||y|}\rho_H\,(x_1)\,[\![Hx,Hy,Hx_2]\!]_{\mathfrak{h}}\\
&-(-1)^{|x_2|(|H|+|x|+|y|+|x_1|)+|H||y|}\rho_H\,(x_2)\,[\![Hx,Hy,Hx_1]\!]_{\mathfrak{h}}\\
&-(-1)^{(|H|+|x|)(|y|+|x_1|+|x_2|)}\mu\left(y,[x_1,x_2]_{\mathfrak{g}}\right)(Hx)+(-1)^{|H||y|}[\![Hx,Hy,H\,[x_1,x_2]_{\mathfrak{g}}]\!]_{\mathfrak{h}}\\
&+(-1)^{(|H|+|y|)(|x_1|+|x_2|)+|H||x|}\mu\left(x,[x_1,x_2]_{\mathfrak{g}}\right)(Hy)\\
=\;& 0
\end{aligned}$$

and

$$\begin{aligned}
&\delta_{\mathrm{II}}(\delta(x,y))\,(x_1,x_2,x_3)\\
=\;&(-1)^{(|x|+|y|)(|x_1|+|x_2|)}D_H\,(x_1,x_2)\,\delta(x,y)x_3-\delta(x,y)[\![x_1,x_2,x_3]\!]_{\mathfrak g}\\
&+(-1)^{(|x|+|y|+|x_1|)(|x_2|+|x_3|)}\mu_H\,(x_2,x_3)\,\delta(x,y)x_1\\
&-(-1)^{(|x|+|y|)(|x_1|+|x_3|)+|x_2||x_3|}\mu_H\,(x_1,x_3)\,\delta(x,y)x_2\\
=\;&(-1)^{(|H|+|x|)(|y|+|x_1|+|x_2|+|x_3|)+|y|(|x_1|+|x_2|)}D_H\,(x_1,x_2)\,\mu\,(y,x_3)\,(Hx)\\
&-(-1)^{(|H|+|y|)(|x_1|+|x_2|+|x_3|)+|x|(|H|+|x_1|+|x_2|)}D_H\,(x_1,x_2)\,\mu\,(x,x_3)\,(Hy)\\
&+(-1)^{(|x_1|+|x_2|)(|H|+|x|+|y|)+|H||y|}D_H\,(x_1,x_2)\,[\![Hx,Hy,Hx_3]\!]_{\mathfrak h}\\
&+(-1)^{(|x_2|+|x_3|)(|H|+|x|+|y|+|x_1|)+(|H|+|x|)(|y|+|x_1|)}\mu_H\,(x_2,x_3)\,\mu\,(y,x_1)\,(Hx)\\
&-(-1)^{(|x_2|+|x_3|)(|H|+|x|+|y|+|x_1|)+|H|(|x|+|x_1|)+|y||x_1|}\mu_H\,(x_2,x_3)\,\mu\,(x,x_1)\,(Hy)\\
&+(-1)^{(|x_2|+|x_3|)(|H|+|x|+|y|+|x_1|)+|H||y|}\mu_H\,(x_2,x_3)\,[\![Hx,Hy,Hx_1]\!]_{\mathfrak h}\\
&-(-1)^{(|H|+|x|)(|y|+|x_1|+|x_2|+|x_3|)+|y|(|x_1|+|x_3|)+|x_2||x_3|}\mu_H\,(x_1,x_3)\,\mu\,(y,x_2)\,(Hx)\\
&-(-1)^{(|H|+|y|)(|x_1|+|x_2|+|x_3|)+|x|(|H|+|x_1|+|x_3|)+|x_2||x_3|}\mu_H\,(x_1,x_3)\,\mu\,(x,x_2)\,(Hy)\\
&+(-1)^{(|x_1|+|x_3|)(|H|+|x|+|y|)+|H||y|+|x_2||x_3|}\mu_H\,(x_1,x_3)\,[\![Hx,Hy,Hx_2]\!]_{\mathfrak h}\\
&-(-1)^{(|H|+|x|)(|y|+|x_1|+|x_2|+|x_3|)}\mu\,(y,[\![x_1,x_2,x_3]\!]_{\mathfrak g})\,(Hx)\\
&+(-1)^{(|H|+|y|)(|x_1|+|x_2|+|x_3|)+|H||x|}\mu\,(x,[\![x_1,x_2,x_3]\!]_{\mathfrak g})\,(Hy)\\
&-(-1)^{|H||y|}[\![Hx,Hy,H[\![x_1,x_2,x_3]\!]_{\mathfrak g}]\!]_{\mathfrak h}\\
=\;&0,
\end{aligned}$$

which implies that $\delta(x,y)$ is a 1-cocycle. This finishes the proof. □

By now, we can establish the cohomology of crossed homomorphisms between Lie–Yamaguti superalgebras as follows. Let $H:\mathfrak g\longrightarrow\mathfrak h$ be a crossed homomorphism from a Lie–Yamaguti superalgebra $(\mathfrak g,[\cdot,\cdot]_{\mathfrak g},[\![\cdot,\cdot,\cdot]\!]_{\mathfrak g})$ to another Lie–Yamaguti superalgebra $(\mathfrak g,[\cdot,\cdot]_{\mathfrak h},[\![\cdot,\cdot,\cdot]\!]_{\mathfrak h})$ with respect to an action (ρ,μ). Define the set of n-cochains to be

$$\mathfrak C^p_H(\mathfrak g,\mathfrak h)=\begin{cases}C^p_{LieY}(\mathfrak g,\mathfrak h), & p\geq 1,\\ \wedge^2\mathfrak g, & p=0.\end{cases}$$

Define $\partial:\mathfrak C^p_H(\mathfrak g,\mathfrak h)\longrightarrow\mathfrak C^{p+1}_H(\mathfrak g,\mathfrak h)$ to be

$$\partial=\begin{cases}\delta^H, & p\geq 1,\\ \delta, & p=0,\end{cases}$$

where the map δ^H is the corresponding coboundary map given by (11)–(14) with coefficients in the representation $(\mathfrak h;\rho_H,\mu_H)$. Then combining with

Proposition 6, we obtain that $(\bigoplus_{n=0}^{\infty} \subseteq_H^n(\mathfrak{g},\mathfrak{h}),\partial)$ is a complex. Denote the set of n-cochains by $\mathcal{Z}_H^n(\mathfrak{g},\mathfrak{h})$ and denote the set of n-cobonudaries by $\mathcal{B}_H^n(\mathfrak{g},\mathfrak{h})$. The resulting n-th cohomology group is given by

$$\mathcal{H}_H^n(\mathfrak{g},\mathfrak{h}) := \mathcal{Z}_H^n(\mathfrak{g},\mathfrak{h})/\mathcal{B}_H^n(\mathfrak{g},\mathfrak{h}), n \geq 0.$$

Definition 9. The cohomology of the cochian complex $(\bigoplus_{n=0}^{\infty} \mathfrak{C}_H^n(\mathfrak{g},\mathfrak{h}),\partial)$ is called the *cohomology of the crossed homomorphism* H.

At the end of this section, we show that a certain homomorphism between two crossed homomorphisms induces a homomorphism between the corresponding cohomology groups. Let H and H' be two crossed homomorphisms from a Lie–Yamaguti superalgebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},[\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}})$ to another Lie–Yamaguti superalgebra $(\mathfrak{h},[\cdot,\cdot]_{\mathfrak{h}},[\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}})$ with respect to an action (ρ,μ). Let $(\psi_{\mathfrak{g}},\psi_{\mathfrak{h}})$ be an even homomorphism from H' to H, where $\psi_{\mathfrak{g}}$ is invertible. For $n \geqslant 2$, define a linear map $p : \mathfrak{C}_{H'}^n(\mathfrak{g},\mathfrak{h}) \longrightarrow \mathfrak{C}_H^n(\mathfrak{g},\mathfrak{h})$ to be

$$\begin{aligned} p_{\mathrm{I}}(f)\left(\mathfrak{X}_1,\cdots,\mathfrak{X}_n\right) &= \psi_{\mathfrak{h}}\left(f\left(\psi_{\mathfrak{g}}^{-1}\left(\mathfrak{X}_1\right),\cdots,\psi_{\mathfrak{g}}^{-1}\left(\mathfrak{X}_n\right)\right)\right), \\ p_{\Pi}(g)\left(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,x\right) &= \psi_{\mathfrak{h}}\left(g\left(\psi_{\mathfrak{g}}^{-1}\left(\mathfrak{X}_1\right),\cdots,\psi_{\mathfrak{g}}^{-1}\left(\mathfrak{X}_n\right),\psi_{\mathfrak{g}}^{-1}(x)\right)\right), \end{aligned}$$

for all $(f,g) \in \mathfrak{C}_{H'}^n(\mathfrak{g},\mathfrak{h})$. Here $\mathfrak{X}_k = x_k \wedge y_k \in \wedge^2\mathfrak{g}, k = 1,2,\cdots,n, x \in \mathfrak{g}$, and we use the notation $\psi_{\mathfrak{g}}^{-1}\left(\mathfrak{X}_k\right) = \psi_{\mathfrak{g}}^{-1}\left(x_k\right) \wedge \psi_{\mathfrak{g}}^{-1}\left(y_k\right), k = 1,2,\cdots,n$. In particular, for $n = 1, p : \mathfrak{C}_{H'}^1(\mathfrak{g},\mathfrak{h}) \longrightarrow \mathfrak{C}_H^1(\mathfrak{g},\mathfrak{h})$ is defined to be

$$p(f)(x) = \psi_{\mathfrak{h}}\left(f\left(\psi_{\mathfrak{g}}^{-1}(x)\right)\right), \quad \forall x \in \mathfrak{g}, f \in \mathfrak{C}_{H'}^1(\mathfrak{g},\mathfrak{h}).$$

Theorem 2. *With the notations above, p is a cochain map from a cochain $\left(\oplus_{n=1}^{\infty}\mathfrak{C}_{H'}^n(\mathfrak{g},\mathfrak{h}),\delta^{H'}\right)$ to $\left(\oplus_{n=1}^{\infty}\mathfrak{C}_H^n(\mathfrak{g},\mathfrak{h}),\delta^H\right)$. Consequently, p induces a homomorphism $p_* : \mathcal{H}_{H'}^n(\mathfrak{g},\mathfrak{h}) \longrightarrow \mathcal{H}_H^n(\mathfrak{g},\mathfrak{h})$ between cohomology groups.*

Proof. For all $(f,g) \in \mathfrak{C}_{H'}^n(\mathfrak{g},\mathfrak{h})$ $(n \geq 2)$, and for all $\mathfrak{X}_k = x_k \wedge y_k \in \wedge^2\mathfrak{g}, k = 1,2,\cdots,n, x \in \mathfrak{g}$, denoting $\varphi = \psi_{\mathfrak{g}}^{-1}$ we have

$$\begin{aligned} & \delta_{\Pi}^H\left(p_{\mathrm{I}}(f),p_{\Pi}(g)\right)\left(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,x\right) \\ = \; & (-1)^{(|y_n|+|x|)(|p_\Pi|+|g|+\varepsilon_{n-1}+|x_n|)+n-1}\mu_H\left(y_n,x\right)p_\Pi(g)\left(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},x_n\right) \\ & -(-1)^{|p_\Pi|+|g|+|y_n||x|+\varepsilon_{n-1}(|x_n|+|x|)+n-1}\mu_H\left(x_n,x\right)p_\Pi(g)\left(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},y_n\right) \\ & +\sum_{k=1}^n(-1)^{|\mathfrak{X}_k|(|p_\Pi|+|g|+\varepsilon_{k-1})+k+1}D_H\left(\mathfrak{X}_k\right)p_\Pi(g)\left(\mathfrak{X}_1,\cdots,\widehat{\mathfrak{X}_k},\cdots,\mathfrak{X}_n,x\right) \\ & +\sum_{k<l}^n(-1)^{\xi_{l-1}+k}p_\Pi(g)\left(\mathfrak{X}_1,\cdots,\widehat{\mathfrak{X}}_k,\cdots,\mathfrak{X}_k\circ\mathfrak{X}_l,\cdots,\mathfrak{X}_n,x\right) \\ & +\sum_{k=1}^n(-1)^{\xi_n+k}p_\Pi(g)\left(\mathfrak{X}_1,\cdots,\widehat{\mathfrak{X}}_k,\cdots,\mathfrak{X}_n,[\![\mathfrak{X}_k,x]\!]_{\mathfrak{g}}\right) \end{aligned}$$

$$\begin{aligned}
&= (-1)^{(|y_n|+|x|)(|g|+\varepsilon_{n-1}+|x_n|)+n-1}\mu_H(y_n,x)\psi_{\mathfrak{h}}\left(g\left(\varphi\left(\mathfrak{X}_1\right),\cdots,\varphi\left(\mathfrak{X}_{n-1}\right),\varphi\left(x_n\right)\right)\right)\\
&\quad -(-1)^{|g|+|y_n||x|+\varepsilon_{n-1}(|x_n|+|x|)+n-1}\mu_H\left(x_n,x\right)\psi_h\left(g\left(\varphi\left(\mathfrak{X}_1\right),\cdots,\varphi\left(\mathfrak{X}_{n-1}\right),\varphi\left(y_n\right)\right)\right)\\
&\quad +\sum_{k=1}^{n}(-1)^{|\mathfrak{X}_k|(|g|+\varepsilon_{k-1}+k+1}D_H\left(\mathfrak{X}_k\right)\psi_{\mathfrak{h}}\left(g\left(\varphi\left(\mathfrak{X}_1\right),\cdots,\widehat{\varphi\left(\mathfrak{X}_k\right)},\cdots,\varphi\left(\mathfrak{X}_n\right),\varphi(x)\right)\right.\\
&\quad +\sum_{k<l}(-1)^{\xi_{l-1}+k}\psi_{\mathfrak{h}}\left(g\left(\varphi\left(\mathfrak{X}_1\right),\cdots,\widehat{\varphi\left(\mathfrak{X}_k\right)},\cdots,\varphi\left(\mathfrak{X}_k\right)\circ\varphi\left(\mathfrak{X}_l\right),\cdots,\varphi\left(\mathfrak{X}_n\right),\varphi(x)\right)\right.\\
&\quad +\sum_{k=1}^{n}(-1)^{\xi_n+k}\psi_{\mathfrak{h}}\left(g\left(\varphi\left(\mathfrak{X}_1\right),\cdots,\widehat{\varphi\left(\mathfrak{X}_k\right)},\cdots,\varphi\left(\mathfrak{X}_n\right),\varphi[\![\mathfrak{X}_k,x]\!]_{\mathfrak{g}}\right)\right)\\
&= \psi_{\mathfrak{h}}\left(\delta_{\mathrm{II}}^{H'}(f,g)\left(\varphi\left(\mathfrak{X}_1\right),\cdots,\varphi\left(\mathfrak{X}_n\right),\varphi(x)\right)\right)\\
&= p_{\mathrm{II}}\left(\delta_{\mathrm{II}}^{H'}(f,g)\right)\left(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,x\right).
\end{aligned}$$

where $\varepsilon_n=|\mathfrak{X}_1|+\cdots+|\mathfrak{X}_n|$ and $\xi_n=|\mathfrak{X}_k|(|\mathfrak{X}_{k+1}|+\cdots+|\mathfrak{X}_n|)$. Note that the third equality holds, since for all $x,y_n\in\mathfrak{g},u\in\mathfrak{h}$, by using that $(\psi_{\mathfrak{g}},\psi_{\mathfrak{h}})$ is a homomorphism from crossed homomorphism H' to crossed homomorphism H, we have

$$\begin{aligned}
&\mu_H\left(y_n,x\right)\psi_{\mathfrak{h}}(u)\\
&= (-1)^{|u|(|y_n|+|x|)+|y_n||H|}[\![\psi_{\mathfrak{h}}(u),Hy_n,Hx]\!]_{\mathfrak{h}}+\mu\left(y_n,x\right)\psi_{\mathfrak{h}}(u)\\
&= (-1)^{|u|(|y_n|+|x|)+|y_n||H'|}[\![\psi_{\mathfrak{h}}(u),\psi_{\mathfrak{h}}\circ H'\circ\varphi\left(y_n\right),\psi_{\mathfrak{h}}\circ H'\circ\varphi(x)]\!]_{\mathfrak{h}}\\
&\quad +\psi_{\mathfrak{h}}\mu\left(\varphi\left(y_n\right),\varphi(x)\right)u\\
&= (-1)^{|u|(|y_n|+|x|)+|y_n||H'|}\psi_{\mathfrak{h}}\left([\![u,H'\circ\varphi\left(y_n\right),H'\circ\varphi(x)]\!]_{\mathfrak{h}}+\mu\left(\varphi\left(y_n\right),\varphi(x)\right)u\right)\\
&= \psi_{\mathfrak{h}}\left(\mu_{H'}\left(\varphi\left(y_n\right),\varphi(x)\right)\right).
\end{aligned}$$

Thus we obtain that $p_{\mathrm{II}}\left(\delta_{\mathrm{II}}^{H'}(f,g)\right)=\delta_{\mathrm{II}}^{H}\left(p_{\mathrm{I}}(f),p_{\mathrm{II}}(g)\right)$ for all $(f,g)\in\mathfrak{C}^n_{H'}(\mathfrak{g},\mathfrak{h})$ $(n\geq 2)$. Similarly, we can show that $p_{\mathrm{I}}\left(\delta_{\mathrm{I}}^{H'}(f,g)\right)=\delta_{\mathrm{I}}^{H}(p_{\mathrm{I}}(f),p_{\mathrm{II}}(g))$ for all $(f,g)\in\mathfrak{C}^n_{H'}(\mathfrak{g},\mathfrak{h})$ $(n\geq 2)$. And moreover, it is easy to see that the case of $n=1$ is still valid. This finishes the proof. □

Example 5. The cohomology of a crossed homomorphism $H:\mathfrak{g}\to\mathfrak{h}$ is defined using the cochain complex $(\bigoplus_{n=0}^{\infty}\mathfrak{C}^n_H(\mathfrak{g},\mathfrak{h}),\partial)$, where
- for $n\geq 1$, $\mathfrak{C}^n_H(\mathfrak{g},\mathfrak{h})=C^n_{\mathrm{LieY}}(\mathfrak{g},\mathfrak{h})$;
- for $n=0$, $\mathfrak{C}^0_H(\mathfrak{g},\mathfrak{h})=\wedge^2\mathfrak{g}$.

The coboundary map ∂ is given by

$$\partial=\begin{cases}\delta^H & \text{if } n\geq 1,\\ \delta & \text{if } n=0,\end{cases}$$

where δ^H is the coboundary map for the representation $(\mathfrak{h};\rho_H,\mu_H)$ induced by H.

5. Deformations of crossed homomorphisms between Lie−Yamaguti superalgebras

In this section, we use the cohomology theory established in the former section to characterize deformations of crossed homomorphisms between Lie−Yamaguti superalgebras.

5.1. Linear deformations of crossed homomorphisms between Lie–Yamaguti superalgebras. In this subsection, we use the cohomology constructed in the former section to characterize the linear deformations of crossed homomorphisms between Lie–Yamaguti superalgebras.

Definition 10. Let $H : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a crossed homomorphism from a Lie–Yamaguti superalgebra $\left(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}}\right)$ to another Lie–Yamaguti superalgebra $\left(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}}\right)$ with respect to an action $(\mathfrak{h};\rho,\mu)$. Let $\mathfrak{H} : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a linear map. If $H_t := H + t\mathfrak{H}$ is still a crossed homomorphism for all t, then we say that $\mathfrak{Y}$ *generates* a linear deformation of the crossed homomorphism H.

It is easy to see that if $\mathfrak{H}$ generates a linear deformation of the crossed homomorphism H, then for all $x, y, z \in \mathfrak{g}$, the equalities

$$\begin{aligned}
\mathfrak{H}[x,y]_{\mathfrak{g}} = \quad & (-1)^{|\mathfrak{H}||x|}\rho(x)(\mathfrak{H}y) - (-1)^{|y|(|\mathfrak{H}|+|x|)}\rho(y)(\mathfrak{H}x) + (-1)^{|\mathfrak{H}||x|}[Hx,\mathfrak{H}y]_{\mathfrak{h}} \\
& +(-1)^{|H|(|\mathfrak{H}|+|x|)}[\mathfrak{H}x, Hy]_{\mathfrak{h}}, \\
\mathfrak{H}[\![x,y,z]\!]_{\mathfrak{g}} = \quad & (-1)^{|\mathfrak{H}|(|x|+|y|)}D(x,y)(\mathfrak{H}z) + (-1)^{(|\mathfrak{H}|+|x|)(|y|+|z|)}\mu(y,z)(\mathfrak{H}x) \\
& -(-1)^{|z|(|\mathfrak{H}|+|y|)+|\mathfrak{H}||x|}\mu(x,z)(\mathfrak{H}y) + (-1)^{|H||y|}[\![\mathfrak{H}x, Hy, Hz]\!]_{\mathfrak{h}} \\
& +(-1)^{|x|(|\mathfrak{H}|+|H|)+|H||y|}[\![Hx,\mathfrak{H}y,Hz]\!]_{\mathfrak{h}} \\
& +(-1)^{|x|(|\mathfrak{H}|+|H|)+|\mathfrak{H}||y|}[\![Hx,Hy,\mathfrak{H}z]\!]_{\mathfrak{h}}
\end{aligned}$$

hold, which means that $\mathfrak{H}$ is a 2-cocycle of the crossed homomorphism H.

Definition 11. Let $H : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a crossed homomorphism from a Lie–Yamaguti superalgebra $\left(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}}\right)$ to another Lie–Yamaguti superalgebra $\left(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}}\right)$ with respect to an action (ρ,μ).

(i) Two linear deformations $H_t^1 = H + t\mathfrak{H}_1$ and $H_t^2 = H + t\mathfrak{H}_2$ are called *equivalent* if there exists an element $\mathfrak{X} \in \wedge^2\mathfrak{g}$ such that $(\mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}(\mathfrak{X}), \mathrm{Id}_{\mathfrak{h}} + tD(\mathfrak{X}))$ is a homomorphism from H_t^2 to H_t^1.

(ii) A linear deformation $H_t = H + t\mathfrak{H}$ of the crossed homomorphism H is called *trivial* if it is equivalent to H.

If two linear deformations H_t^2 and H_t^1 are equivalent, then $\mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}(\mathfrak{X})$ is a homomorphism on the Lie–Yamaguti superalgebra $\mathfrak{g}$, which implies that

$$[[\![\mathfrak{X},y]\!]_{\mathfrak{g}}, [\![\mathfrak{X},z]\!]_{\mathfrak{g}}]_{\mathfrak{g}} = 0, \tag{24}$$

$$\begin{aligned}
&[\![[\![\mathfrak{X},y]\!]_{\mathfrak{g}}, [\![\mathfrak{X},z]\!]_{\mathfrak{g}}, t]\!]_{\mathfrak{g}} + (-1)^{|\mathfrak{X}||z|}[\![[\![\mathfrak{X},y]\!]_{\mathfrak{g}}, z, [\![\mathfrak{X},t]\!]_{\mathfrak{g}}]\!]_{\mathfrak{g}} \\
&\qquad + (-1)^{|\mathfrak{X}|(|y|+|z|)}[\![y, [\![\mathfrak{X},z]\!]_{\mathfrak{g}}, [\![\mathfrak{X},t]\!]_{\mathfrak{g}}]\!]_{\mathfrak{g}} = 0,
\end{aligned} \tag{25}$$

$$[\![[\![\mathfrak{X},y]\!]_{\mathfrak{g}}, [\![\mathfrak{X},z]\!]_{\mathfrak{g}}, [\![\mathfrak{X},t]\!]_{\mathfrak{g}}]\!]_{\mathfrak{g}} = 0 \tag{26}$$

for all $y, z, t \in \mathfrak{g}$.

Note that by equalities (18) and (19), we obtain that, for all $y, z \in \mathfrak{g}$,

$$\rho\left([\![\mathfrak{X},y]\!]_{\mathfrak{g}}\right) D(\mathfrak{X}) = 0, \tag{27}$$

$$(-1)^{|\mathfrak{X}||z|}\mu\left(\llbracket\mathfrak{X},y\rrbracket_{\mathfrak{g}},z\right)D(\mathfrak{X})+(-1)^{|\mathfrak{X}||y|}\mu\left(y,\llbracket\mathfrak{X},z\rrbracket_{\mathfrak{g}}\right)D(\mathfrak{X})+\mu\left(\llbracket\mathfrak{X},y\rrbracket_{\mathfrak{g}},\llbracket\mathfrak{X},z\rrbracket_{\mathfrak{g}}\right)=0 \quad (28)$$

$$\mu\left(\llbracket\mathfrak{X},y\rrbracket_{\mathfrak{g}},\llbracket\mathfrak{X},z\rrbracket_{\mathfrak{g}}\right)D(\mathfrak{X})=0. \quad (29)$$

Moreover, equality (17) yields that, for all $x,y,z\in\mathfrak{g}$,

$$(\mathrm{Id}_{\mathfrak{h}}+tD(x,y))\,(H+t\mathfrak{H}_2)\,z=(-1)^{(|H|+|\mathfrak{H}_1|)(|x|+|y|)}\,(H+t\mathfrak{H}_1)\,(\mathrm{Id}_{\mathfrak{g}}+t\mathfrak{L}(x,y))\,z,$$

which implies that

$$\begin{aligned}\mathfrak{H}_2 z-\mathfrak{H}_1 z=&(-1)^{(|H|+|x|)(|y|+|z|)}\mu(y,z)(Hx)-(-1)^{(|H|+|y|)(|x|+|z|)}\mu(x,z)(Hy)\\&+(-1)^{|H||y|}\llbracket Hx,Hy,Hz\rrbracket_{\mathfrak{h}}\\=&\ \partial(\mathfrak{X})z.\end{aligned} \quad (30)$$

Thus, by (30), we have the following theorem.

Theorem 3. *Let $H:\mathfrak{g}\longrightarrow\mathfrak{h}$ be a crossed homomorphism from a Lie–Yamaguti superalgebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\llbracket\cdot,\cdot,\cdot\rrbracket_{\mathfrak{g}})$ to another Lie–Yamaguti superalgebra $(\mathfrak{h},[\cdot,\cdot]_{\mathfrak{h}},\llbracket\cdot,\cdot,\cdot\rrbracket_{\mathfrak{h}})$ with respect to an action $(\mathfrak{h},\rho,\mu)$. If two linear deformations $H^1_t=H+t\mathfrak{H}_1$ and $H^2_t=H+t\mathfrak{H}_2$ are equivalent, then $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are in the same cohomology class in $\mathcal{H}^2_H(\mathfrak{g},\mathfrak{h})$.*

Definition 12. Let $H:\mathfrak{g}\longrightarrow\mathfrak{h}$ be a crossed homomorphism from a Lie–Yamaguti superalgebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\llbracket\cdot,\cdot,\cdot\rrbracket_{\mathfrak{g}})$ to another Lie–Yamaguti superalgebra $(\mathfrak{h},[\cdot,\cdot]_{\mathfrak{h}},\llbracket\cdot,\cdot,\cdot\rrbracket_{\mathfrak{h}})$ with respect to an action (ρ,μ). If an element $\mathfrak{X}\in\wedge^2\mathfrak{g}$ satisfies (24)–(29) and the following equality

$$D(\mathfrak{X})\left(D(\mathfrak{X})(Hy)-(-1)^{|H||\mathfrak{X}|}H[\mathfrak{X},y]_{\mathfrak{g}}\right)=0,\quad\forall y\in\mathfrak{g},$$

then $\mathfrak{X}$ is called a *Nijenhuis element* associated to H. Denote by $\mathrm{Nij}(H)$ the set of Nijenhuis elements associated to H.

It is easy to see that a trivial deformation of a crossed homomorphism between Lie–Yamaguti superalgebras gives rise to a Nijenhuis element. However, the converse is also true.

Theorem 4. *Let $H:\mathfrak{g}\longrightarrow\mathfrak{h}$ be a crossed homomorphism from a Lie–Yamaguti superalgebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\llbracket\cdot,\cdot,\cdot\rrbracket_{\mathfrak{g}})$ to another Lie–Yamaguti superalgebra $(\mathfrak{h},[\cdot,\cdot]_{\mathfrak{h}},\llbracket\cdot,\cdot,\cdot\rrbracket_{\mathfrak{h}})$ with respect to an action (ρ,μ). Then, for any $\mathfrak{X}\in\mathrm{Nij}(H)$, $H_t:=H+t\mathfrak{H}$ with $\mathfrak{H}=\delta^H(\mathfrak{X})$ is a linear deformation of H. Moreover, this deformation is trivial.*

We need the following lemma to prove the above theorem.

Lemma 3. *Let $H:\mathfrak{g}\longrightarrow\mathfrak{h}$ be a crossed homomorphism from $\mathfrak{g}$ to $\mathfrak{h}$ with respect to an action (ρ,μ). Let $\psi_{\mathfrak{g}}:\mathfrak{g}\longrightarrow\mathfrak{g}$ and $\psi_{\mathfrak{h}}:\mathfrak{h}\longrightarrow\mathfrak{h}$ be two Lie–Yamaguti superalgebra homomorphisms such that (18) and (19) hold.*

Then the linear map $\psi_{\mathfrak{h}}^{-1}\circ H\circ\psi_{\mathfrak{g}}$ is a crossed homomorphism from $\mathfrak{g}$ to $\mathfrak{h}$ with respect to the action (ρ,μ).

Proof. For all $x,y,z\in\mathfrak{g}$, we have

$$\begin{aligned}\left(\psi_{\mathfrak{h}}^{-1}\circ H\circ\psi_{\mathfrak{g}}\right)([x,y]_{\mathfrak{g}}) = \psi_{\mathfrak{h}}^{-1}\Big(&(-1)^{|\psi_{\mathfrak{g}}|(|x|+|H|)+|x||H|}\rho\left(\psi_{\mathfrak{g}}(x)\right)H\left(\psi_{\mathfrak{g}}(y)\right)\\
&-(-1)^{|y|(|x|+|H|+|\psi_{\mathfrak{g}}|)+|H||\psi_{\mathfrak{g}}|}\rho\left(\psi_{\mathfrak{g}}(y)\right)H\left(\psi_{\mathfrak{g}}(x)\right)\\
&+(-1)^{|\psi_{\mathfrak{g}}|(|x|+|H|)+|x||H|}\left[H\circ\psi_{\mathfrak{g}}(x),H\circ\psi_{\mathfrak{g}}(y)\right]_{\mathfrak{h}}\Big)\\
=&(-1)^{|x|(|\psi_{\mathfrak{h}}^{-1}|+|\psi_{\mathfrak{g}}|+|H|)}\rho(x)\left(\psi_{\mathfrak{h}}^{-1}\circ H\circ\psi_{\mathfrak{g}}(y)\right)\\
&-(-1)^{|y|(|x|+|\psi_{\mathfrak{h}}^{-1}|+|\psi_{\mathfrak{g}}|+|H|)}\rho(y)\left(\psi_{\mathfrak{h}}^{-1}\circ H\circ\psi_{\mathfrak{g}}(x)\right)\\
&+(-1)^{|x|(|\psi_{\mathfrak{h}}^{-1}|+|\psi_{\mathfrak{g}}|+|H|)}\left[\psi_{\mathfrak{h}}^{-1}\circ H\circ\psi_{\mathfrak{g}}(x),\psi_{\mathfrak{h}}^{-1}\circ H\circ\psi_{\mathfrak{g}}(y)\right]_{\mathfrak{h}}\end{aligned}$$

and

$$\begin{aligned}&\left(\psi_{\mathfrak{h}}^{-1}\circ H\circ\psi_{\mathfrak{g}}\right)([\![x,y,z]\!]_{\mathfrak{g}})\\
=&\psi_{\mathfrak{h}}^{-1}\Big((-1)^{|y|(|\psi_{\mathfrak{g}}|+|H|)+|x||H|}D\left(\psi_{\mathfrak{g}}(x),\psi_{\mathfrak{g}}(y)\right)H\left(\psi_{\mathfrak{g}}(z)\right)\\
&+(-1)^{(|y|+|z|)(|H|+|x|)+|\psi_{\mathfrak{g}}||z|}\mu\left(\psi_{\mathfrak{g}}(y),\psi_{\mathfrak{g}}(z)\right)H\left(\psi_{\mathfrak{g}}(x)\right)\\
&-(-1)^{|z|(|\psi_{\mathfrak{g}}|+|H|+|y|)+|x||H|}\mu\left(\psi_{\mathfrak{g}}(x),\psi_{\mathfrak{g}}(z)\right)H\left(\psi_{\mathfrak{g}}(y)\right)\\
&+(-1)^{|y|(|\psi_{\mathfrak{g}}|+|H|)}[\![H\circ\psi_{\mathfrak{g}}(x),H\circ\psi_{\mathfrak{g}}(y),H\circ\psi_{\mathfrak{g}}(z)]\!]_{\mathfrak{h}}\Big)\\
=&(-1)^{(|x|+|y|)(|\psi_{\mathfrak{h}}^{-1}|+|\psi_{\mathfrak{g}}|+|H|)}D(x,y)\left(\psi_{\mathfrak{h}}^{-1}\circ H\circ\psi_{\mathfrak{g}}(z)\right)\\
&+(-1)^{(|y|+|z|)(|x|+|\psi_{\mathfrak{h}}^{-1}|+|\psi_{\mathfrak{g}}|+|H|)}\mu(y,z)\left(\psi_{\mathfrak{h}}^{-1}\circ H\circ\psi_{\mathfrak{g}}(x)\right)\\
&-(-1)^{(|x|+|z|)(|\psi_{\mathfrak{h}}^{-1}|+|\psi_{\mathfrak{g}}|+|H|)+|y||z|}\mu(x,z)\left(\psi_{\mathfrak{h}}^{-1}\circ H\circ\psi_{\mathfrak{g}}(y)\right)\\
&+(-1)^{|y|(|\psi_{\mathfrak{h}}^{-1}|+|\psi_{\mathfrak{g}}|+|H|)}[\![\psi_{\mathfrak{h}}^{-1}\circ H\circ\psi_{\mathfrak{g}}(x),\psi_{\mathfrak{h}}^{-1}\circ H\circ\psi_{\mathfrak{g}}(y),\psi_{\mathfrak{h}}^{-1}\circ H\circ\psi_{\mathfrak{g}}(z)]\!]_{\mathfrak{h}},\end{aligned}$$

which implies that the linear map $\psi_{\mathfrak{h}}^{-1}\circ H\circ\psi_{\mathfrak{g}}$ is a crossed homomorphism from $\mathfrak{g}$ to $\mathfrak{h}$ with respect to the action (ρ,μ). □

The proof of Theorem 4: For any Nijenhuis element $\mathfrak{X}\in\mathrm{Nij}(H)$, we define

$$\mathfrak{H}:=\delta(\mathfrak{X}).$$

By Definition 12, for any $t, H_t = H + t\mathfrak{g}$ satisfies that

$$H \circ (\mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}(\mathfrak{X})) = (\mathrm{Id}_{\mathfrak{h}} + tD(\mathfrak{X})) \circ H_t,$$

$$(\mathrm{Id}_{\mathfrak{h}} + tD(\mathfrak{X})) \circ \rho(x) = (-1)^{|x|(|\mathfrak{X}|+|D|)} \rho\left(\mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}(\mathfrak{X})(x)\right) \circ \left(\mathrm{Id}_{\mathfrak{h}} + tD(\mathfrak{X})\right),$$

$$\begin{aligned}&(\mathrm{Id}_{\mathfrak{h}} + tD(\mathfrak{X})) \circ \mu(y,z)\\ &= (-1)^{|D|(|y|+|z|)+|z||\mathfrak{X}|} \rho\left(\mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}(\mathfrak{X})(y), \mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}(\mathfrak{X})(z)\right) \circ \left(\mathrm{Id}_{\mathfrak{h}} + tD(\mathfrak{X})\right),\end{aligned}$$

for all $x, y, z \in \mathfrak{g}$. For t sufficiently small, we see that $\mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}(\mathfrak{X})$ and $\mathrm{Id}_{\mathfrak{h}} + tD(\mathfrak{X})$ are Lie–Yamaguti superalgebra homomorphisms. Thus, we have

$$H_t = (\mathrm{Id}_{\mathfrak{H}} + tD(\mathfrak{X}))^{-1} \circ H \circ (\mathrm{Id}_{\mathfrak{g}} + t\mathfrak{L}(\mathfrak{X})).$$

By Lemma 3, we deduce that H_t is a crossed homomorphism from $\mathfrak{g}$ to $\mathfrak{h}$ for t sufficiently small. Thus $\mathfrak{H} = \delta(\mathfrak{X})$ generates a linear deformation of H. It is easy to see that this deformation is trivial. This completes the proof.

5.2. Formal deformations of crossed homomorphisms between Lie–Yamaguti superalgebras. In this subsection, we study formal deformations of crossed homomorphisms between Lie– Yamaguti superalgebras. Let $\mathbb{K}[[t]]$ be a ring of power series of one variable t. If $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}})$ is a LieYamaguti superalgebra, then there is a Lie–Yamaguti superalgebra structure over the ring $\mathbb{K}[[t]]$ on $\mathfrak{g}[[t]]$ given by

$$\left[\sum_{i=0}^{\infty} x_i t^i, \sum_{j=0}^{\infty} y_j t^j\right] = \sum_{s=0}^{\infty} \sum_{i+j=s} [x_i, y_j]\, t^s,$$

$$[\![\sum_{i=0}^{\infty} x_i t^i, \sum_{j=0}^{\infty} y_j t^j, \sum_{k=0}^{\infty} z_k t^k]\!] = \sum_{s=0}^{\infty} \sum_{i+j+k=s} [\![x_i, y_j, z_k]\!] t^s, \quad \forall x_i, y_j, z_k \in \mathfrak{g}.$$

For any action (ρ, μ) of a Lie–Yamaguti superalgebra $\left(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}}\right)$ on another Lie–Yamaguti superalgebra $\left(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}}\right)$, there is a natural action of the Lie–Yamaguti superalgebra $\mathfrak{g}[[t]]$ on the $\mathbb{K}[[t]]$– Lie–Yamaguti superalgebra $\mathfrak{h}[[t]]$ given by

$$\rho\left(\sum_{i=0}^{\infty} x_i t^i\right)\left(\sum_{k=0}^{\infty} v_k t^k\right) = \sum_{s=0}^{\infty} \sum_{i+k=s} \rho(x_i)\, v_k t^s$$

$$\mu\left(\sum_{i=0}^{\infty} x_i t^i, \sum_{j=0}^{\infty} y_j t^j\right)\left(\sum_{k=0}^{\infty} v_k t^k\right) = \sum_{s=0}^{\infty} \sum_{i+j+k=s} \mu(x_i, y_j)\, v_k t^s, \ \forall x_i, y_j \in \mathfrak{g}, v_k \in V.$$

Let $H : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a crossed homomorphism from a Lie–Yamaguti superalgebra $\left(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}}\right)$ to anther Lie–Yamaguti superalgebra $\left(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}}\right)$

with respect to an action (ρ, μ). Consider the power series

$$H_t = \sum_{i=0}^{\infty} \mathfrak{H}_i t^i, \quad \mathfrak{H}_i \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{h}), \tag{31}$$

that is, $H_t \in \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g}, \mathfrak{h})[[t]] = \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g}, \mathfrak{h}[[t]])$.

Definition 13. Let $H : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a crossed homomorphism from a Lie–Yamaguti superalgebra $\left(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}}\right)$ to anther Lie–Yamaguti superalgebra $\left(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}}\right)$ with respect to an action (ρ, μ). Suppose that H_t is given by (31) with $\mathfrak{H}_0 = H$, and H_t also satisfies

$$H_t[x,y]_{\mathfrak{g}} = (-1)^{|x||H_t|}\rho(x)\,(H_t y) - (-1)^{|y|(|x|+|H_t|)}\rho(y)\,(H_t x) + (-1)^{|x||H_t|}\,[H_t x, H_t y]_{\mathfrak{h}}\,, \tag{32}$$

$$\begin{aligned} H_t[\![x,y,z]\!]_{\mathfrak{g}} &= (-1)^{|H_t|(|x|+|y|)} D(x,y)\,(H_t z) + (-1)^{(|H_t|+|x|)(|y|+|z|)}\mu(y,z)\,(H_t x) \\ &- (-1)^{|z|(|H_t|+|y|)+|x||H_t|}\mu(x,z)\,(H_t y) + (-1)^{|H_t||y|}[\![H_t x, H_t y, H_t z]\!]_{\mathfrak{h}} \end{aligned} \tag{33}$$

for all $x, y, z \in \mathfrak{g}$. We say that H_t is a *formal deformation* of H.

Substituting (31) into (32) and (33) and comparing the coefficients of t^s for all $s \geqslant 0$, we have, for all $x, y, z \in \mathfrak{g}$,

$$\begin{aligned} \sum_{\substack{i+j=s\\ i,j\geq 0}} &\Big((-1)^{|x||\mathfrak{H}_s|}\rho(x)\,(\mathfrak{H}_s y) - (-1)^{|y|(|x|+|\mathfrak{H}_s|)}\mu(y)\,(\mathfrak{H}_s x) \\ &+(-1)^{|x||\mathfrak{H}_j|}[\![\mathfrak{H}_i x, \mathfrak{H}_j y]\!]_{\mathfrak{h}} - \mathfrak{H}_s[\![x,y]\!]_{\mathfrak{g}}\Big)\, t^s = 0 \end{aligned} \tag{34}$$

$$\begin{aligned} \textstyle\sum_{\substack{i+j+k=s\\ i,j,k\geq 0}} \quad &((-1)^{(|x|+|y|)|\mathfrak{H}_s|}D(x,y)\,(\mathfrak{H}_s z) + (-1)^{(|y|+|z|)(|\mathfrak{H}_s|+|x|)}\mu(y,z)\,(\mathfrak{H}_s x) \\ &-(-1)^{|z|(|\mathfrak{H}_s|+|y|)+|x||\mathfrak{H}_s|}\mu(x,z)\,(\mathfrak{H}_s y)) \\ &+(-1)^{|y||\mathfrak{H}_k|}[\![\mathfrak{H}_i x, \mathfrak{H}_j y, \mathfrak{H}_k z]\!]_{\mathfrak{h}} - \mathfrak{H}_s[\![x,y,z]\!]_{\mathfrak{g}})\, t^s = 0 \end{aligned} \tag{35}$$

Proposition 7. *Let $H : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a crossed homomorphism from a Lie–Yamaguti superalgebra $\left(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}}\right)$ to another Lie–Yamaguti superalgebra $\left(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}}\right)$ with respect to an action $(\mathfrak{h}; \rho, \mu)$. If $H_t = \sum_{i=0}^{\infty} \mathfrak{H}_i t^i$ is a formal deformation of H, then $\delta^H \mathfrak{H}_1 = 0$, i.e., $\mathfrak{H}_1 \in \mathfrak{C}^1_H(\mathfrak{g}, \mathfrak{h})$ is a 1-cocycle of H.*

Proof. When $s=1$, the equalities (34) and (35) are equivalent to

$$\begin{aligned}\mathfrak{H}_1[x,y]_{\mathfrak{g}} =&(-1)^{|x||\mathfrak{H}_1|}\rho(x)\left(\mathfrak{H}_1 y\right)-(-1)^{|y|(|x|+|\mathfrak{H}_1|)}\rho(y)\left(\mathfrak{H}_1 x\right)\\ &+(-1)^{|x||\mathfrak{H}_1|}\left[Hx,\mathfrak{H}_1 y\right]_{\mathfrak{h}}+\left[\mathfrak{H}_1 x,Hy\right]_{\mathfrak{h}},\\ \mathfrak{H}_1[\![x,y,z]\!]_{\mathfrak{g}} =&(-1)^{\mathfrak{H}_1(|x|+|y|)}D(x,y)\left(\mathfrak{H}_1 z\right)+(-1)^{(|y|+|z|)(|\mathfrak{H}_1|+|x|)}\mu(y,z)\left(\mathfrak{H}_1 x\right)\\ &-(-1)^{|z|(|H_1|+|y|)+|x||\mathfrak{H}_1|}\mu(x,z)\left(\mathfrak{H}_1 y\right)+[\![\mathfrak{H}_1 x,Hy,Hz]\!]_{\mathfrak{h}}\\ &+(-1)^{|\mathfrak{H}_1||x|}[\![Hx,\mathfrak{H}_1 y,Hz]\!]_{\mathfrak{h}}+(-1)^{|y|(|\mathfrak{H}_1|+|x|)}[\![Hx,Hy,\mathfrak{H}_1 z]\!]_{\mathfrak{h}},\\ &\forall x,y,z\in\mathfrak{g}.\end{aligned}$$

which implies that $\delta^H\left(\mathfrak{H}_1\right)=0$, i.e., $\mathfrak{H}_1$ is a 1-cocycle of δ^H. □

Definition 14. Let $H:\mathfrak{g}\longrightarrow\mathfrak{h}$ be a crossed homomorphism from a Lie–Yamaguti superalgebra $\left(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},[\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}}\right)$. to another Lie–Yamaguti superalgebra $\left(\mathfrak{h},[\cdot,\cdot]_{\mathfrak{h}},[\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}}\right)$ with respect to an action $(\mathrm{h};\rho,\mu)$. The 1-cocycle $\mathfrak{H}_1$ is called the *infinitesimal* of the formal deformation $H_t=\sum_{i=0}^{\infty}\mathfrak{H}_i t^i$ of H.

In the following section, we will introduce the idea of comparable formal deformations of crossed homomorphisms between Lie–Yamaguti superalgebras.

Definition 15. Let $H:\mathfrak{g}\longrightarrow\mathfrak{h}$ be a crossed homomorphism from a Lie–Yamaguti superalgebra $\left(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},[\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}}\right)$ to another Lie–Yamaguti superalgebra $\left(\mathfrak{h},[\cdot,\cdot]_{\mathfrak{h}},[\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}}\right)$ with respect to an action (ρ,μ). Two formal deformations $\bar{H}_t=\sum_{i=0}^{\infty}\overline{\mathfrak{H}}_i i^i$ and $H_t=\sum_{i=0}^{\infty}\mathfrak{H}_i t^i$, where $\overline{\mathfrak{H}}_0=\mathfrak{H}_0=H$, are said to be *equivalent* if there exist $\mathfrak{X}\in\wedge^2\mathfrak{g},\phi_i\in\mathfrak{gl}(\mathfrak{g})$ and $\varphi_i\in\mathfrak{gl}(\mathfrak{h}), i\geqslant 2$, such that for

$$\phi_t=\mathrm{Id}_{\mathfrak{g}}+t\mathfrak{L}(\mathfrak{X})+\sum_{i=2}^{\infty}\phi_i t^i,\quad \varphi_t=\mathrm{Id}_{\mathfrak{h}}+tD(\mathfrak{X})+\sum_{i=2}^{\infty}\varphi_i t^i, \tag{36}$$

the following conditions are satisfied:

$$[\phi_t(x),\phi_t(y)]_{\mathfrak{g}}=\phi_t[x,y]_{\mathfrak{g}},\quad [\![\phi_t(x),\phi_t(y),\phi_t(z)]\!]_{\mathfrak{g}}=\phi_t[\![x,y,z]\!]_{\mathfrak{g}},\quad \forall x,y,z\in\mathfrak{g},$$

$$[\varphi_t(u),\varphi_t(v)]_{\mathfrak{h}}=\varphi_t[u,v]_{\mathfrak{h}},\ [\![\varphi_t(u),\varphi_t(v),\varphi_t(w)]\!]_{\mathfrak{h}}=\varphi_t[\![u,v,w]\!]_{\mathfrak{h}},\ \forall u,v,w\in\mathfrak{h},$$

$$\varphi_t\rho(x)u=\rho\left(\phi_t(x)\right)(\varphi_t(u)),\ \varphi_t\mu(x,y)u=\mu\left(\phi_t(x),\phi_t(y)\right)(\varphi_t(u)),\ \forall x,y\in\mathfrak{g},u\in\mathfrak{h}$$

and

$$H_t\circ\varphi_t=\phi_t\circ\bar{H}_t \tag{37}$$

as $\mathbb{K}[[t]]$-module maps.

The following theorem is the second key conclusion in this section.

Theorem 5. *Let $H : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a crossed homomorphism from a Lie–Yamaguti superalgebra $\left(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}}\right)$ to another Lie–Yamaguti superalgebra $\left(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}}\right)$ with respect to an action $(\mathfrak{h};\rho,\mu)$. If two formal deformations $\bar{H}_t = \sum_{i=0}^{\infty} \bar{\mathfrak{H}}_i t^i$ and $H_t = \sum_{i=0}^{\infty} \mathfrak{H}_i t^i$ are equivalent, then their infinitesimals are in the same cohomology classes.*

Proof. Let (ϕ_t, φ_t) be the maps defined by (36), which makes two deformations $\overline{H}_t = \sum_{i=0}^{\infty} \bar{\mathfrak{H}}_i t^i$ and $H_t = \sum_{i=0}^{\infty} \mathfrak{H}_i t^i$ equivalent. By (37), we have

$$\begin{aligned}\overline{\mathfrak{H}_1} z &= \mathfrak{H}_1 z + (-1)^{|x|(|y|+|z|)}\mu(y,z)(Hx) - (-1)^{|y||z|}\mu(x,z)(Hy)\\ &\quad + [\![Hx, Hy, Hz]\!]_{\mathfrak{h}} = \mathfrak{H}_1 z + \partial(\mathfrak{X})(z),\end{aligned}$$

for all $z \in \mathfrak{h}$, which implies that $\overline{\mathfrak{H}}_1$ and $\mathfrak{H}_1$ are in the same cohomology class. □

5.3. Order n deformations of crossed homomorphisms between Lie–Yamaguti superalgebras. A deformation of order n of a crossed homomorphism is extensible in this subsection if and only if this cohomology class in the second cohomology group is trivial. We also present a special cohomology class associated with an order n deformation of a crossed homomorphism. Because of this, we refer to this particular cohomology class as the obstruction class of an extensible deformation of order n.

Definition 16. Let $H : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a crossed homomorphism from a Lie–Yamaguti superalgebra $\left(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}}\right)$ to another Lie–Yamaguti superalgebra $\left(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}}\right)$ with respect to an action $(\mathfrak{h};\rho,\mu)$. If $H_t = \sum_{i=0}^{n} \mathfrak{H}_i t^i$ with $\mathfrak{H}_0 = H, \mathfrak{H}_i \in \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{h}), i = 1, 2, \cdots, n$, defines a $\mathbb{K}[t]/\left(t^{n+1}\right)$ module from $\mathfrak{g}[t]/\left(t^{n+1}\right)$ to the Lie–Yamaguti superalgebra $\mathfrak{h}[t]/\left(t^{n+1}\right)$ satisfying

$$H_t[x,y]_{\mathfrak{g}} = (-1)^{|x||H_t|}\rho(x)\left(H_t y\right) - (-1)^{|y|(|x|+|H_t|)}\rho(y)\left(H_t x\right) + (-1)^{|x||H_t|}\left[H_t x, H_t y\right]_{\mathfrak{h}} \tag{38}$$

$$\begin{aligned}H_t[\![x,y,z]\!]_{\mathfrak{g}} &= (-1)^{|H_t|(|x|+|y|)}D(x,y)\left(H_t z\right) + (-1)^{(|y|+|z|)(|H_t|+|x|)}\mu(y,z)\left(H_t x\right)\\ &\quad - (-1)^{|z|(|H_t|+|y|)+|H_t||x|}\mu(x,z)\left(H_t y\right) + (-1)^{|H_t||y|}[\![H_t x, H_t y, H_t z]\!]_{\mathfrak{h}}\end{aligned} \tag{39}$$

for all $x, y, z \in \mathfrak{g}$, we say that H_t is an *order n deformation* of H.

Remark 3. The left hand sides of (38) and (39) hold in the Lie–Yamaguti superalgebra $\mathfrak{g}[t]/\left(t^{n+1}\right)$ and the right hand sides of (38) and (39) make sense since H_t is a $\mathbb{K}[t]/\left(t^{n+1}\right)$ module map.

Definition 17. Let $H : \mathfrak{g} \longrightarrow \mathfrak{b}$ be a crossed homomorphism from a Lie–Yamaguti superalgebra $\left(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}}\right)$ to another Lie–Yamaguti superalgebra $\left(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}}\right)$ with respect to an action (ρ,μ). Let $H_t =$

$\sum_{i=0}^{n} \mathfrak{H}_i t^i$ be an order n deformation of H. If there exists a 1-cochain $\mathfrak{H}_{n+1} \in \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g}, \mathfrak{h})$ such that $\widetilde{H}_t = T_t + \mathfrak{H}_{n+1} t^{n+1}$ is an order $n+1$ deformation of H, then we say that H_t is *extendable.*

The following theorem is the third key conclusion in this section.

Theorem 6. *Let $H : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a crossed homomorphism from a Lie–Yamaguti superalgebra $\left(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}}\right)$ to another Lie–Yamaguti superalgebra $\left(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}}\right)$ with respect to an action $(\mathrm{b}; \rho, \mu)$. Let $H_t = \sum_{i=0}^{n} \mathfrak{H}_i t^i$ be an order n deformation of H. Then H_t is extendable if and only if the cohomology class $\left[\mathrm{Ob}^H\right] \in \mathcal{H}_H^2(\mathfrak{g}, \mathfrak{h})$ is trivial, where $\mathrm{Ob}^H = \left(\mathrm{Ob}_{\mathrm{I}}^H, \mathrm{Ob}_{\mathrm{II}}^H\right) \in \mathfrak{C}_H^2(\mathfrak{g}, \mathfrak{h})$ is defined by*

$$\mathrm{Ob}_{\mathrm{I}}^H(x_1, x_2) = \sum_{\substack{i+j=n+n+1 \\ 0<i,j<n+1}} [\mathfrak{H}_i x, \mathfrak{H}_j y]_{\mathfrak{h}},$$

$$\mathrm{Ob}_{\mathrm{II}}^H(x_1, x_2, x_3) = \sum_{\substack{i+j+k=n+1 \\ 0<i,j,k<n+1}} [\![\mathfrak{H}_i x_1, \mathfrak{H}_j x_2, \mathfrak{H}_k x_3]\!], \quad \forall x_1, x_2, x_3 \in \mathfrak{g}.$$

Proof. Let $\widetilde{H}_t = \sum_{i=0}^{n+1} \mathfrak{H}_i t^i$ be an extension of H_t, then for all $x, y, z \in \mathfrak{g}$,

$$\begin{aligned} \widetilde{H}_t[x,y]_{\mathfrak{g}} =& (-1)^{|H_t||x|}\rho(x)\left(\widetilde{H}_t y\right) - (-1)^{|y|(|H_t|+|x|)}\rho(y)\left(\widetilde{H}_t x\right) \\ &+ (-1)^{|H_t||x|}\left[\widetilde{H}_t x, \widetilde{H}_t y\right]_{\mathfrak{h}}, \end{aligned} \tag{40}$$

$$\begin{aligned} \widetilde{H}_t[\![x,y,z]\!]_{\mathfrak{g}} =& (-1)^{|\widetilde{H}_t|(|x|+|y|)} D(x,y)\left(\widetilde{H}_t z\right) + (-1)^{(|\widetilde{H}_t|+|x|)(|y|+|z|)}\mu(y,z)\left(\widetilde{H}_t x\right) \\ &- (-1)^{|z|)(|\widetilde{H}_t|+|y|)+|\widetilde{H}_t||x|}\mu(x,z)\left(\widetilde{H}_t y\right) + (-1)^{|\widetilde{H}_t||y|}[\![\widetilde{H}_t x, \widetilde{H}_t y, \widetilde{H}_t z]\!]_{\mathfrak{h}}. \end{aligned} \tag{41}$$

Expanding the equality (40) and comparing the coefficients of t^n yields that

$$\begin{aligned} \mathfrak{H}_{n+1}[x,y]_{\mathfrak{g}} = \;& (-1)^{|\mathfrak{H}_{n+1}||x|}\rho(x)\left(\mathfrak{H}_{n+1} y\right) - (-1)^{|y|(|\mathfrak{H}_{n+1}|+|x|)}\rho(y)\left(\mathfrak{H}_{n+1} x\right) \\ &+ \textstyle\sum_{i+j=n+1}(-1)^{|\mathfrak{H}_j||x|}[\mathfrak{H}_i x, \mathfrak{H}_j y]_{\mathfrak{h}}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &(-1)^{|\mathfrak{H}_{n+1}||x|}\rho(x)\left(\mathfrak{H}_{n+1} y\right) - (-1)^{|y|(|\mathfrak{H}_{n+1}|+|x|)}\rho(y)\left(\mathfrak{H}_{n+1} x\right) + [\mathfrak{H}_{n+1} x, Hy]_{\mathfrak{h}} \\ &+ (-1)^{|\mathfrak{H}_{n+1}||x|}[x, \mathfrak{H}_{n+1} y]_{\mathfrak{h}} - \mathfrak{H}_{n+1}[x,y]_{\mathfrak{g}} + \sum_{\substack{i+j=n+1 \\ 0<i,j<n+1}} (-1)^{|\mathfrak{H}_j||x|}[\mathfrak{H}_i x, \mathfrak{H}_j y]_{\mathfrak{h}} = 0, \end{aligned}$$

i.e.,

$$\mathrm{Ob}_{\mathrm{I}}^H + \delta_{\mathrm{I}}^T(\mathfrak{H}_{n+1}) = 0. \tag{42}$$

Similarly, expanding the equality (41) and comparing the coefficients of t^n yields that

$$\mathrm{Ob}_{\mathrm{II}}^{H} + \delta_{\mathrm{II}}^{T}\left(\mathfrak{H}_{n+1}\right) = 0. \tag{43}$$

From (42) and (43), we get

$$\mathrm{Ob}^{H} = \delta^{H}\left(-\mathfrak{H}_{n+1}\right).$$

Thus, the cohomology class $\left[\mathrm{Ob}^{H}\right]$ is trivial.

Conversely, suppose that the cohomology class $\left[\mathrm{Ob}^{H}\right]$ is trivial. Then there exists $\mathfrak{H}_{n+1} \in \mathfrak{C}_{H}^{1}(\mathfrak{g},\mathfrak{h})$, such that $\mathrm{Ob}^{H} = -\delta^{H}\left(\mathfrak{H}_{n+1}\right)$.

Set $\widetilde{H}_t = H_t + \mathfrak{H}_{n+1}t^{n+1}$. Then, for all $0 \leqslant s \leqslant n+1$, and for all $x, y, z \in \mathfrak{g}$, $\widetilde{H}_t$ satisfies

$$\sum_{i+j=s}\Big(\mathfrak{H}_s[x,y]_{\mathfrak{g}} - \Big((-1)^{|\mathfrak{H}_s||x|}\rho(x)\left(\mathfrak{H}_s y\right) - (-1)^{|y|(|\mathfrak{H}_s|+|x|)}\rho(y)\left(\mathfrak{H}_s x\right)$$
$$+(-1)^{|\mathfrak{H}_j||x|}\left[\mathfrak{H}_i x, \mathfrak{H}_j y\right]_{\mathfrak{h}}\Big)\Big) = 0$$

$$\sum_{i+j+k=s}\left(\mathfrak{H}_s[\![x,y,z]\!]_{\mathfrak{g}}\right.$$
$$-\Big((-1)^{|\mathfrak{H}_s|(|x|+|y|)}D(x,y)\left(\mathfrak{H}_s z\right) + (-1)^{(|\mathfrak{H}_s|+|x|)(|y|+|z|)}\mu(y,z)\left(\mathfrak{H}_s x\right)$$
$$-(-1)^{|z|)(|\mathfrak{H}_s|+|y|)+|\mathfrak{H}_s||x|}\mu(x,z)\left(\mathfrak{H}_s y\right) + (-1)^{|\mathfrak{H}_k|x|}[\![\mathfrak{H}_i x, \mathfrak{H}_j y, \mathfrak{H}_k z]\!]_{\mathfrak{h}}\Big)\Big) = 0,$$

which implies that $\widetilde{H}_t$ is an order $n+1$ deformation of H. Hence it is an extension of H_t. □

Definition 18. Let $H : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a crossed homomorphism from a Lie–Yamaguti superalgebra $\left(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot\cdot,\cdot]\!]_{\mathfrak{g}}\right)$ to another Lie–Yamaguti superalgebra $\left(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}}\right)$ with respect to an action $(\mathfrak{h};\rho,\mu)$, and $H_t = \sum_{i=0}^{n}\mathfrak{H}_i t^i$ be an order n deformation of H. Then the cohomology class $\left[\mathrm{Ob}^{H}\right] \in \mathcal{H}_{H}^{2}(\mathfrak{g},\mathfrak{h})$ defined in Theorem 6 is called the *obstruction class* of H_t *being extendable.*

Corollary 1. *Let $H : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a crossed homomorphism from a Lie–Yamaguti superalgebra $\left(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{g}}\right)$ to another Lie–Yamaguti superalgebra $\left(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, [\![\cdot,\cdot,\cdot]\!]_{\mathfrak{h}}\right)$ with respect to an action $(\mathfrak{h};\rho,\mu)$. If $\mathcal{H}_{H}^{2}(\mathfrak{g},\mathfrak{h}) = 0$, then every 1-cocycle in $\mathcal{Z}_{H}^{1}(\mathfrak{g},\mathfrak{h})$ is the infinitesimal of some formal deformation of H.*

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