ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 29, Number 1, June 2025
Available online at https://ojs.utlib.ee/index.php/ACUTM

Some classes of matrix transforms related to speed-Maddox spaces over ultrametric fields

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Abstract. Let $\mathbb{K}$ be a complete, non-trivially valued, ultrametric (or non-archimedean) field, and $\lambda = \{\lambda_n\}$ – a sequence in $\mathbb{K}$ with the property $0 < |\lambda_n| \nearrow \infty, n \to \infty$, i.e., the speed of convergence. In the present paper, the concepts of speed-Maddox space, paranormed zero-convergence, paranormed convergence and paranormed boundedness with speed λ (or shortly, paranormed λ-zero-convergence, paranormed λ-convergence and paranormed λ-boundedness) over $\mathbb{K}$ have been recalled. Let μ be another speed in $\mathbb{K}$. Necessary and sufficient conditions are found for a matrix A over $\mathbb{K}$ to transform the set of all paranormally λ-bounded sequences into the set of all paranormally μ-bounded, all paranormally μ-convergent or all paranormally μ-zero-convergent sequences.

1. Introduction

Let, throughout the paper, $\mathbb{K}$ be a complete, non-trivially valued, ultrametric (or non-archimedean) field. We also suppose that sequences, infinite series and infinite matrices have entries in $\mathbb{K}$, and indices and summation indices run from 0 to ∞, unless otherwise stated. Given an infinite matrix $A = (a_{nk})$ and a sequence $x = \{x_k\}$ by the *A-transform* of x, we mean the sequence $A(x) = \{(Ax)_n\}$, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k, n = 0, 1, 2, \ldots,$$

it being assumed that the series on the right converge. If $(Ax)_n \to s$, $n \to \infty$, we say that x is *A-summable* or *summable A* to s. In the present paper,

Received January 28, 2025.
2020 *Mathematics Subject Classification.* 40C05, 40D05, 40H05, 46S10.
Key words and phrases. Maddox spaces, matrix transforms, paranormed spaces, paranormed boundedness, paranormed convergence, paranormed zero-convergence.
https://doi.org/10.12697/ACUTM.2025.29.05
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we consider the matrix transformations between subsets of certain Maddox spaces defined by speeds of convergence.

Let $p = \{p_n\}$ be a sequence of strictly positive real numbers, and let

$$c_0(p) = \{x = \{x_n\} \,:\, \lim_n |x_n|^{p_n} = 0\},$$

$$c(p) = \{x = \{x_n\} \,:\, \lim_n |x_n - l|^{p_n} = 0 \text{ for some } l \in \mathbb{K}\},$$

$$m(p) = \{x = \{x_n\} \,:\, |x_n|^{p_n} = O(1)\}.$$

Earlier $c_0(p)$, $c(p)$ and $m(p)$ were defined over the field of complex numbers, and called as Maddox spaces (see, for example, [5, 6]). For a bounded sequence p, the Maddox spaces are also linear paranormed spaces. A short overview on these spaces has been given, for example, in [2] and [7]. Further, throughout the paper, we suppose that p is bounded. Then, similarly to Corollary 2.11 of [7] it is easy to prove that

$$c_0(p) \subset c_0,\ c(p) \subset c,\ m \subset m(p),$$

where m, c, c_0, respectively, denote the ultrametric Banach spaces of bounded, convergent and null sequences under the ultrametric norm

$$\|x\| = \sup_k |x_k|,\ x = \{x_k\} \in m, c, c_0.$$

We note that in the ultrametric setup the Maddox spaces are studied by Natarajan (see, for example, [8]).

Let further in the present paper, $\lambda = \{\lambda_n\}$ in $\mathbb{K}$ be a speed of convergence, i.e., a sequence with the property

$$0 < |\lambda_n| \nearrow \infty, n \to \infty.$$

The speed-Maddox spaces over $\mathbb{K}$ were defined as follows (see [13]):

$$(c_0(p))^\lambda = \{x = \{x_n\} \,:\, \lim_{n\to\infty} x_n = s\,(\text{say})\,\text{and}\,\{\lambda_n(x_n - s)\} \in c_0(p)\},$$

$$(c(p))^\lambda = \{x = \{x_n\} \,:\, \lim_{n\to\infty} x_n = s\,(\text{say})\,\text{and}\,\{\lambda_n(x_n - s)\} \in c(p)\},$$

$$(m(p))^\lambda = \{x = \{x_n\} \,:\, \lim_{n\to\infty} x_n = s\,(\text{say})\,\text{and}\,\{\lambda_n(x_n - s)\} \in m(p)\}.$$

We note that these spaces over the field of complex numbers are introduced in [12].

Following [13], a convergent sequence $x = \{x_n\}$ in $\mathbb{K}$ with the limit $\lim_{n\to\infty} x_n = s$ is said to be

a) *paranormally zero-convergent with speed* λ (or shortly, *paranormally* λ*-zero-convergent*) if $x \in (c_0(p))^\lambda$,

b) *paranormally convergent with speed* λ (or shortly, *paranormally* λ*-convergent*) if $x \in (c(p))^\lambda$,

c) *paranormally bounded with speed* λ (or shortly, *paranormally* λ*-bounded*) if $x \in (m(p))^\lambda$.

If $p_n \equiv 1$, then

$$c_0(p) = c_0,\ c(p) = c,\ m(p) = m.$$

Consequently in this case the paranormed convergence and the paranormed boundedness with speed reduce to the ordinary convergence and boundedness with speed studied in classical case by Kangro [3, 4] (see also [1], Chapters 8 and 9), and in ultrametric setup by Natarajan [9, 10, 11].

In [13], we started to study matrix transforms related to speed-Maddox spaces in ultrametric setup. Now we continue these studies. If X, Y are sequence spaces, we write $A = (a_{nk}) \in (X, Y)$ if $\{(Ax)_n\} \in Y$, whenever $x = \{x_k\} \in X$. Let q be another sequence of strictly positive real numbers, μ – another speed in $\mathbb{K}$, and A – a matrix with entries in $\mathbb{K}$. We shall prove necessary and sufficient conditions for A to transform $(m(p))^\lambda$ into $(c_0(q))^\mu$, $(c(q))^\mu$ or $(m(q))^\mu$, i.e., we characterize the matrix classes $((m(p))^\lambda, (c_0(q))^\mu)$, $((m(p))^\lambda, (c(q))^\mu)$ and $((m(p))^\lambda, (m(q))^\mu)$.

2. Auxiliary results

In this section, we recall a few results, which are used in proving the main results of the paper. For presenting these results throughout this section by $p = \{p_k\}$ and $q = \{q_k\}$ we denote bounded sequences with $p_k > 0$ and $q_k > 0$.

Lemma 1. *A matrix $A \in (m(p), c)$ if and only if*

$$\sup_n \left(\sup_k |a_{nk}|\, M^{\frac{1}{p_k}} \right) < \infty \text{ for all } M > 0, \tag{1}$$

$$\lim_{n\to\infty} \left(\sup_k |a_{nk} - a_k|\, M^{\frac{1}{p_k}} \right) = 0 \text{ for all } M > 0. \tag{2}$$

Moreover,

$$\lim_n (Ax)_n = \sum_k a_k x_k \tag{3}$$

for every $x = (x_k) \in m(p)$, where

$$\lim_{n\to\infty} a_{nk} = a_k,\ k = 0, 1, 2, \ldots. \tag{4}$$

Proof. We prove only (3), leaving other parts to the reader, since the proof is analogous to the proof of statement 21 of Theorem 4.13 in [7], or to the proof of statement 11 of Theorem, p. 231–234 in [2] for the classical case if $q_k \equiv 1$.

First we note that the existence of the limits in (4) follows from (2). To prove (3), let $x = (x_k)$ be an arbitrary sequence in $m(p)$. Then $|x_k|^{p_k} < M$, $k = 0, 1, 2, \ldots$ for some $M > 0$, from which it follows that

$$|x_k| < M^{\frac{1}{p_k}},\ k = 0, 1, 2, \ldots \text{ for some } M > 0. \tag{5}$$

Now,

$$(Ax)_n = \sum_{k=0}^{\infty}(a_{nk} - a_k)x_k + \sum_{k=0}^{\infty} a_k x_k \tag{6}$$

for every $x = (x_k) \in m(p)$. As

$$\left|\sum_{k=0}^{\infty}(a_{nk} - a_k)x_k\right| \leq \sup_k |a_{nk} - a_k|\,|x_k| \leq \sup_k |a_{nk} - a_k|\, M^{\frac{1}{p_k}}$$

for every $x \in m(p)$ by (5), then

$$\lim_{n\to\infty}\left|\sum_{k=0}^{\infty}(a_{nk} - a_k)x_k\right| = 0$$

for every $x \in m(p)$ by (2). Therefore, due to (6), equality (3) holds for every $x \in m(p)$, completing the proof. □

As the following results can be proved like analogues in the classical case (see [2], statements 7, 11 and 15 of Theorem in p. 231–234, or [7], statements 17, 21 and 25 of Theorem 4.13), we omit the proofs of them.

Lemma 2. *A matrix $A \in (m(p), c(q))$ if and only if condition* (1) *holds, and*

$$\lim_{n\to\infty}\left(\sup_k |a_{nk} - a_k|\, M^{\frac{1}{p_k}}\right)^{q_n} = 0 \text{ for all } M > 0, \tag{7}$$

where a_k is defined by (4).

Lemma 3. *A matrix $A \in (m(p), c_0(q))$ if and only if*

$$\lim_{n\to\infty}\left(\sup_k |a_{nk}|\, M^{\frac{1}{p_k}}\right)^{q_n} = 0 \text{ for all } M > 0. \tag{8}$$

Lemma 4. *A matrix $A \in (m(p), m(q))$ if and only if*

$$\sup_n\left(\sup_k |a_{nk}|\, M^{\frac{1}{p_k}}\right)^{q_n} < M \text{ for all } M > 0. \tag{9}$$

3. Main results

Throughout this section, let $\lambda = \{\lambda_k\}$, $\mu = \{\mu_k\}$ be speeds of convergence over $\mathbb{K}$, $p = \{p_k\}$, $q = \{q_k\}$ – bounded sequences of strictly positive real numbers, and $A = (a_{nk})$ – a matrix over $\mathbb{K}$. For proving the main results, we need one preliminary lemma.

Lemma 5. *If*

$$\lim_{k\to\infty} |\lambda_k|^{p_k} = \infty, \tag{10}$$

then for every $v = \{v_k\} \in m(p)$ there exists $x = \{x_k\} \in (m(p))^{\lambda}$ with $\lim_{k\to\infty} x_k := s$, such that

$$v_k = \lambda_k (x_k - s),\ k = 0, 1, 2, \ldots. \tag{11}$$

Proof. Let $v = \{v_k\}$ be an arbitrary sequence in $m(p)$. Then $|v_k|^{p_k} = O(1)$ by definition, and

$$\lim_{k\to\infty} \left|\frac{v_k}{\lambda_k}\right|^{p_k} = 0$$

by (10). This implies

$$\lim_{k\to\infty} \frac{v_k}{\lambda_k} = 0.$$

Defining now $x = \{x_k\}$ by

$$x_k = \frac{v_k}{\lambda_k} + s,\ k = 0, 1, 2, \ldots, \tag{12}$$

for some $s \in \mathbb{K}$, we obtain that relation (11) holds with $\lim_{k\to\infty} x_k = s$, i.e., $x \in (m(p))^\lambda$. □

Remark 1. If $\inf_k p_k > 0$, then condition (10) holds.

To formulate the main results of the paper, we need the matrix $B = (b_{nk})$ defined by

$$b_{nk} := \frac{\mu_n(a_{nk} - a_k)}{\lambda_k}, n, k = 0, 1, 2, \ldots,$$

provided that the limits (4) exist. Also, we need the special sequences

$$e_k := \{0, ..., 0, 1, 0, ...\},$$

where 1 is in the k-th position only ($k = 0, 1, 2, \ldots$), and

$$e := (1, 1, \ldots, 1, \ldots).$$

We note that $e_k, e \in (m(p))^\lambda$.

Theorem 1. *Let condition* (10) *be satisfied. Then* $A \in ((m(p))^\lambda, (c(q))^\mu)$ *if and only if*

$$A(e), A(e_k) \in (c(q))^\mu,\ k = 0, 1, 2, \ldots, \tag{13}$$

$$\sup_n \left(\sup_k \left|\frac{a_{nk}}{\lambda_k}\right| M^{\frac{1}{p_k}}\right) < \infty \ \textit{for all}\ M > 0, \tag{14}$$

$$\lim_{n\to\infty} \left(\sup_k \left|\frac{a_{nk} - a_k}{\lambda_k}\right| M^{\frac{1}{p_k}}\right) = 0 \ \textit{for all}\ M > 0, \tag{15}$$

$$\sup_n \left(\sup_k |b_{nk}|\, M^{\frac{1}{p_k}}\right) < \infty \ \textit{for all}\ M > 0, \tag{16}$$

$$\lim_{n\to\infty} \left(\sup_k |b_{nk} - b_k|\, M^{\frac{1}{p_k}}\right) = 0 \ \textit{for all}\ M > 0, \tag{17}$$

where

$$\lim_{n\to\infty} b_{nk} = b_k \ (say),\ k = 0, 1, 2, \ldots. \tag{18}$$

Proof. Necessity. Let $A = (a_{nk}) \in ((m(p))^{\lambda}, (c(q))^{\mu})$. Hence condition (13) holds, since $e, e_k \in (m(p))^{\lambda}$. As in this case $A(e_k), A(e) \in c$, then condition (4) holds, and

$$\text{there exists } \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = a \text{ (say)}. \tag{19}$$

Let $x = \{x_k\}$ be an arbitrary sequence in $(m(p))^{\lambda}$. Then, defining $\{v_k\}$ by (11), where $s = \lim_{k\to\infty} x_k$, we get (12), which implies

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k} v_k + s \sum_{k=0}^{\infty} a_{nk}. \tag{20}$$

As $\{(Ax)_n\} \in c$, from (20) we obtain by (19) that the matrix

$$A_{\lambda} := \left(\frac{a_{nk}}{\lambda_k}\right)$$

transforms this sequence $\{v_k\} \in m(p)$ into c. Using Lemma 5, we can conclude that for every $\{v_k\} \in m(p)$ there exists $x = \{x_k\} \in (m(p))^{\lambda}$ with $\lim_{k\to\infty} x_k := s$, such that relation (11) holds. Hence $A_{\lambda} \in (m(p), c)$. Therefore, by Lemma 1 and condition (19), we have that conditions (14), (15) hold, and

$$\eta := \lim_{n\to\infty} (Ax)_n = \sum_{k=0}^{\infty} \frac{a_k}{\lambda_k} v_k + sa \tag{21}$$

for every $x = \{x_k\} \in (m(p))^{\lambda}$. Using (20) and (21) we can write

$$(Ax)_n - \eta = \sum_{k=0}^{\infty} \frac{(a_{nk} - a_k)}{\lambda_k} v_k + s\left(\sum_{k=0}^{\infty} a_{nk} - a\right),$$

and so

$$\mu_n\left((Ax)_n - \eta\right) = \sum_{k=0}^{\infty} b_{nk} v_k + s\mu_n\left(\sum_{k=0}^{\infty} a_{nk} - a\right) \tag{22}$$

for every $x \in (m(p))^{\lambda}$. By assumption, $\{(Ax)_n\} \in (c(q))^{\mu}$ for every $x \in (m(p))^{\lambda}$, which implies

$$\{\mu_n\left((Ax)_n - \eta\right)\} \in c(q) \tag{23}$$

for every $x \in (m(p))^{\lambda}$. Moreover,

$$\left\{\mu_n\left(\sum_{k=0}^{\infty} a_{nk} - a\right)\right\} \in c(q), \tag{24}$$

because $A(e) \in (c(q))^{\mu}$. Consequently, from (22) we obtain $B = (b_{nk}) \in (m(p), c(q))$. Therefore conditions (16) and (17) hold by Lemma 2. We note that the existence of the limits b_k in (18) follows from (17).

Sufficiency. We suppose that conditions (13) – (17) hold. Then the limits in (4) exist, condition (19) holds by (13), and relation (20) holds for every $x \in (m(p))^{\lambda}$, where $\lim_{k\to\infty} x_k = s$ and v_k is defined by (11). We note that

$$\lim_{n\to\infty} \frac{a_{nk}}{\lambda_k} = \frac{a_k}{\lambda_k},\ k = 0, 1, 2, \ldots$$

by (4). Thus, with the help of Lemma 1 we have by (14) and (15) that $A_\lambda \in (m(p), c)$, and (21) holds for every $x \in (m(p))^{\lambda}$. In that case also (22) holds for every $x \in (m(p))^{\lambda}$ by (20) and (21). Further, relation (24) holds by (13), and, using Lemma 2, we can conclude that $B \in (m(p), c(q))$ by (16) and (17). Therefore, from (22) we conclude that (23) is also satisfied for every $x \in (m(p))^{\lambda}$, completing the proof of the theorem. □

Theorem 2. *Let condition* (10) *be satisfied. Then* $A \in ((m(p))^{\lambda}, (c_0(q))^{\mu})$ *if and only if conditions* (14) *and* (15) *hold, and*

$$A(e), A(e_k) \in (c_0(q))^{\mu},\ k = 0, 1, 2, \ldots, \tag{25}$$

$$\lim_{n\to\infty} \left(\sup_k |b_{nk}|\, M^{\frac{1}{p_k}} \right)^{q_n} = 0 \ for\ all\ M > 0.$$

Theorem 3. *Let condition* (10) *be satisfied. Then* $A \in ((m(p))^{\lambda}, (m(q))^{\mu})$ *if and only if conditions* (14) *and* (15) *hold, and*

$$A(e), A(e_k) \in (m(q))^{\mu},\ k = 0, 1, 2, \ldots, \tag{26}$$

$$\sup_n \left(\sup_k |b_{nk}|\, M^{\frac{1}{p_k}} \right)^{q_n} < \infty \ for\ all\ M > 0,$$

Theorems 2 and 3 may be proved in a similar fashion as the proof of Theorem 1. Therefore we omit them. We only note that for the proof of Theorem 2, instead of Lemma 2 we need to use Lemma 3 and it is necessary to replace (13) by (25), and in the proof of Theorem 3, instead of Lemma 2 we need to use Lemma 4 and it is necessary to replace (13) by (26).

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