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Properties of non-Berwaldian Randers metric of Douglas type on 4-dimensional hypercomplex Lie groups

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Abstract. In this paper, we first obtain the non-Berwaldian Randers metrics of Douglas type on 4-dimensional hypercomplex simply connected Lie groups. Then we give the S-curvature formulas for the non-Berwaldian Randers metric of Douglas type on these spaces. We also give some geometric properties and results on these spaces. We show that there is not any non-Berwaldian Randers metric of Douglas type on these Lie groups which have vanishing S-curvature and these spaces are never naturally reductive. Finally, we determine the geodesic vectors of Randers metrics on 4-dimensional hypercomplex simply connected Lie groups.

1. Introduction

The most natural quaternionic analogues of complex manifolds are called hypercomplex manifolds. Hyper manifolds are one of the most important manifolds in geometry, which is why many mathematicians have studied this type of manifolds. These manifolds also have many applications in physical sciences, such as black holes, supersymmetric σ-models and topological quantum field theory [1, 5].

Recently, there have been many studies on hypercomplex Lie groups. In [6], the formulas of the flag curvature are given and it is shown that, in some directions, the flag curvature of the Randers metrics and the sectional curvature of the hyper-Hermitian metrics have the same sign. In [11], the authors classify Einstein-like metrics on hypercomplex four-dimensional Lie groups and then they obtain the exact form of all harmonic maps on these

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spaces. They also calculate the energy of an arbitrary left-invariant vector field X on these spaces and determine all critical points for their energy functional restricted to vector fields of the same length. In [13], we consider invariant infinite series (α, β)-metrics. Then we describe all geodesic vectors of these spaces on the left invariant hypercomplex four dimensional simply connected Lie groups.

In this paper, we obtain the non-Berwaldian Randers metrics of Douglas type on 4-dimensional hypercomplex simply connected Lie groups. Then we give the S-curvature formulas for the non-Berwaldian Randers metric of Douglas type on these spaces. We also give some geometric properties and results about S-curvature, geodesic orbit (g. o.), naturally reductive, weakly symmetric and generalized Berwald spaces.

2. Preliminaries

Finsler geometry has its genesis in integrals of the form

$$\int_a^b F\Big(x^1, x^2, \ldots, x^n; \frac{dx^1}{dt}, \frac{dx^2}{dt}, \ldots, \frac{dx^n}{dt}\Big)dt.$$

The function $F(x^1, \ldots, x^n; y^1, \ldots, y^n)$ is positive unless all the y^i are zero. Finsler geometry also asserts itself in applications, most notably in theory of relativity, control theory and mathematical biology [13]. Let M be an n-dimensional C^∞ manifold and $TM = \cup_{x\in M} T_xM$ the tangent bundle. A Finsler metric on a manifold M is a non-negative function $F : TM \to \mathbb{R}$ with the following properties:

(1) F is smooth on the slit tangent bundle $TM_0 := TM \setminus \{0\}$;
(2) $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_xM$ and $\lambda > 0$;
(3) the $n \times n$ Hessian matrix

$$[g_{ij}] = \frac{1}{2}\Big[\frac{\partial^2 F^2}{\partial y^i \partial y^j}\Big]$$

is positive definite at every point $(x, y) \in TM_0$.

A Randers metric on a smooth n-dimensional manifold M consists of a Riemannian metric $\alpha = \sqrt{a_{ij}dx^i \otimes dx^j}$ on M and a 1-form $\beta := b_i dx^i$. Using α and β we define a function F on TM as follows [4]:

$$F(x, y) = \alpha(x, y) + \beta(x, y), \quad x \in M, y \in T_xM,$$

where F is a Finsler structure if and only if $||\beta|| := \sqrt{b_i b^i} < 1$, $b^i := a^{ij} b_j$ and (a^{ij}) is the inverse of the matrix (a_{ij}). Let $x \in M$. Then the Riemannian metric induces an inner product in the cotangent space T^*_xM in a standard way. An easy computation shows that $\langle dx^i, dx^j \rangle = a^{ij}$. This inner product defines a linear isomorphism between T^*_xM and T_xM. Through this inner product the 1-form β corresponds to a smooth vector field U on M [4]. Let $U = u^i\partial/\partial x_i$, then $u^i = \sum_{i=1}^{n} a^{ij}b_j = b^i$ and, for every $y \in T_xM$, we have

$$\langle y, U\rangle = \beta(x, y).$$

It is obvious that $||\beta|| = ||U|| < 1$. So we can write:

$$F(x, y) = \alpha(y) + \langle U, y\rangle, \quad x \in M, y \in T_xM,$$

where $\langle,\rangle$ is the inner product induced by the Riemannian metric α and $\alpha(U|_x) < 1$ for every $x \in M$.

A Finsler structure F is said to be of Berwald type if the Chern connection coefficients Γ^i_{jk} in natural coordinates have no y dependence. Indeed, a Finsler space (M, F) is called a Berwald space if the geodesic spray coefficients

$$G^i(x, y) = \frac{1}{4}g^{il}(x, y)\Big([F^2]_{x^k y^l}(x, y)y^k - [F^2]_{x^l}(x, y)\Big)$$

are quadratic in $y \in TM\{0\}$. We note that Berwald spaces are just a bit more general than Riemannian and locally Minkowskian spaces. They provide examples that are more properly Finslerian, but only slightly so. Now let

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\right).$$

We call $D := D^i_{jkl}dx^j \otimes dx^k \otimes dx^l$ the Douglas tensor. Douglas metric is characterized by the curvature equation $D = 0$. We note that all Berwald spaces are Douglas spaces, but there are many non-Berwald Douglas metrics.

3. Randers metric of Douglas type on 4-dimensional hypercomplex Lie groups

In this section, we study the structure of non-Berwaldian Randers metrics of Douglas type on four dimensional hypercomplex simply connected Lie groups. An almost complex structure on a real smooth manifold M is a tensor field J such that $J^2 = \mathbf{-1}$, where $\mathbf{1}$ denotes the identity transformation of T_xM. Assume that (M, J) is an almost complex manifold. If the Lie bracket of any two holomorphic vector fields is again a holomorphic vector field, then the almost complex structure is said to be integrable. Also, for any two vector fields X and Y, the Nijenhuis tensor N is defined as

$$N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY].$$

Definition 1. A manifold M, with three complex structures I, J and K which are globally defined and integrable, is called a *hypercomplex manifold* if

$$I^2 = J^2 = K^2 = \mathbf{-1}, \quad IJ = K = -JI.$$

We recall that, for a 4-dimensional manifold M, a hypercomplex structure on M is a family $H = \{J_\alpha\}_{\alpha=1,2,3}$ of fiber wise endomorphisms of TM satisfying

$$-J_2J_1 = J_1J_2 = J_3, \quad J_\alpha^2 = -Id_{TM}, \quad N_\alpha = 0, \quad \alpha = 1, 2, 3,$$

where N_α is the Nijenhuis tensor corresponding to J_α.

We note that, a Riemannian metric $\langle , \rangle$ on a hypercomplex manifold (M, H) is called *hyper-Hermitian* if for all vector fields X and Y on M and for all $\alpha = 1, 2, 3$ the following equation is satisfied:

$$\langle J_\alpha X, J_\alpha Y\rangle = \langle X, Y\rangle.$$

Definition 2. A hypercomplex structure $H = \{J_\alpha\}_{\alpha=1,2,3}$ on a Lie group G is said to be *left invariant* if, for any $t \in G$,

$$J_\alpha = Tl_t \circ J_\alpha \circ Tl_{t^{-1}},$$

where Tl_t is the differential function of the left translation l_t.

Suppose that G is a simply connected 4-dimensional real Lie group with a left invariant hyper-Hermitian metric. Assume that $\mathfrak{g}$ is the Lie algebra of G. In [2], Barberis showed that $\mathfrak{g}$ is either Abelian or isomorphic to one of the following Lie algebras, where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of $\mathfrak{g}$.

Case 1. The non-zero Lie brackets are

$$[e_2, e_3] = e_4, \quad [e_3, e_4] = e_2, \quad [e_4, e_2] = e_3, \quad e_1 : \text{central}.$$

Also, the non-zero connection components are

$$\nabla_{e_2}e_3 = \frac{1}{2}e_4, \quad \nabla_{e_2}e_4 = -\frac{1}{2}e_3, \quad \nabla_{e_3}e_2 = -\frac{1}{2}e_4,$$
$$\nabla_{e_3}e_4 = \frac{1}{2}e_2, \quad \nabla_{e_4}e_2 = \frac{1}{2}e_3, \quad \nabla_{e_4}e_3 = -\frac{1}{2}e_2.$$

Case 2. The non-zero Lie brackets are

$$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_2, \quad [e_1, e_4] = e_2, \quad [e_2, e_4] = -e_1. \tag{1}$$

Also, the non-zero connection components are given by

$$\begin{aligned} &\nabla_{e_1}e_1 = -e_3, \quad \nabla_{e_1}e_3 = e_1, \quad \nabla_{e_2}e_2 = -e_3, \\ &\nabla_{e_2}e_3 = e_2, \quad \nabla_{e_4}e_1 = -e_2, \quad \nabla_{e_4}e_2 = e_1. \end{aligned} \tag{2}$$

Case 3. The non-zero Lie brackets are

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = e_3, \quad [e_1, e_4] = e_4. \tag{3}$$

Also, the non-zero connection components are given by

$$\nabla_{e_2}e_1 = -e_2, \quad \nabla_{e_2}e_2 = e_1, \quad \nabla_{e_3}e_1 = -e_3,$$
$$\nabla_{e_3}e_3 = e_1, \quad \nabla_{e_4}e_1 = -e_4, \quad \nabla_{e_4}e_4 = e_1.$$

Case 4. The non-zero Lie brackets are

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = \frac{1}{2}e_3, \quad [e_1, e_4] = \frac{1}{2}e_4, \quad [e_3, e_4] = \frac{1}{2}e_2. \tag{4}$$

Also, the non-zero connection components are given by

$$\begin{aligned}
&\nabla_{e_2} e_1 = -e_2, \quad \nabla_{e_2} e_2 = e_1, \quad \nabla_{e_2} e_3 = -\frac{1}{4}e_4,\\
&\nabla_{e_2} e_4 = \frac{1}{4}e_3, \quad \nabla_{e_3} e_1 = -\frac{1}{2}e_3, \quad \nabla_{e_3} e_2 = -\frac{1}{4}e_4,\\
&\nabla_{e_3} e_3 = \frac{1}{2}e_1, \quad \nabla_{e_3} e_4 = \frac{1}{4}e_2, \quad \nabla_{e_4} e_1 = -\frac{1}{2}e_4,\\
&\nabla_{e_4} e_2 = \frac{1}{4}e_3, \quad \nabla_{e_4} e_3 = -\frac{1}{4}e_2, \quad \nabla_{e_4} e_4 = \frac{1}{2}e_1.
\end{aligned}$$

In [6], the authors showed that, Lie groups of Case 1, admit only Berwaldian Randers metrics of Douglas type. Lie groups of Case 2, admit both Berwaldian and non-Berwaldian Randers metrics of Douglas type. Lie groups of Cases 3 and 4 admit only non-Berwaldian Randers metrics of Douglas type.

Let U be a left invariant vector field. Here we obtain the exact form of non-Berwaldian Randers metrics of Douglas type F on simply connected four-dimensional hypercomplex Lie groups.

Theorem 1. *For a non-Abelian 4-dimensional hypercomplex simply connected Lie group G, a non-Berwaldian Randers metric of Douglas type F defined by a left invariant hyper-Hermitian metric g and a left invariant vector field U, the Randers metrics are given for three types below, where $y = \sum_{i=1}^{4} a_i e_i$ is a vector in the Lie algebra $\mathfrak{g}$, $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of $\mathfrak{g}$ and $a_i, m, n \in \mathbb{R}$.*

Type A. For Lie algebra $\mathfrak{g}$, corresponding to Case 2 and $U = me_3 + ne_4$, we have

$$F(y) = \sqrt{\sum_{i=1}^{4} a_i^2} + ma_3 + na_4, \text{ with } 0 < \sqrt{m^2 + n^2} < 1, m \neq 0. \tag{5}$$

Type B. For Lie algebra $\mathfrak{g}$, corresponding to Case 3 and $U = me_1$, we have

$$F(y) = \sqrt{\sum_{i=1}^{4} a_i^2} + ma_1, \text{ with } |m| < 1, m \neq 0. \tag{6}$$

Type C. For Lie algebra $\mathfrak{g}$*, corresponding to Case* 4 *and* $U = me_1$*, we have*

$$F(y) = \sqrt{\sum_{i=1}^{4} a_i^2} + ma_1, \text{ with } |m| < 1, m \neq 0. \tag{7}$$

Proof. Type A. Let the Lie algebra $\mathfrak{g}$ belong to Case 2. Then, using Theorem 3.1 in [6], we have $U = me_3 + ne_4$ with $0 < \sqrt{m^2 + n^2} < 1, m \neq 0$. Thus, for every $y = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$, $a_i \in \mathbb{R}$, we have

$$\begin{aligned} F(y) =& \sqrt{\langle y, y\rangle} + \langle U, y\rangle \\ =& \sqrt{\langle a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4, a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4\rangle} \\ &+ \langle me_3 + ne_4, a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4\rangle \\ =& \sqrt{\sum_{i=1}^{4} a_i^2} + ma_3 + na_4. \end{aligned}$$

For *Types B* and *C* we have similar proofs. □

We note that in *Type A*, if $m = 0$, the Randers metric is of Berwald type.

4. S-Curvature of left invariant Randers metrics on hypercomplex 4-dimensional Lie groups

In this section we give an explicit formula for S-curvature of left invariant Randers metrics on hypercomplex 4-dimensional Lie groups. We note that, S-Curvature is a quantity to measure the rate of change of the volume form of a Finsler space along geodesic. S-curvature is a non-Riemannian quantity, indeed, any Riemannian manifold has vanishing S-curvature. For a Lie group G with a left invariant Randers metric F which is defined by an inner product $\langle, \rangle$ on $\mathfrak{g}$, the Lie algebra of G and a left invariant vector field U, the S-curvature is given by (see [4])

$$S(e, y) = \frac{n+1}{2}\left(\frac{\big\langle [U, y], \langle y, U\rangle U - y\big\rangle}{F(y)} - \big\langle [U, y], U\big\rangle\right). \tag{8}$$

Theorem 2. *Let* (M, F) *be a Randers metric of Douglas type which is not Berwaldian and defined by the Riemannian metric* $\langle, \rangle$ *on the non-commutative* 4*-dimensional hypercomplex simply connected Lie group* G*. Assume that* U *is a left invariant vector field. Then the* S*-curvature is*

for Type A, with the non-Berwaldian Douglas metric (5)*,*

$$S(e, y) = \frac{5}{2}\left(\frac{m(a_1^2 + a_2^2)}{\sqrt{\sum_{i=1}^{4} a_i^2} + ma_3 + na_4}\right), \tag{9}$$

for Type B, with the non-Berwaldian Douglas metric (6),

$$S(e,y) = -\frac{5}{2}\left(\frac{m(a_2^2+a_3^2+a_4^2)}{\sqrt{\sum_{i=1}^4 a_i^2}+ma_1}\right), \tag{10}$$

for Type C, with the non-Berwaldian Douglas metric (7),

$$S(e,y) = -\frac{5}{4}\left(\frac{m(2a_2^2+a_3^2+a_4^2)}{\sqrt{\sum_{i=1}^4 a_i^2}+ma_1}\right), \tag{11}$$

where $y = \sum_{i=1}^4 a_i e_i$ *is a vector in the Lie algebra* $\mathfrak{g}$ *and* $a_i, m, n \in \mathbb{R}$.

Proof. Type A. As already mentioned, in this type U is of the form

$$U = me_3 + ne_4, \quad \sqrt{m^2+n^2} < 1, m \neq 0 \text{ and } m, n \in \mathbb{R}.$$

Then, for every $y = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4,\ a_i \in \mathbb{R}$, we obtain that

$$\begin{aligned} [U,y] &= [me_3 + ne_4, a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4] \\ &= (na_2 - ma_1)e_1 - (ma_2 + na_1)e_2, \\ \big\langle [U,y], U\big\rangle &= 0, \end{aligned} \tag{12}$$

and

$$\begin{aligned} \langle y, U\rangle U - y = &-a_1e_1 - a_2e_2 \\ &+ \big((m^2-1)a_3 + mna_4\big)e_3 + \big(mna_3 + (n^2-1)a_4\big)e_4. \end{aligned} \tag{13}$$

The equations (12) and (13) imply that

$$\langle [U,y], \langle y,U\rangle U - y\rangle = m(a_1^2 + a_2^2). \tag{14}$$

By replacing equations (5) and (12) to (14) in (8) gives us the result (9).

Type B. Now, U is of the form

$$U = me_1, \quad |m| < 1, m \neq 0 \text{ and } m \in \mathbb{R}.$$

Then, for every $y = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4,\ a_i \in \mathbb{R}$, we obtain that

$$\begin{aligned} [U,y] &= [me_1, a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4] \\ &= m(a_2e_2 + a_3e_3 + a_4e_4), \\ \big\langle [U,y], U\big\rangle &= 0, \end{aligned} \tag{15}$$

and

$$\langle y, U\rangle U - y = a_1(m^2-1)e_1 - a_2e_2 - a_3e_3 - a_4e_4,$$

which implies that

$$\langle [U,y], \langle y,U\rangle U - y\rangle = -m(a_2^2 + a_3^2 + a_4^2). \tag{16}$$

Replacing equations (6), (15) and (16) in the equation (8) gives us the result (10).

Type C. Now, U is of the form

$$U = me_1, \quad |m| < 1, m \neq 0 \text{ and } m \in \mathbb{R}.$$

Then for every $y = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4,\ a_i \in \mathbb{R}$, we obtain that

$$\begin{aligned} [U, y] &= [me_1, a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4] \\ &= \frac{m}{2}(2a_2e_2 + a_3e_3 + a_4e_4), \\ \big\langle [U, y], U \big\rangle &= 0, \end{aligned} \tag{17}$$

and

$$\langle y, U\rangle U - y = a_1(m^2 - 1)e_1 - a_2e_2 - a_3e_3 - a_4e_4,$$

which implies that

$$\langle [U, y], \langle y, U\rangle U - y\rangle = -\frac{m}{2}(2a_2^2 + a_3^2 + a_4^2). \tag{18}$$

Replacing equations (7), (17) and (18) in (8) gives us the result (11). □

5. Some results on 4-dimensional hypercomplex simply connected Lie groups

In the following we have the important result about vanishing S-curvature of Randers metrics on hypercomplex Lie groups.

Theorem 3. *There is not any non-Berwaldian Randers metric $F = \alpha + \beta$ of Douglas type on the 4-dimensional simply connected hypercomplex Lie groups which have vanishing S-curvature.*

Proof. Type A. Let F have the vanishing S-curvature. Then $ad(U)$ is skew symmetric with respect to the inner product $\langle , \rangle$ such that $U = me_3 + ne_4$ and $m \neq 0$. Equivalently, we have

$$\langle [U, x], y\rangle + \langle [U, y], x\rangle = 0, \quad x, y \in \mathfrak{g}. \tag{19}$$

Now by replacing (x, y) in (19) by (e_1, e_1), where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of $\mathfrak{g}$, and using (1), we get $m = 0$, that is a contradiction.

Type B and *Type C*. By an argument similar to *Type A*, let F have the vanishing S-curvature such that $U = me_1$ and $m \neq 0$. If we replace (x, y) in (19) by (e_2, e_2) and use (3) and (4), we have $m = 0$ and $U = 0$, which gives us the contradiction. □

We recall that a Finsler metric F is said to be Ricci-quadratic if its Ricci curvature $Ric(x, y)$ is quadratic with respect to y.

Theorem 4. *There is not any Randers metric $F = \alpha + \beta$ of Type A, Type B and Type C on the 4-dimensional simply connected hypercomplex Lie group which is Ricci-quadratic.*

Proof. Type A. Let $(H, F = \alpha + \beta)$ be Ricci-quadratic. Then, by using Theorem 7.9 of [4], we conclude that F is Berwald type, and by using Corollary 3.2 in [6], we get a contradiction. For *Type B* and *Type C*, the proof is similar to *Type A*. □

Remark 1. Consider Randers metrics of *type A* with the condition $m = 0$. Then F is of Berwald type and this implies that $U = ne_4$ and for all $x \in \mathfrak{g}$ we have $\nabla_x U = 0$. Thus, by using equation (2), we get $U = 0$, which is a contradiction. So there is not any Randers metric $F = \alpha + \beta$ of type (A) with condition $m = 0$ on the 4-dimensional simply connected hypercomplex Lie group which is Ricci-quadratic.

Let $M = G/K$ be a homogeneous manifold with an invariant Riemannian metric g. Then M is said to be *naturally reductive* if it admits an $Ad(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ satisfying the condition

$$\big\langle [x, y]_{\mathfrak{m}}, z \big\rangle + \big\langle y, [x, z]_{\mathfrak{m}} \big\rangle = 0, \quad x, y, z \in \mathfrak{m}, \tag{20}$$

where $\langle , \rangle$ is the bilinear form on $\mathfrak{m}$ induced by g and $[,]_{\mathfrak{m}}$ is the projection to $\mathfrak{m}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. If K is equal to $\{e\}$ then $\mathfrak{m} = \mathfrak{g}$ and this gives us that the condition (20) reduces to the condition

$$\big\langle [x, y], z \big\rangle + \big\langle y, [x, z] \big\rangle = 0, \quad X, Y, Z \in \mathfrak{g}, \tag{21}$$

for a bi-invariant Riemannian metric on G.

There are two definitions of naturally reductive Finsler metrics. The first definition has been given by Deng and Hou in [3] and the second one by Latifi in [8]. In the following, we bring these two definitions.

A homogeneous manifold G/K with an invariant Finsler metric F is called *naturally reductive* if there exists an invariant Riemannian metric g on G/K such that $(G/K, g)$ is naturally reductive and the connections of g and F coincide [3].

A homogeneous manifold G/K with an invariant Finsler metric F is called *naturally reductive* if there exists an $Ad(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ such that

$$g_y([x, u]_{\mathfrak{m}}, v) + g_y(u, [x, v]_{\mathfrak{m}}) + 2C_y([x, y]_{\mathfrak{m}}, u, v) = 0,$$

where $y \neq 0, x, u, v \in \mathfrak{m}$ [8].

Theorem 5. *Simply connected* 4*-dimensional hypercomplex homogeneous Randers manifolds of Douglas type* (H, F) *are never naturally reductive.*

Proof. Type A. Assume that the Douglas metric F is naturally reductive. Then, by Theorem 4.1 of [10], the underling Riemannian metric g is naturally reductive. Thus, by (21), we get

$$\big\langle [x, y], z \big\rangle + \big\langle y, [x, z] \big\rangle = 0, \quad \forall x, y, z \in \mathfrak{g}.$$

Now, if we replace (x, y, z) by (e_1, e_2, e_4) in the above equation, where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of $\mathfrak{g}$, and use equation (1), then we have the following contradiction:

$$\langle [e_1, e_2], e_4 \rangle + \langle e_2, [e_1, e_4] \rangle = 1 = 0.$$

For *Type* and *Type C* we have a similar proof. □

We recall that, a homogeneous Finsler space (M, F) is said to be a *g. o. space* if there exists a transitive group G of isometries such that every geodesic in M is of the form $exp(tX)p$ with $X \in \mathfrak{g}$ and $p \in M$. In [8], the author proved that if (M, F) is a g. o. Finsler space then the S-curvature vanishes.

Proposition 1. *Simply connected 4-dimensional hypercomplex homogeneous non-Berwaldian Randers manifolds of Douglas type (H, F) are not g. o. Finsler spaces.*

Proof. Type A. Let (H, F) be a g. o. Finsler space. Then, by Corollary 5.3 in [8], the S-curvature vanishes which is a contradiction with Theorem 3. For *Type B* and *Type C* we have a similar proof. □

Recall that, a connected Finsler space (M, F) is said to be a *weakly symmetric space* if, for every two points r and s in M, there exists an isometry η in the complete group of isometries $I(M, F)$ such that $\eta(r) = s$. Furthermore, a weakly symmetric Finsler space must be a g. o. Finsler space. Now we have the following corollary by using Proposition 1.

Corollary 1. *Simply connected 4-dimensional hypercomplex homogeneous non-Berwaldian Randers manifolds of Douglas type (H, F) are never weakly symmetric spaces.*

A Finsler space that has vanishing S-curvature can be seen as a generalized Berwald space. So, by using Theorem 3, we have the following result.

Corollary 2. *There is not any non-Berwaldian Randers metric $F = \alpha+\beta$ of Douglas type on the 4-dimensional simply connected hypercomplex Lie groups for which it is a generalized Berwald space.*

Theorem 6. *There is no simply connected hypercomplex Lie group with metric g that admits a bi-invariant Randers metric of Douglas type.*

Proof. Type A. Let the left invariant metric given in equation (5) be also right invariant. By using Theorem 3.2 in [9], the Randers metric F is bi-invariant if and only if for every $x, y, z \in \mathfrak{g}$, we have

$$\langle [x, y], z \rangle + \langle y, [x, z] \rangle = 0.$$

Now, if we replace (x, y, z) by (e_1, e_2, e_4) in the above equation, where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of $\mathfrak{g}$, and use (1), then we have the following contradiction:

$$\big\langle [e_1, e_2], e_4 \big\rangle + \big\langle e_2, [e_1, e_4] \big\rangle = 1 = 0.$$

For *Type B* and *Type C* we have a similar proof. □

One of the important concepts of Finsler geometry is actually geodesics. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. We recall that, a Finsler space is called *homogeneous* if the group of isometries of this space acts transitively. Assume that $(G/K, F)$ is a homogeneous Finsler space and let e be the identity of G. Then a vector $0 \neq y \in \mathfrak{g}$ is called a *geodesic vector* if the curve $exp(ty).eK$ is a geodesic of $(G/K, F)$. In the Riemannian setting, it was shown that a vector $0 \neq x \in \mathfrak{g}$ is a geodesic vector if and only if [7]

$$\langle [x, y], x \rangle = 0, \quad \forall y \in \mathfrak{g}. \tag{22}$$

Theorem 7. *Let H be a 4-dimensional hypercomplex simply connected Lie group which is equipped with a left invariant Randers metric F. Then we have*

(1) *for Case 1, $0 \neq y$ is a geodesic vector of (H, F) if and only if $y \in span\{e_1\}$;*
(2) *for Case 2, $0 \neq y$ is a geodesic vector of (H, F) if and only if $y \in span\{e_3, e_4\}$;*
(3) *for Case 3, $0 \neq y$ is a geodesic vector of (H, F) if and only if $y \in span\{e_1\}$;*
(4) *for Case 4, $0 \neq y$ is a geodesic vector of (H, F) if and only if $y \in span\{e_1\}$.*

Proof. We only prove *Case* 2. The proof of other cases is similar. By using Theorem 2.2. in [12] and equation (22), $0 \neq y$ is a geodesic vector if and only if $\langle [y, z], y \rangle = 0$, for all $z \in \mathfrak{H}$, where $\mathfrak{H}$ is the Lie algebra of H. Let $y = \sum_{i=1}^{4} a_i e_i$ and $z \in \{e_1, e_2, e_3, e_4\}$. So we have

$$\left\langle \Big[\sum_{i=1}^{4} a_i e_i, e_i \Big], \sum_{i=1}^{4} a_i e_i \right\rangle = 0.$$

Then we obtain the following system of equations:

$$\begin{cases} a_1 a_3 + a_2 a_4 = 0, \\ a_1 a_4 - a_2 a_3 = 0, \\ a_1^2 + a_2^2 = 0. \end{cases}$$

By solving the above system of equations we get $a_1 = a_2 = 0$ and this proves the claim.

□

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