ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 29, Number 2, December 2025
Available online at https://ojs.utlib.ee/index.php/ACUTM

# On $\lambda$-ideal statistical convergence in fuzzy cone normed spaces

Reena Antal

Abstract. In this paper, we have presented and explored the $\lambda$-ideal statistical convergence for sequences on fuzzy cone normed spaces. The related topological and geometrical properties are demonstrated with examples. Through analyzing the criteria based on $\lambda$-ideal statistical convergence on these spaces, we aim to establish a comprehensive set of equivalent conditions for sequences that exhibit $\lambda$-ideal statistical convergence.

## 1. Introduction

Researchers are engaged in improving existing techniques and technology and creating new ones. Convergence is a concept used to acquire information from existing data. There are several procedures for convergence, both analytical and non-analytical. Alternative procedures will be tried when classical methods fail to provide solutions. Statistical convergence [10] is applied when the usual convergence concept is unable to give a solution. The study of summability theory and sequence convergence has been one of the most significant and active areas in mathematics for many decades. It serves as a tool to solve numerous open problems in the wide area of sequence spaces and summability theory, as well as in various other applications. Generalizations of statistical convergence are emerging in many articles provided by various authors (see [1, 2, 4, 15, 16, 18, 22, 25]).

In 1984, Katsaras introduced the concept of fuzzy norms, which has since gained significant attention in mathematics and related fields [17]. Later, in 1992, Felbin further expanded on this concept by defining fuzzy norm in the setup of a linear space along a related metric of Kaleva and Seikkala type [11].

Received December 27, 2024.
2020 *Mathematics Subject Classification.* 40A35, 40A05, 46S40.
*Key words and phrases.* Fuzzy cone normed space, ideal, statistical convergence.
https://doi.org/10.12697/ACUTM.2025.29.10
Corresponding author: Reena Antal

This work contributes to the understanding and application of fuzzy norms in various mathematical contexts. Another approach used in defining a fuzzy norm which is connected with a metric given by Kramosil and Michalek [21], was introduced by Cheng and Moderson [6] in 1994. Following their definition, Bag and Samanta [3] proposed differently the concept of fuzzy norm. Meanwhile, the concept of metric space has been widely generalized in several forms, including the cone metric space which has been structured and presented by Huang and Zhang [14] in 2007 as a metric space which is analyzed in an ordered Banach space. After conducting their research, several scholars have provided various mathematical conceptual advancements in cone metric spaces ([5, 7, 8, 9]). The fundamental prospective for fuzzy cone metric space is comprehensively investigated and presented by Öner et al. [24], which leads to a broad scope of more complex mathematical structures. In 2017, Tamang and Bag [26] presented the fuzzy cone norm as an expansion of the fuzzy norm, using a real Banach space instead of $\mathbb{R}$ to define a fuzzy cone normed linear space. This expansion has significant implications for both classical and fuzzy mathematics. As these fields continue to progress, the examination of convergence results within the structure of fuzzy cone normed linear spaces, well emphasized in [13, 19], is poised to play a crucial role in furthering our understanding of these mathematical concepts.

Our paper aims to set up the generalized statistical ideal convergence on a fuzzy cone normed linear space. We believe that the results of this paper can be useful in further developments of fuzzy mathematics.

Throughout the paper, we take $\mathbb{B}$ for a real Banach space.

**Definition 1** ([14])**.** A set $Q$, $Q \subseteq \mathbb{B}$, is defined to be a *cone* if the following axioms hold:

(i) $Q$ is nonempty and closed,
(ii) if $\alpha, \beta \in \mathbb{R}, \alpha, \beta \geq 0$ and $u, v \in Q$, then $\alpha u + \beta v \in Q$,
(iii) if $u \in Q$ and $-u \in Q$, then $u = 0$.

Every cone $Q$ in $\mathbb{B}$ induces a partial ordering $\preceq$ on $\mathbb{B}$, defined by $a \preceq b$ if and only if $b - a \in Q$. We write $a \prec b$ if $a \preceq b$ and $a \neq b$. But $a \ll b$ will stand for $b - a \in int(Q)$. Throughout the article we assume all cones to have nonempty interior.

**Definition 2** ([26])**.** A *fuzzy cone normed space* (FCNS) is a 3-tuple $(\mathbb{S}, \aleph_C, \circledast)$, where $\mathbb{S}$ is a vector space, $Q$ is a cone in $\mathbb{B}$, $\circledast$ is a continuous $t$-norm and $\aleph_C$ is a fuzzy set on $\mathbb{S} \times (0, \infty)$ satisfying for every $s, t \in \mathbb{S}$ and $u, v > 0$:

(i) $\aleph_C > 0$,
(ii) $\aleph_C(s, u) = 1$ for $u > 0$ iff $s = 0$,
(iii) $\aleph_C(\alpha s, u) = \aleph_C\left(s, \frac{u}{|\alpha|}\right)$, for $\alpha \neq 0$,
(iv) $\aleph_C(s, u) \circledast \aleph_C(t, v) \leq \aleph_C(s + t, u + v)$,

(v) $\aleph_C(s,.) : int(Q) \to [0,1]$,
(vi) $\lim_{\|u\|\to\infty} \aleph_C(s,u) = 1$.

Then, $\aleph_C$ is known as a fuzzy cone norm.

**Example 1.** Consider $\mathbb{B} = \mathbb{R}$ and $Q = [0,\infty)$ as a normal cone. Let $\alpha_1 \circledast \alpha_2 = \alpha_1\alpha_2$ for all $\alpha_1, \alpha_2 \in [0,1]$. For every $u \in int(Q)$ and $s \in \mathbb{S}$, where $\mathbb{S} = \mathbb{R}$, take $\aleph_C(s,u) = e^{\frac{-\sqrt{|s|}}{\sqrt{u}}}$. Then, the 3-tuple $(\mathbb{S}, \aleph_C, \circledast)$ becomes a FCNS.

**Example 2.** Let $(\mathbb{S}, \|\,.\,\|)$ be a normed space, let $Q \subset \mathbb{B}$ and let $\alpha_1 \circledast \alpha_2 = \alpha_1\alpha_2$ for all $\alpha_1, \alpha_2 \in [0,1]$. For every $u \in int(Q)$ and $s \in \mathbb{S}$, take $\aleph_C(s,u) = \frac{u}{u+\|s\|}$. Then the 3-tuple $(\mathbb{S}, \aleph_C, \circledast)$ becomes a FCNS.

**Definition 3** ([26])**.** Let $(\mathbb{S}, \aleph_C, \circledast)$ be a FCNS with a fuzzy cone norm $\aleph_C$. A sequence $\{s_k\}$ in $\mathbb{S}$ is called *convergent to $s_0 \in \mathbb{S}$ with respect to the fuzzy cone norm* $\aleph_C$ if, for every $\varepsilon \in (0,1)$ and $u \in int(Q)$, we can find a positive integer $k_0$ such that

$$\aleph_C(s_k - s_0, u) \le 1 - \varepsilon \text{ for all } k > k_0.$$

In this case we write $\aleph_C$ - $\lim_{k\to\infty} s_k = s_0$ or $s_k \xrightarrow{\aleph_C} s_0$.

A sequence $\{s_k\}$ is *statistically convergent* to $s_0$, provided the set $A(\varepsilon) = \{k \in \mathbb{N} : |s_k - s_0| > \varepsilon\}$ has zero natural density [12]. The natural density of any set $M \subseteq \mathbb{N}$, is given by $d(M) = \lim_{n\to\infty} \frac{1}{n} \mid \{m \le n : m \in M\} \mid$.

**Definition 4** ([13])**.** Let $(\mathbb{S}, \aleph_C, \circledast)$ be a FCNS with a fuzzy cone norm $\aleph_C$. A sequence $\{s_k\}$ in $\mathbb{S}$ is called *statistically convergent to $s_0 \in \mathbb{S}$ with respect to the fuzzy cone norm* $\aleph_C$ if, for every $\varepsilon \in (0,1)$ and $u \in int(Q)$, we have

$$d(\{k \in \mathbb{N} : \aleph_C(s_k - s_0, u) \le 1 - \varepsilon\}) = 0.$$

In this case we write $St_{\aleph_C}$ - $\lim_{k\to\infty} s_k = s_0$ or $s_k \xrightarrow{St_{\aleph_C}} s_0$ .

Kostryko et al. [20] have introduced and investigated ideal convergence ($\mathcal{I}$-convergence) of sequences as an extension of statistical convergence proposed by Fast [10].

**Definition 5** ([20])**.** A family $\mathcal{I}$ which is subset of $P(\mathbb{S})$, where $\mathbb{S} \ne \emptyset$ and $P(\mathbb{S})$ is the power set of the set $\mathbb{S}$, is called an *ideal* on $\mathbb{S}$ provided that (a) $\emptyset \in \mathcal{I}$, (b) $U, V \in \mathcal{I} \Rightarrow U \cup V \in \mathcal{I}$, (c) for $U \in \mathcal{I}, V \subset U \Rightarrow V \in \mathcal{I}$.

Moreover, any non-trivial ideal $\mathcal{I}$ ($\mathcal{I} \ne \emptyset, \mathbb{S} \notin \mathcal{I}$) becomes *admissible ideal* on $\mathbb{S}$, if all possible singleton sets of $\mathbb{S}$ are collected in $\mathcal{I}$ *i.e.* $\mathcal{I} \supset \{\{s\} : s \in \mathbb{S}\}$.

**Example 3.** The collection $\mathcal{I}_f$ of finite subsets of $\mathbb{N}$ is an admissible ideal.

**Definition 6** ([20])**.** A non-empty collection $\mathbb{F} \subset P(\mathbb{S})$, where $\mathbb{S} \neq \emptyset$, is called a *filter* on $\mathbb{S}$ provided (a) $\emptyset \notin \mathbb{F}$, (b) $U, V \in \mathbb{F} \Rightarrow U \cap V \in \mathbb{F}$, (c) for $U \in \mathbb{F}, U \subset V \Rightarrow V \in \mathbb{F}$.

Each ideal $\mathcal{I}$ is connected to a filter $\mathbb{F}(\mathcal{I})$, the relation is expressed by $\mathbb{F}(\mathcal{I}) = \{K \subseteq \mathbb{S} : K^c \in \mathcal{I}\}$.
$\mathcal{I}$ will denote an admissible ideal throughout the article.

**Definition 7** ([20])**.** A sequence $\{s_k\}$ in $\mathbb{S}$ is called *ideal convergent* ($\mathcal{I}$-convergent) to $s_0 \in \mathbb{S}$ if, for every $\varepsilon > 0$, one has $\{k \in \mathbb{N} : |s_k - s_0| \geq \varepsilon\} \in \mathcal{I}$. Here $s_0$ is called an $\mathcal{I}$-limit of the sequence $\{s_k\}$.

**Definition 8** ([20])**.** A sequence $\{s_k\}$ in $\mathbb{S}$ is called *ideal statistically convergent* ($\mathcal{I}$-*St*-convergent) to $s_0 \in \mathbb{S}$ if, for every $\varepsilon > 0$ and $\delta > 0$, one has

$$\left\{n \in \mathbb{N} : \frac{1}{n} |\, \{n \in \mathbb{N} : |s_k - s_0| \geq \varepsilon\} \,| \geq \delta \right\} \in \mathcal{I}.$$

Here, $s_0$ is called $\mathcal{I}$-*St*-limit of the sequence $\{s_k\}$.

Güler[13] introduced the structure of ideal convergence within the framework of fuzzy cone normed spaces.

**Definition 9** ([13])**.** Let $(\mathbb{S}, \aleph_C, \circledast)$ be a FCNS with a fuzzy cone norm $\aleph_C$. A sequence $\{s_k\}$ in $\mathbb{S}$ is called *ideal convergent to* $s_0 \in \mathbb{S}$ *with respect to the fuzzy cone norm* $\aleph_C$ if, for every $\varepsilon \in (0,1)$ and $u \in int(Q)$, we have

$$\{k \in \mathbb{N} : \aleph_C(s_k - s_0, u) \leq 1 - \varepsilon\} \in \mathcal{I}.$$

In this case we write $\mathcal{I}_{\aleph_C}\text{-}\lim_{k\to\infty} s_k = s_0$ or $s_k \xrightarrow{\mathcal{I}_{\aleph_C}} s_0$.

The aim of the paper is associating the theory of fuzzy cone normed spaces with $\lambda$-ideal statistical convergence highlighting more advanced theoretical concepts. The $\lambda$-statistical convergence is a generalized form of sequence convergence which was incorporated by Mursaleen [23] with the help of a non-decreasing sequence $\lambda = \{\lambda_n\}$, which tends to $\infty$ such that $\lambda_{n+1} \leq 1 + \lambda_n$ and $\lambda_1 = 1$. Additionally, the generalized de la Vallée-Poussin mean is given by

$$t_n(s) = \frac{1}{\lambda_n} \sum_{j \in I_n} s_j, \text{ where } I_n = [1 + n - \lambda_n, n].$$

Throughout the paper, we use $I_n$ for $[1 + n - \lambda_n, n]$.

## 2. Main results

We define and investigate $\lambda$-ideal statistical convergence of sequences in the setup of fuzzy cone normed spaces (FCNS) to establish the interesting findings.

**Definition 10.** Let $(\mathbb{S}, \aleph_C, \circledast)$ be a FCNS with a fuzzy cone norm $\aleph_C$. A sequence $\{s_k\}$ in $\mathbb{S}$ is called *$\mathcal{I}$-statistically convergent to $s_0 \in \mathbb{S}$ with respect to the fuzzy cone norm* $\aleph_C$ if, for every $\varepsilon \in (0,1)$, $\delta > 0$ and $u \in int(Q)$, one has

$$\left\{k \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \aleph_C(s_k - s_0, u) \leq 1 - \varepsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case we write $St\text{-}\mathcal{I}_{\aleph_C}\text{-}\lim_{k\to\infty} s_k = s_0$ or $s_k \xrightarrow{St\text{-}\mathcal{I}_{\aleph_C}} s_0$.

**Definition 11.** Let $(\mathbb{S}, \aleph_C, \circledast)$ be a FCNS with a fuzzy cone norm $\aleph_C$. A sequence $\{s_k\}$ in $\mathbb{S}$ is called *$\lambda$-statistically convergent to $s_0 \in \mathbb{S}$ with respect to the fuzzy cone norm* $\aleph_C$ if, for every $\varepsilon \in (0,1)$ and $u \in int(Q)$, one has

$$\lim_{n\to\infty} \frac{1}{\lambda_n}|\{k \in I_n : \aleph_C(s_k - s_0, u) \leq 1 - \varepsilon\}| = 0.$$

In this case we write $\lambda\text{-}St\text{-}\lim_{k\to\infty} s_k = s_0$ or $s_k \xrightarrow{\lambda\text{-}St} s_0$.

**Definition 12.** Let $(\mathbb{S}, \aleph_C, \circledast)$ be a FCNS with a fuzzy cone norm $\aleph_C$. A sequence $\{s_k\}$ in $\mathbb{S}$ is called *$\mathcal{I}$-$\lambda$-statistically convergent to $s_0 \in \mathbb{S}$ with respect to the fuzzy cone norm* $\aleph_C$ if, for every $\varepsilon \in (0,1)$, $\delta > 0$ and $u \in int(Q)$, one has

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : \aleph_C(s_k - s_0, u) \leq 1 - \varepsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case we write $St_\lambda\text{-}\mathcal{I}_{\aleph_C}\text{-}\lim_{k\to\infty} s_k = s_0$ or $s_k \xrightarrow{St_\lambda\text{-}\mathcal{I}_{\aleph_C}} s_0$.

**Theorem 1.** *Let $\{s_k\}$ be a sequence from a FCNS $(\mathbb{S}, \aleph_C, \circledast)$ which is $\mathcal{I}$-$\lambda$-statistically convergent with respect to the fuzzy cone norm $\aleph_C$. Then the limit is unique.*

*Proof.* Let $St_\lambda\text{-}\mathcal{I}_{\aleph_C}\text{-}\lim_{k\to\infty} s_k = a$ and $St_\lambda\text{-}\mathcal{I}_{\aleph_C}\text{-}\lim_{k\to\infty} s_k = b$.
We select $\nu \in (0,1)$ for $\varepsilon \in (0,1)$ with $(1-\nu) \circledast (1-\nu) > 1 - \varepsilon$. Then, for $\delta > 0$ and $u \in int(Q)$, define

$$A_1 = \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\left|\left\{k \in I_n : \aleph_C\left(s_k - a, \frac{u}{2}\right) \leq 1 - \nu\right\}\right| \geq \delta\right\} \in \mathcal{I},$$

$$A_2 = \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\left|\left\{k \in I_n : \aleph_C\left(s_k - b, \frac{u}{2}\right) \leq 1 - \nu\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

We have

$$\frac{1}{\lambda_n}|\{k \in I_n : k \in A_1\}| + \frac{1}{\lambda_n}|\{k \in I_n : k \in A_2\}| \geq \frac{1}{\lambda_n}|\{k \in I_n : A_1 \cup A_2\}|.$$

For $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in \mathbb{N} : k \in A_1 \cup A_2\}| \geq \delta\right\} \in \mathcal{I}.$$

Assume $0 < \delta_1 < 1$ with $0 < 1 - \delta_1 < \delta$.
Consider $A = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} |\{k \in I_n : k \in A_1 \cup A_2\}| \geq \delta_1 \right\} \in \mathcal{I}$. For $n \in A$,

$$\frac{1}{\lambda_n} |\{k \in I_n : k \in A_1 \cup A_2\}| < 1 - \delta_1,$$

$$\frac{1}{\lambda_n} |\{k \in I_n : k \notin A_1 \cup A_2\}| \geq \delta_1.$$

Therefore,

$$\{k \in I_n : k \notin A_1 \cup A_2\} \neq \emptyset.$$

For $k \notin A_1 \cup A_2$, we have

$$\begin{aligned} \aleph_C(a - b, u) &\geq \aleph_C \left( s_k - a, \frac{u}{2} \right) \circledast \aleph_C \left( s_k - b, \frac{u}{2} \right) \\ &> (1 - \nu) \circledast (1 - \nu) \\ &> 1 - \varepsilon. \end{aligned}$$

Clearly, $A^c \subset \{k \in I_n : \aleph_C(a - b, u) > 1 - \varepsilon\} = B$. If $n \notin A$, then $\delta_1 \leq \frac{|A_1{}^c \cap A_2{}^c|}{n} \leq \frac{|B^c|}{n}$. Now, we get

$$\frac{1}{\lambda_n} \left| \{k \in I_n : k \in P\} \right| < 1 - \delta_1 < \delta.$$

Thus,

$$A^c \subset \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \left| \{k \in I_n : k \in P\} \right| \geq \delta \right\}.$$

As $A \in \mathcal{I}$, we have $\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \left| \{k \in I_n : k \in P\} \right| \geq \delta \right\} \in \mathcal{I}$. Hence $a = b$. □

**Theorem 2.** *Let $\{r_k\}$ and $\{s_k\}$ be two sequences from a FCNS $(\mathbb{S}, \aleph_C, \circledast)$. Then*
*(i) if $St_\lambda$ -$\mathcal{I}_{\aleph_C}$ - $\lim_{k\to\infty} r_k = r_0$ and $St_\lambda$ -$\mathcal{I}_{\aleph_C}$ - $\lim_{k\to\infty} s_k = s_0$, then*
*$St_\lambda$ -$\mathcal{I}_{\aleph_C}$ - $\lim_{k\to\infty} (r_k + s_k) = r_0 + s_0$,*
*(ii) if $St_\lambda$-$\mathcal{I}_{\aleph_C}$ - $\lim_{k\to\infty} s_k = s_0$, then $St_\lambda$ -$\mathcal{I}_{\aleph_C}$ - $\lim_{k\to\infty} \alpha s_k = \alpha s_0$ where $\alpha \in \mathbb{R}$.*

*Proof.* $(i)$ Let $St_\lambda$ -$\mathcal{I}_{\aleph_C}$ - $\lim_{k\to\infty} r_k = r_0$ and $St_\lambda$ -$\mathcal{I}_{\aleph_C}$ - $\lim_{k\to\infty} s_k = s_0$.
We select $\nu \in (0, 1)$ for $\varepsilon \in (0, 1)$ with $(1 - \nu) \circledast (1 - \nu) > 1 - \varepsilon$. Then, for $\delta > 0$ and $u \in int(Q)$, define

$$A_1 = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \aleph_C \left( r_k - r_0, \frac{u}{2} \right) \leq 1 - \nu \right\} \right| \geq \delta \right\} \in \mathcal{I},$$

$$A_2 = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \aleph_C \left( s_k - s_0, \frac{u}{2} \right) \leq 1 - \nu \right\} \right| \geq \delta \right\} \in \mathcal{I}.$$

We have

$$\frac{1}{\lambda_n} \left| \{k \in I_n : A_1 \cup A_2\} \right| \leq \frac{1}{\lambda_n} \left| \{k \in I_n : k \in A_1\} \right| + \frac{1}{\lambda_n} \left| \{k \in I_n : k \in A_2\} \right|.$$

For $\delta > 0$, we get

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \left|\{k \in \mathbb{N} : k \in A_1 \cup A_2\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Take $0 < \delta_1 < 1$ with $0 < 1 - \delta_1 < \delta$.
Consider $A = \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : k \in A_1 \cup A_2\}| \geq \delta_1\right\} \in \mathcal{I}$. For $n \in A$,

$$\frac{1}{\lambda_n}|\{k \in I_n : k \in A_1 \cup A_2\}| < 1 - \delta_1,$$

$$\frac{1}{\lambda_n}|\{k \in I_n : k \notin A_1 \cup A_2\}| \geq \delta_1.$$

Therefore,

$$\{k \in I_n : k \notin A_1 \cup A_2\} \neq \emptyset.$$

For $k \notin A_1 \cup A_2$, we have

$$\begin{aligned} \aleph_C((r_k + s_k) - (r_0 + s_0), u) &\geq \aleph_C\left(r_k - r_0, \frac{u}{2}\right) \circledast \aleph_C\left(s_k - s_0, \frac{u}{2}\right) \\ &> (1 - \nu) \circledast (1 - \nu) \\ &> 1 - \varepsilon. \end{aligned}$$

Clearly, $A^c \subset \{k \in I_n : \aleph_C((r_k + s_k) - (r_0 + s_0), u) > 1 - \varepsilon\} = P$. If $n \notin A$, then $\delta_1 \leq \frac{|{A_1}^c \cap {A_2}^c|}{n} \leq \frac{|P^c|}{n}$. Now we get

$$\frac{1}{\lambda_n}\left|\{k \in I_n : k \in P\}\right| < 1 - \delta_1 < \delta,$$

and thus

$$A^c \subset \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\left\{k \in I_n : k \in P\right\}| \geq \delta\right\}.$$

As $A \in \mathcal{I}$, we have $\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\left|\{k \in I_n : k \in P\}\right| \geq \delta\right\} \in \mathcal{I}$.
Hence $St_\lambda$-$\mathcal{I}_{\aleph_C}$-$\lim\limits_{k \to \infty}(r_k + s_k) = r_0 + s_0$.

*(ii)* (a) If $\alpha = 0$, then the result is obvious.
(b) If $|\alpha| > 1$. For $u \in int(Q)$, $\nu \in (0, 1)$ and $\delta > 0$. Put

$$A_1 = \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\left|\left\{k \in I_n : \aleph_C\left(s_k - s_0, \frac{u}{2}\right) \leq 1 - \nu\right\}\right| \geq \delta\right\} \in \mathcal{I},$$

$$A_2 = \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\left|\left\{k \in I_n : \aleph_C\left(\alpha s_k - \alpha s_0, \frac{u}{2}\right) \leq 1 - \nu\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

As $\aleph_C$ is fuzzy cone norm, we have $\aleph_C\left(\alpha s_k - \alpha s_0, \frac{u}{2}\right) = \aleph_C\left(s_k - s_0, \frac{u}{|\alpha|}\right)$. Since $\aleph_C$ is a non-decreasing function and $\frac{u}{|\alpha|} \leq u$ for $|\alpha| > 1$, the equality $A_1 = A_2$ must hold.

As $St_\lambda$-$\mathcal{I}_{\aleph_C}$-$\lim_{k\to\infty} s_k = s_0$ which gives $A_1 \in \mathcal{I}$ and hence $A_2 \in \mathcal{I}$. Therefore, $St_\lambda$-$\mathcal{I}_{\aleph_C}$-$\lim_{k\to\infty} \alpha s_k = \alpha s_0$.

(c) If $|\alpha| < 1$, then for $u \in int(Q)$, $\nu \in (0,1)$ and $\delta > 0$, define $A_1$ and $A_2$ as in part (b). Since $\aleph_C$ is a non-decreasing function and $\frac{u}{|\alpha|} \geq u$ for $|\alpha| > 1$,

$$\begin{aligned}\aleph_C\left(s_k - s_0, \frac{u}{|\alpha|}\right) &\geq \aleph_C\left(s_k - s_0, u\right) \circledast \aleph_C\left(0, \frac{u}{|\alpha|} - u\right)\\ &= \aleph_C\left(s_k - s_0, u\right)\\ &> 1 - \nu.\end{aligned}$$

Then $A_2 \subseteq A_1$, which implies $A_2 \in \mathcal{I}$. □

**Theorem 3.** *If* $\aleph_C$ - $\lim_{k\to\infty} s_k = s_0$, *then* $St_\lambda$ -$\mathcal{I}_{\aleph_C}$ - $\lim_{k\to\infty} s_k = s_0$.

*Proof.* If a sequence $\{s_k\}$ from $\mathbb{S}$ is $\aleph_C$ - $\lim_{k\to\infty} s_k = s_0$, then, for every $\varepsilon \in (0,1)$ and $u \in int(Q)$, we can find a positive integer $k_0$ with $\aleph_C(s_k - s_0, u) \leq 1 - \varepsilon$ for all $k > k_0$. Then

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : \aleph_C(s_k - s_0, u) \leq 1 - \varepsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

□

The example below demonstrates that the converse of the above theorem is not true.

**Example 4.** Let $\mathbb{B} = \mathbb{R}^2$ and $Q = \{a = (a_1, a_2) : a_1, a_2 \geq 0\} \subset \mathbb{B}$ is a normal cone. Consider $\mathbb{S} = \mathbb{R}$, $\alpha_1 \circledast \alpha_2 = \alpha_1\alpha_2$ for all $\alpha_1, \alpha_2 \in [0,1]$. For every $u \in int(Q)$ and $s \in \mathbb{S}$, take $\aleph_C(s,u) = e^{\frac{-|s|}{\|u\|}}$. Then the 3-tuple $(\mathbb{S}, \aleph_C, \circledast)$ is a FCNS. Let $\mathcal{I} = \{K \subset \mathbb{N} : d(K) = 0\}$. Take a sequence $\{s_k\}$ in the FCNS $(\mathbb{S}, \aleph_C, \circledast)$ as

$$s_k = \begin{cases} k & n - [\sqrt{\lambda_n}] + 1 \leq k \leq n, \quad \text{where } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Consider $J = \{k \in I_n : \aleph_C(s_k, u) \leq 1 - \varepsilon\}$ for every $\varepsilon \in (0,1)$. Then we get

$$\begin{aligned}J &= \{k \in I_n : e^{\frac{-|s_k|}{\|u\|}} \leq 1 - \varepsilon\}\\ &= \{k \in I_n : |s_k| > 0\}\\ &= \{k \in I_n : s_k = k\}.\end{aligned}$$

Thus,

$$\frac{1}{\lambda_n}|\{k \in I_n : k \in J\}| \to 0.$$

Hence $St_\lambda$ - $\mathcal{I}_{\aleph_C}$ - $\lim_{k\to\infty} s_k = 0$.

**Theorem 4.** *Let $\{s_k\}$ be any sequence from a FCNS $(\mathbb{S}, \aleph_C, \circledast)$. If each sub-sequence $\{s_{k_i}\}$ is $St_\lambda$ -$\mathcal{I}_{\aleph_C}$-convergent to $s_0$, then $\{s_k\}$ is also $St_\lambda$ -$\mathcal{I}_{\aleph_C}$-convergent to $s_0$.*

*Proof.* Suppose $\{s_k\}$ is also $St_\lambda$-$\mathcal{I}_{\aleph_C}$-convergent to $s_0$. For $\nu \in (0,1)$, $\delta > 0$ and $u \in int(Q)$, we have

$$A = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \aleph_C \left( s_k - s_0, \frac{u}{2} \right) \leq 1 - \nu \right\} \right| \geq \delta \right\} \in \mathcal{I}.$$

Now $A$ must be an infinite set since $\mathcal{I}$ is an admissible ideal.
Consider $A = \{k_1 < k_2 < k_3 < \cdots < k_i < \dots\} \subseteq I_n$. Let $y_i = x_{k_i}$ for $i \in \mathbb{N}$, which is not $St_\lambda$-$\mathcal{I}_{\aleph_C}$-convergent to $s_0$, this leads to a contradiction. However, the example below clears that the converse of the above mentioned result is not true. □

**Example 5.** Let $\mathbb{B} = \mathbb{R}^2$ then $Q = \{a = (a_1, a_2) : a_1, a_2 \geq 0\} \subset \mathbb{B}$ is a normal cone. Consider $\mathbb{S} = \mathbb{R}$, $\alpha_1 \circledast \alpha_2 = \alpha_1 \alpha_2$ and $\aleph_C : X \times int(Q) \to [0,1]$ which is given by $\aleph_C(s,u) = e^{\frac{-|s|}{\|u\|}}$ for all $s \in \mathbb{S}$ and $u \in int(Q)$. Take $\mathcal{I} = \{M \subset \mathbb{N} : d(M) = 0\}$. Take a sequence $\{s_k\}$ in $\mathbb{S}$ as

$$s_k = \begin{cases} 2 & n - [\sqrt{\lambda_n}] + 1 \leq k \leq n, \text{ where } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{s_k\}$ is also $St_\lambda$-$\mathcal{I}_{\aleph_C}$-convergent to 0 but the sub-sequence $\{s_{k_i}\} = \{2\}$ of $\{s_k\}$ is not $St_\lambda$-$\mathcal{I}_{\aleph_C}$-convergent to 0.

**Theorem 5.** *Let $\{s_k\}$ be any sequence in a FCNS $(\mathbb{S}, \aleph_C, \circledast)$. The sequence $\{s_k\}$ is $St_\lambda$ -$\mathcal{I}_{\aleph_C}$-convergent on $\mathbb{S}$ iff there exists $\aleph_C$-convergent sequence $\{r_k\}$ satisfying*

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \left| \{ k \in I_n : s_k \neq r_k \} \right| \geq \delta \right\} \in \mathcal{I}.$$

*Proof.* Let $St_\lambda$-$\mathcal{I}_{\aleph_C}$-$\lim_{k\to\infty} s_k = s_0$. For each $u \in int(Q)$, $\varepsilon \in (0,1)$ and $\delta > 0$, consider

$$F_j = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \aleph_C(s_k - s_0, u) \geq \frac{1}{j} \right\} \right| \right\} ; j \in \mathbb{N}.$$

Then $F_j \in F(\mathcal{I})$, where $\mathcal{I}$ is an admissible ideal. There is $A \subset \mathbb{N}$ such that $A \subset F(\mathcal{I})$ and $A - F_j$ is finite for every $j \in \mathbb{N}$. Thus, $s_k \to s_0$ *i.e.* for every $\varepsilon \in (0,1)$, $u \in int(Q)$, we can find an integer $k_0 > 0$ such that $\aleph_C(s_k - s_0, t) > 1 - \varepsilon$ for all $k \geq k_0$ and $k \in A$.
Define a sequence $\{r_k\}$ in $\mathbb{S}$ as

$$r_k = \begin{cases} s_k & k \in A \\ s_0 & \text{otherwise} \end{cases} \quad \text{where } n \in \mathbb{N}.$$

The sequence $\{r_k\}$ is $\aleph_C$-convergent to $s_0$.
Then we have $\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} |\{k \in I_n : s_k \neq r_k\}| \geq \delta\right\} \in \mathcal{I}$.

Conversely, let $\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} |\{k \in I_n : s_k \neq r_k\}| \geq \delta\right\} \in \mathcal{I}$. Take $St_\lambda$ - $\mathcal{I}_{\aleph_C}$ - $\lim_{k\to\infty} s_k = 0$ and $\aleph_C$ - $\lim_{k\to\infty} s_k = s_0$. Then, for every $\varepsilon \in (0,1)$, $u \in int(Q)$, we get $\{k \in I_n : \aleph_C(s_k - s_0, u) \leq 1 - \varepsilon\} \subseteq \{k \in I_n : \aleph_C(r_k - s_0, u) \leq 1 - \varepsilon\} \cup \{k \in I_n : x_k \neq y_k\}$. Since $\aleph_C$ - $\lim_{k\to\infty} s_k = s_0$, the set $\{k \in I_n : \aleph_C(s_k - s_0, u) \leq 1-\varepsilon\} \subseteq \{k \in I_n : \aleph_C(r_k - s_0, u) \leq 1-\varepsilon\}$ involves almost finitely many terms. Also, by supposition, $\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} |\{k \in I_n : s_k \neq r_k\}| \geq \delta\right\} \in \mathcal{I}$. Hence we have $\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} |\{k \in I_n : \aleph_C(r_k - s_0, u) \leq 1 - \varepsilon\}| \geq \delta\right\} \in \mathcal{I}$. Therefore, $St_\lambda$ - $I_{\aleph_C}$ - $\lim_{k\to\infty} s_k = s_0$. □

Consider $S(\mathcal{I}_{\aleph_C})$ and $S_\lambda(\mathcal{I}_{\aleph_C})$ as the collections of ideal statistically and $\lambda$-ideal statistically convergent sequences with respect to $\aleph_C$, respectively.

**Theorem 6.** *(i) If* $\lim_{n\to\infty} inf \dfrac{\lambda_n}{n} > 0$, *then* $S(I_{\aleph_C}) \subset S_\lambda(\mathcal{I}_{\aleph_C})$.
*(ii) If* $\lim_{n\to\infty} \inf \dfrac{\lambda_n}{n} = 0$ *and* $\mathcal{I}$*-strongly admissible ideal, then* $S(\mathcal{I}_{\aleph_C}) \subsetneqq S_\lambda(\mathcal{I}_{\aleph_C})$.

*Proof.* $(i)$ For $\varepsilon \in (0,1)$,

$$\{k \leq n : \aleph_C(s_k - s_0, t) \leq 1 - \varepsilon\} \subseteq \{k \in I_n : \aleph_C(s_k - s_0, t) \leq 1 - \varepsilon\}.$$

Also,

$$\begin{aligned}\frac{1}{n}|\{k \leq n : \aleph_C(s_k - s_0, t) \leq 1 - \varepsilon\}| &\geq \frac{1}{n}|\{k \in I_n : \aleph_C(s_k - s_0, t) \leq 1 - \varepsilon\}| \\ &\geq \frac{\lambda_n}{n}\frac{1}{\lambda_n}|\{k \in I_n : \aleph_C(s_k - s_0, t) \leq 1 - \varepsilon\}|.\end{aligned}$$

If $\lim_{n\to\infty} inf \dfrac{\lambda_n}{n} = a$, where $a > 0$. As $\{s_k\} \in S(\mathcal{I}_{\aleph_C})$, then for $\delta > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \in \mathbb{N} : \aleph_C(s_k - s_0, t) \leq 1 - \varepsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

Therefore,

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : \aleph_C(s_k - s_0, t) \leq 1 - \varepsilon\}| \geq \delta\right\} \in \mathcal{I},$$

*i.e.*, $\{s_k\} \in S_\lambda(\mathcal{I}_{\aleph_C})$.

$(ii)$ Define a sequence $s = \{s_k\}$ as

$$s_k = \begin{cases} u & k \in I_{n(i)}, \, i = 1, 2, 3, \ldots \\ 0 & \text{otherwise}, \end{cases} \quad \text{where } n \in \mathbb{N},$$

where $u \in X$, $\|u\| = 1$. Then $\{s_k\}$ is statistically convergent and $\{s_k\} \in S(\mathcal{I}_{\aleph_C})$. But $\{s_k\} \notin S_\lambda(\mathcal{I}_{\aleph_C})$. □

**Theorem 7.** *If* $\lim_{n\to\infty} inf \dfrac{\lambda_n}{n} = 1$, *then* $S_\lambda(\mathcal{I}_{\aleph_C}) \subset S(\mathcal{I}_{\aleph_C})$.

*Proof.* As $\lim_{n\to\infty} inf \dfrac{\lambda_n}{n} = 1$, we select $n_0 \in \mathbb{N}$ so that it satisfies $\left|\frac{\lambda_n}{n} - 1\right| < \frac{\delta}{2}$ for all $n \geq n_0$, where $\delta > 0$. For $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\frac{1}{n}|\{k \leq n : \aleph_C(s_k - s_0, u) \leq 1-\varepsilon\}| \\
&= \frac{1}{n}|\{k \leq n-\lambda_n : \aleph_C(s_k - s_0, u) \leq 1-\varepsilon\}| \\
&\quad + \frac{1}{n}|\{k \in I_n : \aleph_C(s_k - s_0, u) \leq 1-\varepsilon\}| \\
&\leq \frac{n-\lambda_n}{n} + \frac{\lambda_n}{n}\frac{1}{\lambda_n}|\{k \in I_n : \aleph_C(s_k - s_0, u) \leq 1-\varepsilon\}| \\
&\leq 1 - (1 - \frac{\delta}{2}) + \frac{1}{\lambda_n}|\{k \in I_n : \aleph_C(s_k - s_0, u) \leq 1-\varepsilon\}| \\
&= \frac{\delta}{2} + \frac{1}{\lambda_n}|\{k \in I_n : \aleph_C(s_k - s_0, u) \leq 1-\varepsilon\}|, \ \forall\, n \geq n_0.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \aleph_C(s_k - s_0, u) \leq 1-\varepsilon\}| \geq \delta\} \\
&\quad \subset \{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : \aleph_C(s_k - s_0, u) \leq 1-\varepsilon\}| \geq \frac{\delta}{2}\} \cup \{1,2,3,\ldots,n_0\}.
\end{aligned}$$

If $\lambda\text{-}St\text{-}\lim_{k\to\infty} s_k = s_0$, then the set on the R.H.S. belongs to $\mathcal{I}$. Hence $\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \aleph_C(s_k - s_0, u) \leq 1-\varepsilon\}| \geq \delta\} \in \mathcal{I}$. Therefore, $\{s_k\}$ is $St\text{-}\mathcal{I}_{\aleph_C}$-convergent to $s_0$. □

## 3. Conclusion

In the paper, we have introduced a convergence structure called λ-ideal statistical convergence on cone metric spaces. The computational techniques with a single structure may not always be sufficient to produce better results alone although merging of two or more are able to provide much improved results. Thus, the significance of introducing ideal convergence in this structure is that resultant computational techniques will give a novel mathematical tool to deal with the convergence problems that have been motivated on the basis of practical approach by factual incompleteness, indeterminacy and inconsistency of the data. Moreover, λ-ideal statistical convergence on cone metric spaces can be explored for the different setups like double sequences, triple sequences, ideals, difference sequences and many more.

## Acknowledgements

The author is grateful to the referees and the editor for their corrections and suggestions, which have greatly improved the readability of the paper.

## References

[1] R. Antal, M. Chawla, V. Kumar, and B. Hazarika, *On $\triangle^m$-statistical convergence double sequences in intuitionistic fuzzy normed spaces*, Proyecciones **41** (2022), 697–713. DOI

[2] S. Aytar, *Rough statistical convergence*, Numer. Funct. Anal. Optim. **29** (2008), 291–303. DOI

[3] T. Bag and S. K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **11** (2003), 687–706. URL

[4] V. K. Bhardwaj and I. Bala, *On weak statistical convergence*, Int. J. Math. Math. Sci. **2007** (2007), 1–9. URL

[5] H. Çakallı, *On variations of quasi-Cauchy sequences in cone metric spaces*, Filomat **30** (2016), 603–610. DOI

[6] S. C. Cheng and J. N. Mordesen, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Calcutta. Math. Soc. **86** (1994), 429–436.

[7] N. R. Das, R. Dey, and B. C. Tripathy, *Statistically convergent and statistically Cauchy sequence in a cone metric space*, TWMS J. Pure Appl. Math. **5** (2014), 59–65. URL

[8] I. A. Demırcı, Ö. Kışı, and M. Gürdal, *On quasi I-statistical convergence of triple sequences in cone metric spaces*, Math. Pannon. **29** (2023), 87–95. DOI

[9] I. A. Demirci, Ö. Kişi, and M. Gürdal, *Rough $I^2$-statistical convergence in cone metric spaces in certain details*, Bull. Math. Anal. Appl. **15** (2023), 7–23. URL

[10] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

[11] C. Felbin, *Finite dimensional fuzzy normed linear space*, Fuzzy Sets Syst. **48** (1992), 239–248. DOI

[12] J. A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301–313.

[13] A. C. Güler, *I-convergence in fuzzy cone normed spaces*, Sahand Commun. Math. **18** (2021), 45–57. DOI

[14] L. G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332** (2007), 1468–1476. DOI

[15] M. I. Idrisi, K. Saini, M. Ahmad, and S. Aslam, *A study of fuzzy anti-$\lambda$-ideal convergent triple sequence spaces*, J. Appl. Anal. **29** (2023), 353–358. DOI

[16] M. I. Idrisi, K. Saini, and N. Rao, *On some anti $\lambda$-ideal convergent fuzzy double sequence spaces which contains $2l\infty$, $2c$ and $2c_0$*, Filomat **38** (2024), 8861–8868. URL

[17] A. K. Katsaras, *Fuzzy topological vector spaces II*, Fuzzy Sets Syst. **12** (1984), 143–154. DOI

[18] M. Kirişci and A. Karaisa, *Fibonacci statistical convergence and Korovkin type approximation theorems*, J. Inequal. Appl. **2017** (2017), 1–15. DOI

[19] Ö. Kişi and M. Gürdal, *New results on lacunary ideal convergence in fuzzy cone normed spaces*, Sigma J. Eng. Nat. Sci. **41** (2023), 1255–1263. URL

[20] P. Kostyrko, T. Šalát, and W. Wilczyński, *I-convergence*, Real analysis exchange **26** (2000), 669–686. URL

[21] I. Kramosil and J. Michálek, *Fuzzy metrics and statistical metric spaces*, Kybernetika **11** (1975), 336–344. URL

[22] M. Kucukaslan and M. Yilmazturk, *On deferred statistical convergence of sequences*, Kyungpook Math. J. **56** (2016), 357–366. DOI

[23] M. Mursaleen, *$\lambda$-statistical convergence*, Math. Slovaca, **50** (2000), 111–115. URL
[24] T. Öner, M. B. Kandemir, and B. Tanay, *Fuzzy cone metric spaces*, J. Nonlinear Sci. Appl. **8** (2015), 610–616. DOI
[25] Reena, Meenakshi, and V. Kumar, *Generalized statistical convergence of order $\alpha$ in random $n$-normed space*, Adv. Appl. Math. Sci. **18** (2019), 715–729. URL
[26] P. Tamang and T. Bag, *Some results on finite dimensional fuzzy cone normed linear space*, Ann. Fuzzy Math. Inform. 13 (2017), 123–134. URL

Department of Mathematics, Chandigarh University, Mohali, Punjab, India
*E-mail address*: reena.antal@gmail.com