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Tensor product of partial modules

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Abstract. In this article partial modules over rings and tensor product of partial modules and its properties are studied. Left and right partial modules, partial bimodules and their homomorphisms are defined. Next, partial quotient modules are defined and the fundamental homomorphism theorem for partial modules is proven. Also, the tensor product of partial modules and the tensor product of homomorphisms of partial modules is defined. Some properties of the tensor product, the existence of hom-functors and tensor functors are proven. Finally it is shown that the hom-functor and the tensor functor are adjoint functors.

1. Introduction

In this article we study partial modules over associative rings. We do not assume that the ring multiplication is commutative nor that there exists a unit element.

Partial operations arise in several very natural situations. For example, subtraction of natural numbers is a partial operation on natural numbers. Even though partial structures have been widely studied, see e.g. surveys [1, 3, 4], it seems that there are only a few specific works on partial modules. These are [6, 8, 9, 10], which consider partial modules from a category theoretical viewpoint. In this article we define partial modules in a more general way than in [10]. We note, that in this paper, if we want to emphasize that some module is classical, i.e., its action is defined everywhere, we call them *global*.

The aim of this paper is to write down the analougous definitions and results for partial modules from a previous paper by the authors [7], which considered partial acts over semigroups. Also, many results are generalizations of known results for global modules, but there are several nuances, when proving them for partial modules, which are studied and emphasized.

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In this paper, we first define partial modules, homomorphisms of partial modules and give some examples of partial modules. Then we define a partial quotient module of a partial module and prove the fundamental homomorphism theorem for partial modules. Next we define and construct a tensor product of partial modules and a tensor product of homomorphisms of partial modules. We prove some of their properties. We also show how one can make a tensor product into a global module. Then we define tensor functors of partial modules, see how these behave on short exact sequences and finally we prove the adjunction of a tensor functor and a hom-functor.

2. Definition of a partial module

We start by defining partial modules and their homomorphisms.

Definition 2.1. Let R be a ring and M be a set. A triple $(M;+,\cdot)$, where $+: M\times M\to M$ is a mapping and $\cdot : M\times R\to M$ is a partial mapping, is called a *partial right R-module* if the following conditions hold:

(M1) the pair $(M;+)$ is an abelian group with zero element 0;
(M2) $\forall r\in R\colon\quad \exists\, 0\cdot r \,\wedge\, 0\cdot r=0$;
(M3) $\forall m\in M\colon\quad \exists\, m\cdot 0_R \,\wedge\, m\cdot 0_R=0$, where 0_R is a zero element of the ring R;
(M4) $\forall m,m'\in M\,\forall r\in R\colon\quad \exists\, m\cdot r \,\wedge\, \exists\, m'\cdot r \,\wedge\, \exists\,(m+m')\cdot r \Rightarrow$ $(m+m')\cdot r=m\cdot r+m'\cdot r$;
(M5) $\forall m\in M\,\forall r,r'\in R\colon\quad \exists\, m\cdot r \,\wedge\, \exists\, m\cdot r' \,\wedge\, \exists\, m\cdot(r+r') \Rightarrow$ $m\cdot(r+r')=m\cdot r+m\cdot r'$;
(M6) $\forall m\in M\,\forall r,r'\in R\colon\quad \exists\, m\cdot r \,\wedge\, \exists\,(m\cdot r)\cdot r' \Rightarrow$ $\exists\, m\cdot(rr') \,\wedge\, (m\cdot r)\cdot r'=m\cdot(rr')$.

The partial mapping $\cdot$ is called a *partial right R-action*. We also denote the partial right R-module $(M;+,\cdot)$ by M_R. We define *partial left R-modules* and *R-actions* dually.

Note that any abelian group M can be considered as a partial (right) R-module over any ring R, if we define the partial action to be defined only if $m=0$ or $r=0$ and in this case $m\cdot r=0$ (here $m\in M$ and $r\in R$). Such an action will be called a *zero action* on M.

It is possible that a partial module M_R satisfies a stronger condition (M4′) than (M4) or a stronger condition (M5′) than (M5):

(M4′) $\forall m,m'\in M\,\forall r\in R\colon\quad \exists\, m\cdot r \,\wedge\, \exists\, m'\cdot r \Rightarrow$ $\exists\,(m+m')\cdot r \,\wedge\, (m+m')\cdot r=m\cdot r+m'\cdot r$;
(M5′) $\forall m\in M\,\forall r,r'\in R\colon\quad \exists\, m\cdot r \,\wedge\, \exists\, m\cdot r' \Rightarrow$ $\exists\, m\cdot(r+r') \,\wedge\, m\cdot(r+r')=m\cdot r+m\cdot r'$.

If conditions (M4′) and (M5′) hold simultaneously for a partial module M_R we say that this partial module is a *lax partial module* (this term was introduced in [6]). Generally, we do not assume that both conditions (M4′)

and (M5′) hold, except for Section 4, which is about factorizing partial modules.

Definition 2.2. Let R and S be rings. A quadruple $(M;+,\star,\cdot)$ is called a *partial* (S,R)*-bimodule* if ${}_SM = (M;+,\star)$ is a partial left S-module, $M_R = (M;+,\cdot)$ is a partial right R-module and for all $m \in M$, $r \in R$ and $s \in S$ we have

$$\exists\, s \star m \wedge \exists\, m \cdot r \Rightarrow \exists\,(s \star m)\cdot r \wedge \exists\, s \star (m \cdot r) \wedge (s \star m)\cdot r = s \star (m \cdot r).$$

We denote a partial (S,R)-bimodule by ${}_SM_R$. We will usually omit the symbols of partial actions $\star$ and $\cdot$, if there is no threat of confusion.

Definition 2.3. Let R be a ring and M_R and N_R be partial right R-modules. A mapping $f : M \to N$ is called a *homomorphism of partial* R*-modules* if for every $m, m' \in M$ and for every $r \in R$ the equality

$$f(m + m') = f(m) + f(m')$$

holds and also the condition

$$\exists\, mr \Rightarrow \exists\, f(m)r \wedge f(mr) = f(m)r,$$

holds.

We define homomorphisms of partial left modules dually.

Note that for every homomorphism $f\colon M_R \to N_R$ of partial (right) R-modules, we may consider its *kernel*

$$\ker f := \{m \in M \mid f(m) = 0\}$$

and its *image*

$$\operatorname{im} f := \{f(m) \mid m \in M\}.$$

Proposition 2.4. *Let M_R and N_R be partial right R-modules and let $f : M_R \to N_R$ be a homomorphism of partial right R-modules. Then the kernel* $\ker f$ *and the image* $\operatorname{im} f$ *are also partial right R-modules.*

Proof. It is easy to see that the kernel $\ker f$ is a partial right R-module with respect to the action of the partial module M_R. For the image $\operatorname{im} f$, we note that for all $m \in M$, $r, r' \in R$, if there exist products $f(m)r \in \operatorname{im} f$ and $(f(m)r)r' \in \operatorname{im} f$, then there also exists the product $f(m)(rr') \in N_R$ and therefore $f(m)(rr') = (f(m)r)r' \in \operatorname{im} f$. □

Definition 2.5. Let R and S be rings and ${}_SM_R$ and ${}_SN_R$ partial (S,R)-bimodules. A mapping $f : M \to N$ is called a *homomorphism of partial* (S,R)*-bimodules* if $f : {}_SM \to {}_SN$ is a homomorphism of partial left S-modules and $f : M_R \to N_R$ is a homomorphism of partial right R-modules.

Next we also introduce full homomorphisms and strong homomorphisms. We need these stronger notions of homomorphisms for some upcoming results.

Definition 2.6. Let M_R and N_R be partial right R-modules and a mapping $f : M_R \to N_R$ be a homomorphism of partial right R-modules. The mapping f is called a *full homomorphism* if for every $m \in M$ and $r \in R$ we have

$$\exists\, f(m)r \Rightarrow (\exists\, m' \in M : f(m) = f(m') \wedge \exists\, m'r).$$

Definition 2.7. [5, page 81] Let M_R and N_R be partial right R-modules and a mapping $f : M_R \to N_R$ be a homomorphism of partial right R-modules. The homomorphism f is called *strong* if for all $m \in M$ and $r \in R$ we have

$$\exists\, f(m)r \Rightarrow \exists\, mr.$$

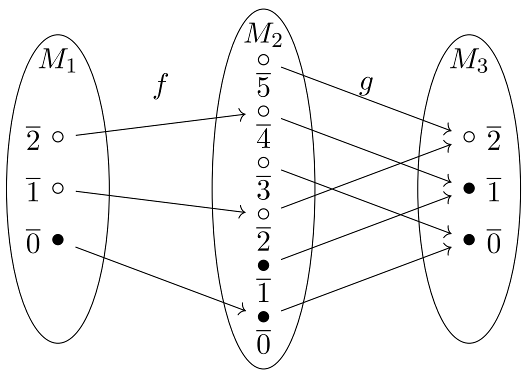
We note that every strong homomorphism is a full homomorphism and every full homomorphism is a homomorphism.

We can form different categories with partial modules and their homomorphisms.

Proposition 2.8. *Partial right (left) R-modules ((S,R)-bimodules) with the respective (strong) homomorphisms of partial modules form a category.*

We denote the category of partial right R-modules (where morphisms are homomorphisms of partial modules) by PMod_R, the category of partial left R-modules by ${}_R\mathsf{PMod}$ and the category of partial (S,R)-bimodules by ${}_S\mathsf{PMod}_R$. We also denote the category of partial right R-modules, where morphisms are strong homomorphisms by $\mathsf{PMod}^{\mathsf{s}}_R$. Similarly we have the category ${}_R\mathsf{PMod}^{\mathsf{s}}$. For the categories of partial modules satisfying condition (M4$'$) we will use notations PMod'_R and ${}_R\mathsf{PMod}'$. Clearly, the category of global R-modules Mod_R is a full subcategory of PMod_R, PMod'_R and $\mathsf{PMod}^{\mathsf{s}}_R$.

Example 2.9. Consider the abelian groups $M_1 = M_3 = \mathbb{Z}_3$ and $M_2 = \mathbb{Z}_6$ with respect to addition and the ring $R = \mathbb{Z}$. Let M_1 be a partial R-module with zero action. Let M_2 and M_3 be partial R-modules with zero action and additionally $\overline{1} \cdot 1 = \overline{1}$. Now we have the following homomorphisms $f\colon M_1 \to M_2$ and $g\colon M_2 \to M_3$ shown in the following diagram (the nodes, which can be expressed as $m = mr$ are filled, all other nodes are empty):



It is easy to see that f and g are both full homomorphisms. In M_3 we have

$$\overline{1} = \overline{1} \cdot 1 = g(\overline{4}) \cdot 1 = g(f(\overline{2})) \cdot 1 = (g \circ f)(\overline{2}) \cdot 1.$$

Yet there exists no $m \in M_1$ such that $(g \circ f)(m) = \overline{1}$ and also exists a product $m1$. Hence, $g \circ f$ is not full. Therefore, partial modules with full homomorphisms do not form a category.

Proposition 2.10. *The category* PMod_R *is a complete category.*

Proof. The category PMod_R has products. Indeed, let $(M_R^i)_{i\in I}$ be a family of partial right R-modules. The Cartesian product

$$\prod_{i\in I} M_R^i = \left\{(m_i)_{i\in I} \mid m_i \in M^i\right\}$$

is a partial R-module if we define the product $(m_i)_{i\in I} \cdot r$ if and only if there exists the product $m_i r$ for every $i \in I$. In such case we take $(m_i)_{i\in I} \cdot r = (m_i r)_{i\in I}$. The canonical projections

$$p_j : \prod_{i\in I} M_R^i \to M_R^j, \quad (m_i)_{i\in I} \mapsto m_j$$

are homomorphisms of partial modules. It is easy to see that the partial module $\prod_{i\in I} M_R^i$ with the canonical projections p_j, is the product (in the category-theoretic sense) of the family $(M_R^i)_{i\in I}$.

The category PMod_R also has equalizers. Let M_R, N_R be partial right R-modules and $f, g : M_R \to N_R$ homomorphisms of partial modules. The set

$$E = \{m \in M \mid f(m) = g(m)\}$$

is a partial right R-module with respect to the R-action of the partial module M_R. The mapping

$$e : E_R \to M_R, \quad m \mapsto m$$

is a homomorphism of partial modules. It is straightforward to check that the partial module E_R with the mapping e is the equalizer of the homomorphisms f and g.

In conclusion, since the category PMod_R has products and equalizers, it is a complete category by Theorem 2.8.1 in [2]. □

It turns out that the category PMod'_R has several nice properties.

Proposition 2.11. *The category* PMod'_R *is a complete category.*

Proof. The same constructions for products and equalizers as in the category PMod_R also work for the category PMod'_R. □

Corollary 2.12. *The category* PMod'_R *is an additive category.*

Proof. Let M_R and N_R be partial right R-modules satisfying condition (M4′) and $f, g : M_R \to N_R$ be their homomorphisms. Let us show that

$$f + g : M_R \to N_R, \quad m \mapsto f(m) + g(m),$$

is also a homomorphism between partial modules. Take $m \in M$ and $r \in R$ so that the product mr exists. Then there exist products $f(m)r = f(mr)$ and $g(m)r = g(mr)$. The partial module N_R satisfies condition (M4′), therefore there exists the product $(f(m) + g(m))r$ and we can write

$$\begin{aligned}(f+g)(mr) &= f(mr) + g(mr) = f(m)r + g(m)r = (f(m) + g(m))r \\ &= (f+g)(m)r.\end{aligned}$$

Hence, $f + g$ is also a homomorphism between M_R and N_R. It is easy to see that $\mathrm{Hom}(M_R, N_R)$ is an abelian group. Therefore, the category PMod'_R is preadditive.

We note that the zero object of the category PMod'_R is $M_R = \{0\}$ with zero action.

In conclusion, the category PMod'_R is an additive category. □

Finally we define polite partial bimodules.

Definition 2.13. Let S and R be rings. A partial (S, R)-bimodule $(M;+,\star,\cdot)$ is called *left polite* if for all $m \in M \setminus \{0\}$, $r \in R \setminus \{0\}$ and $s \in S \setminus \{0\}$ we have

$$\exists\, m \cdot r \,\wedge\, \exists\, s \star (m \cdot r) \Rightarrow \exists\, s \star m \,\wedge\, \exists\, (s \star m) \cdot r.$$

A partial bimodule ${}_SM_R$ is called *right polite* if the converse of this implication holds. We say that ${}_SM_R$ is *polite* if it is left polite and right polite.

3. Examples of partial modules

In this section we give some examples of partial modules over rings.

Example 3.1. The abelian group $(\mathbb{Z}; +)$ is a partial right module over the ring $(\mathbb{Q}; +, \cdot)$ if the action is the multiplication of integers. In other words, we say that $z \cdot q$, where $z \in \mathbb{Z}$ and $q \in \mathbb{Q}$, is defined if and only if the product zq is an integer and then we take $z \cdot q = zq$. For all $q \in \mathbb{Q}$ we have $0q = 0 \in \mathbb{Z}$ and for all $z \in \mathbb{Z}$ we have $z0 = 0 \in \mathbb{Z}$. Therefore $\mathbb{Z}_{\mathbb{Q}}$ satisfies conditions (M2) and (M3) in the definition of a partial module. Thanks to the field properties of $\mathbb{Q}$, conditions (M4)–(M6) also hold for $\mathbb{Z}_{\mathbb{Q}}$.

Example 3.2. We can also consider a generalization of the previous example. Let R be a ring and M be a subgroup of $(R; +)$. Then M_R is a partial right R-module if for all $m \in M$ and for all $r \in R$ the product $m \cdot r$ is defined if and only if $m \cdot r \in M$. In this situation, we take $m \cdot r = mr$.

Example 3.3. Let S be a ring, $R = \mathrm{Mat}_n(S)$ and M be a subgroup of the abelian group $(\mathrm{Mat}_{m,n}(S);+)$. We obtain a partial right R-module M_R if for all $A \in M$ and for all $B \in R$ the product $A \cdot B$ is defined if and only if $AB \in M$ in the usual sense of matrix multiplication. Notice that the dimensions of the matrices A and B are suitable for the multiplication. For every $B \in R$ we have $0B = 0 \in M$ and if $0 \in R$ we have $A0 = 0 \in M$ for every $A \in M$. Hence M_R satisfies conditions (M2) and (M3) in the definition of a partial module. Matrix multiplication is associative and distributive, therefore conditions (M4)–(M6) are also satisfied.

We notice that there are many different suitable subgroups M. For example, all same-dimensional matrices, where there are zeros at positions with fixed indices and at all other positions there are arbitrary elements of S, form an additive subgroup.

Example 3.4. Let V be a vector space over a field K and U be a subspace of V. Consider the ring $R = (\mathrm{End}(V);+,\circ)$ with pointwise addition and composition of mappings. We obtain a partial left R-module ${}_RU$ if for all $f \in R$ and $u \in U$ the product $f \cdot u$ is defined if and only if $f(u) \in U$. In this case we take $f \cdot u = f(u)$. Denote the zero mapping $\mathbf{0}\colon V \to V$, $v \mapsto 0$. Since $\mathbf{0} \in R$, $\mathbf{0}(u) = 0 \in U$ for all $u \in U$ and $f(0) = 0 \in U$ for all $f \in R$, the partial module ${}_RU$ satisfies conditions (M2) and (M3) in the definition of a partial module. Let us show that the other conditions also hold.

(M4) If $u, u' \in U$ and $f \in R$ are such that $f(u) \in U$ and $f(u') \in U$, then
$$f(u+u') = f(u) + f(u') \in U.$$
(M5) If for $f, f' \in R$ and $u \in U$ we have $f(u) \in U$ and $f'(u) \in U$, we also have
$$(f+f')(u) = f(u) + f'(u) \in U.$$
(M6) If $f(u) \in U$ and $g(f(u)) \in U$, then $(g \circ f)(u) = g(f(u)) \in U$.

Example 3.5. The previous example also works if K is a ring, V is a (global) K-module and U is a submodule of the module V.

Notice that all these examples even satisfy conditions (M4′) and (M5′).

Example 3.6. Take $R = \mathbb{Z}$ and $M = \mathbb{Z}_3$. We can consider a partial right R-module $(M,*)$ with the following action. For all $z \in R$ the products $\overline{0} * z = \overline{0}$ and $\overline{1} * z = \overline{z}$ are defined. In the case of $\overline{2} \in M$ we say, that $\overline{2} * z$, $z \in R$, is defined if and only if $z = 0$ and take $\overline{2} * 0 = \overline{0}$. Conditions (M1)-(M5) hold due to the properties of integers. Let us check condition (M6). Let the products $x * z_1$ and $(x * z_1) * z_2$, where $z_1, z_2 \in R$ and $x \in M$, be defined. There are two possibilities. If $z_1 = 0$, then $z_1z_2 = 0$ and therefore $x * (z_1z_2)$ is defined. If $z_1 \neq 0$, then $x \neq \overline{2}$ and thus $x * (z_1z_2)$ is defined again. Hence condition (M6) holds. Notice that condition (M4′) does not hold for this partial module. Clearly the product $\overline{1} * 4 = \overline{1}$ exists, but

$$\overline{1} * 4 + \overline{1} * 4 \neq (\overline{1} + \overline{1}) * 4 = \overline{2} * 4$$

since the product $\overline{2} * 4$ does not exist.

Example 3.7. Reverse the roles of M and R in the previous example and take $R = \mathbb{Z}_3$ and $M = \mathbb{Z}$. We can consider a partial right R-module $(M, *)$ with the following action. The product $m * r$, where $m \in M$, $r \in R$, exists if $m = 0$ or $r = \overline{0}$ and in these cases we take $m * r = 0$. The product $m * r$ also exists if $m = 1$ and $r = \overline{1}$ and in this case we take $m * r = 1$. All other options are undefined. Again, conditions (M1)–(M5) are satisfied due to the properties of integers. For condition (M6) assume that the products $m * r_1$ and $(m * r_1) * r_2$, $m \in M$, $r_1, r_2 \in R$, exist. If $m = 0$ or $r_1 = \overline{0}$ or $r_2 = \overline{0}$, then clearly there exists $m * (r_1 r_2) = 0$. The other option is $m = 1$ and $r_1 = r_2 = \overline{1}$. Then also $m * (r_1 r_2) = 1$ exists. Hence condition (M6) is satisfied. In this example we see that the partial module in question does not satisfy condition (M5′). Obviously the product $1 * \overline{1}$ exists, but

$$1 * \overline{1} + 1 * \overline{1} \neq 1 * (\overline{1} + \overline{1}) = 1 * \overline{2},$$

since the product $1 * \overline{2}$ is not defined.

4. Factorizing and the fundamental homomorphism theorem for partial modules

In this section, we consider factorizing partial modules and then formulate and prove the fundamental homomorphism theorem for partial modules. We start with defining a partial quotient module of a partial module. Recall that lax partial modules are partial modules that satisfy conditions (M4′) and (M5′) simultaneously.

Definition 4.1 ([5], p. 82)**.** Let R be a ring, M_R a lax partial right R-module and $N \subseteq M$ a subset that forms a global R-module under the operations of M_R. We call the partial module

$$M/N := \{[m] \mid m \in M\},$$

where

$$[m] := m + N = \{m + n \mid n \in N\},$$

a *partial quotient module* of a partial module M_R by a global module N if for all $m, m' \in M$

$$[m] + [m'] := [m + m']$$

and for all $m \in M$ and $r \in R$ the product $[m]r$ is defined if and only if there exists $m' \in M$ such that $[m] = [m']$ and the product $m'r$ is defined. In such a case we define

$$[m]r := [m'r].$$

Notice that for every partial module there exists at least one global submodule and it is $\{0\}$. The following lemma, which is a direct generalization of the well-known description of equality in the quotient module M/N, holds.

Lemma 4.2. *Let M_R be a lax partial R-module, $N \subseteq M$ a global R-module under the operations of M_R and $m, m' \in M$. Then for $[m]$,$[m'] \in M/N$ we have $[m] = [m']$ if and only if $m - m' \in N$.*

We show that a partial quotient module M/N is indeed a partial R-module.

Lemma 4.3. *If M_R is a lax partial R-module and $N \subseteq M$ is a global R-module with respect to the operations of M_R, then M/N is a lax partial right R-module.*

Proof. First we show that the operations on a partial quotient module are well-defined. The fact that addition is well defined follows from the fact that the operation of quotient groups is well defined.

Let $m, m', l, l' \in M$ be such that $[m] = [m']$ and $[l] = [l']$. Then, according to Lemma 4.2, we have $m - m' \in N$ and $l - l' \in N$. Also, let $r \in R$. Then

$$\exists [m]r \Leftrightarrow (\exists m'' \in M : [m'] = [m] = [m''] \wedge \exists m''r) \Leftrightarrow \exists [m']r.$$

Hence the product $[m]r$ exists if and only if the product $[m']r$ exists and in that case $[m]r = [m''r] = [m'r]$. It means that the partial R-action on the partial quotient module is also well defined.

It is easy to see that a partial quotient module M/N satisifies conditions (M1)–(M3) in the definition of a partial module. Let us show that conditions (M4′), (M5′) and (M6) also hold. Let $[m], [m'] \in M/N$ and $r \in R$ be such that the products $[m]r$ and $[m']r$ exist. Thus there exist $l, l' \in M$ such that the products lr, $l'r$ are defined and $[l] = [m]$, $[l'] = [m']$. Since the condition (M4′) holds for the partial module M_R, the product $(l + l')r = lr + l'r$ also exists. Therefore the product $[l + l']r = ([l] + [l'])r = ([m] + [m'])r$ is defined and the equalities

$$\begin{aligned}[m]r + [m']r = [lr] + [l'r] = [lr + l'r] = [(l + l')r] = [l + l']r = ([l] + [l'])r \\ = ([m] + [m'])r\end{aligned}$$

hold. Hence condition (M4′) holds.

Now let $[m] \in M/N$ and $r, r' \in R$ be such that the products $[m]r$ and $[m]r'$ are defined. Then there exist $l, l' \in M$ such that the products lr, lr' exist and $[l] = [m] = [l']$. By Lemma 4.2, we have $n \coloneqq l - l' \in N$. Since N is a global module, the product nr' exists. Also the product $(n + l')r'$ exists because M_R satisfies condition (M4′). Therefore

$$nr' + l'r' = (n + l')r' = (l - l' + l')r' = lr',$$

and hence the product lr' is defined. Since condition (M5′) holds for M_R, the product $l(r+r') = lr + lr'$ exists. So the product $[l](r+r') = [m](r+r')$ also exists and the equalities

$$[m](r + r') = [l](r + r') = [l(r + r')] = [lr + lr'] = [lr] + [lr'] = [l]r + [l]r' \\ = [m]r + [m]r'$$

hold. Therefore M/N satisfies condition (M5′).

Finally, let $[m] \in M$ and $r, r' \in R$ be such that the products $[m]r$, $([m]r)r'$ exist. Then there exist $l, l' \in M$ such that the products lr, $l'r'$ exist and $[l] = [m]$, $[l'] = [m]r = [lr] = [l]r$. By Lemma 4.2, we have $n := l' - lr \in N$. Since N is a global module, the products nr' and $(-n)r'$ exist and $(-n)r' = -nr' \in N$. Also the product $(l' - n)r'$ exists because M_R satisfies condition (M4′). Now

$$l'r' + (-n)r' = (l' - n)r' = (l' - l' + lr)r' = (lr)r'$$

and hence the product $(lr)r'$ is defined. Therefore the product $l(rr')$ exists by condition (M6). Hence the product $[l](rr') = [m](rr')$ is defined and

$$([m]r)r' = [lr]r' = [(lr)r'] = [l(rr')] = [l](rr') = [m](rr').$$

It means that the partial quotient module M/N satisfies condition (M6). So M/N is indeed a partial right R-module, which satisfies conditions (M4′) and (M5′). □

It is easy to see that the following two lemmas hold.

Lemma 4.4. *Let R be a ring, M_R and N_R be partial right R-modules and $f : M_R \to N_R$ a strong homomorphism of partial modules. Then* $\ker f$ *is a global R-module with respect to the operations of M_R.*

Lemma 4.5. *Let R be a ring, M_R be a lax partial right R-module and $N \subseteq M$ a global R-module under the operations of M_R. Then the mapping*

$$\pi_N : M \to M/N, \quad m \mapsto [m],$$

which is called a natural projection, *is a surjective full homomorphism of partial modules.*

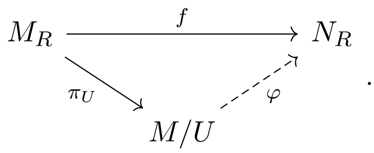
Now we are ready to formulate and prove the *fundamental homomorphism theorem for partial modules.*

Theorem 4.6. *Let R be a ring, M_R, N_R be partial right R-modules and $f : M_R \to N_R$ a strong homomorphism of partial R-modules. Assume that M_R is a lax partial module. Let $U \subseteq M$ be such that $U \subseteq \ker f$ and U is a global R-module with respect to the operations of M_R. Then there exists a unique strong homomorphism of partial modules $\varphi : M/U \to N$ such that*

$$f = \varphi \circ \pi_U.$$

Also the equalities $\operatorname{im} \varphi = \operatorname{im} f$ *and* $\ker \varphi = (\ker f)/U$ *hold.*

This theorem can be illustrated by the following commutative diagram:



Proof. Let R be a ring, M_R, N_R partial right R-modules and $f : M_R \to N_R$ a strong homomorphism of partial R-modules. Assume that M_R satisfies conditions (M4′) and (M5′). Let $U \subseteq M$ be such that $U \subseteq \ker f$ and U is a global R-module with respect to the operations of M_R. Define a mapping

$$\varphi : M/U \to N, \quad [m] \to f(m).$$

It is easy to see that φ is a well-defined homomorphism of abelian groups. Let $[m] \in M/N$. If $r \in R$ is such that the product $[m]r$ exists, then there exists $m'' \in M$ so that the product $m''r$ is defined and the equalities $[m] = [m'']$, $[m] = [m''r]$ hold. Therefore the product $f(m'')r$ also exists and

$$\varphi([m]r) = \varphi([m''r]) = f(m''r) = f(m'')r = \varphi([m''])r = \varphi([m])r.$$

Even more, if $[m] \in M/N$ and $r \in R$ are such that the product $\varphi([m])r = f(m)r$ exists, then there also exists the product mr because f is a strong homomorphism. Therefore φ is a strong homomorphism of partial R-modules. Clearly $f = \varphi \circ \pi_U$ holds. The uniqueness of the homomorphism φ follows from the fundamental homomorphism theorem for abelian groups.

Clearly the equalities

$$\operatorname{im} \varphi = \{\varphi([m]) = f(m) \mid m \in M\} = \operatorname{im} f$$

and

$$\begin{aligned} \ker \varphi &= \{[m] \mid m \in M, \varphi([m]) = 0\} = \{m + U \mid m \in M, f(m) = 0\} \\ &= \{m + U \mid m \in \ker f\} = (\ker f)/U \end{aligned}$$

hold. This completes the proof. □

Finally, we note that using the construction of quotient abelian groups does not seem to work for partial submodules and therefore a global submodule is used for the factorization. However, the authors believe that the construction for partial quotient modules with fewer assumptions exists, but it likely differs from the construction of quotient abelian groups.

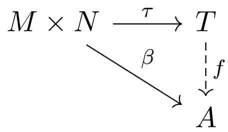
5. Tensor product of partial modules

In this section we consider a tensor product of partial modules and its properties. Note that in this section we do not assume any stronger conditions than in Definition 2.1, if not necessary. We start by defining a tensorial mapping and a tensor product.

Definition 5.1. Let R be a ring, ${}_RN$ a partial left and M_R a partial right R-module. Let A be an abelian group. A mapping $\beta : M \times N \to A$ is called *R-tensorial* if the following conditions hold:

(1) $\forall m_1, m_2 \in M\, \forall n \in N\colon \quad \beta(m_1 + m_2, n) = \beta(m_1, n) + \beta(m_2, n)$;
(2) $\forall m \in M\, \forall n_1, n_2 \in N\colon \quad \beta(m, n_1 + n_2) = \beta(m, n_1) + \beta(m, n_2)$;
(3) $\forall m \in M\, \forall n \in N\, \forall r \in R\colon \quad \exists\, mr \wedge \exists\, rn \Rightarrow \beta(mr, n) = \beta(m, rn)$.

Definition 5.2. Let R be a ring, ${}_RN$ a partial left and M_R a partial right R-module. An abelian group T with R-tensorial mapping $\tau : M \times N \to T$ is called a *tensor product* of partial modules M_R and ${}_RN$ if for every abelian group A and for every R-tensorial mapping $\beta : M \times N \to A$ there exists a unique group homomorphism $f : T \to A$ such that $\beta = f \circ \tau$, i.e. the diagram



commutes.

It is easy to see that a tensor product of partial modules is unique up to an isomorphism.

Proposition 5.3. *If (T, τ) and (T', τ') are two tensor products of partial modules M_R and ${}_RN$ then there exists a group isomorphism $f : T \to T'$ such that $\tau' = f \circ \tau$.*

Therefore we can use the following notions for the tensor product (T, τ) of partial modules M_R and ${}_RN$:

$$M \otimes_R N := T, \quad \otimes := \tau, \quad m \otimes n := \tau(m, n).$$

We call the expressions $m \otimes n$, $m \in M$, $n \in N$, *elementary tensors.*

Next we construct the tensor product of two partial modules. Let R be a ring, ${}_RN$ a partial left and M_R a partial right R-module. We may consider the tensor product of abelian groups $M \otimes_{\mathbb{Z}} N$ (see [11])[1]. Also, consider the subgroup $H \subseteq M \otimes_{\mathbb{Z}} N$, which is the smallest subgroup generated by the set

$$\{mr \otimes_{\mathbb{Z}} n - m \otimes_{\mathbb{Z}} rn \mid m \in M, n \in N, r \in R, \exists mr, \exists rn\}.$$

Now denote the quotient group by

$$T := (M \otimes_{\mathbb{Z}} N)/H$$

and define a mapping

$$\tau : M \times N \to T, \qquad (m, n) \mapsto [m \otimes_{\mathbb{Z}} n].$$

[1] We will use the subscript $\mathbb{Z}$ to emphasize considering tensors in the tensor product of abelian groups, e.g. $m \otimes_{\mathbb{Z}} n \in M \otimes_{\mathbb{Z}} N$.

Proposition 5.4. *The pair (T, τ) is a tensor product of partial modules M_R and ${}_RN$.*

Proof. Let us consider the pair $((M \otimes_{\mathbb{Z}} N)/H, \tau)$. Then for all $m \in M$, $n \in N$ and $r \in R$ if the products mr and rn exist, we have

$$\tau(mr, n) = [mr \otimes_{\mathbb{Z}} n] = [m \otimes_{\mathbb{Z}} rn] = \tau(m, rn).$$

Since $\otimes_{\mathbb{Z}}$ is additive in both arguments, the mapping τ is R-tensorial.

Let now A be an abelian group and $\beta : M \times N \to A$ an R-tensorial mapping. Consider the well-defined homomorphism of abelian groups

$$\bar{\varphi}\colon M \otimes_{\mathbb{Z}} N \to A, \qquad m \otimes_{\mathbb{Z}} n \mapsto \beta(m, n).$$

Let $m \in M$, $n \in N$, $r \in R$ be such that mr and rn exist. Then

$$\begin{aligned}\bar{\varphi}(mr \otimes_{\mathbb{Z}} n - m \otimes_{\mathbb{Z}} rn) &= \bar{\varphi}(mr \otimes_{\mathbb{Z}} n) - \bar{\varphi}(m \otimes_{\mathbb{Z}} rn) = \beta(mr, n) - \beta(m, rn)\\ &= \beta(m, rn) - \beta(m, rn) = 0.\end{aligned}$$

Hence $H \subseteq \ker \bar{\varphi}$, and, by the fundamental homomorphism theorem of groups, there exists a unique homomorphism of groups $\varphi\colon T \to A$ such that $\beta = \varphi \circ \tau$. □

Since the tensor product is unique up to an isomorphism, we can use the previous construction of the tensor product from now on. Next we give a description for the elements of the tensor product.

Proposition 5.5. *Let R be a ring, M_R a partial right and ${}_RN$ a partial left R-module. Every element ν of the tensor product $M \otimes_R N$ can be expressed as*

$$\nu = \sum_{k=1}^{k^*} m_k \otimes n_k,$$

where $k^* \in \mathbb{N}$ and $m_k \in M$, $n_k \in N$ for all $k = 1, \dots, k^*$. The following properties hold:

(1) $\forall m_1, m_2 \in M \, \forall n \in N : \quad (m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$;
(2) $\forall m \in M \, \forall n_1, n_2 \in N : \quad m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$;
(3) $\forall m \in M \, \forall n \in N \, \forall r \in R : \quad \exists\, mr \wedge \exists\, rn \Rightarrow mr \otimes n = m \otimes rn$;
(4) *$0 \otimes 0$ is the zero element of the abelian group $M \otimes_R N$ and $0 \otimes 0 = m \otimes 0 = 0 \otimes n$ for all $m \in M$ and $n \in N$;*
(5) $\forall m \in M \, \forall n \in N : \quad -(m \otimes n) = (-m) \otimes n = m \otimes (-n)$.

Proof. Let us consider the tensor product $M \otimes_R N$ of partial R-modules M_R and ${}_RN$ and its element ν. Since $\nu \in (M \otimes_{\mathbb{Z}} N)/H$, where H is the group used in Proposition 5.4, we can express ν as

$$\nu = \left[\sum_{k=1}^{k^*} m_k \otimes_{\mathbb{Z}} n_k\right] = \sum_{k=1}^{k^*} [m_k \otimes_{\mathbb{Z}} n_k] = \sum_{k=1}^{k^*} m_k \otimes n_k.$$

Next we show that the properties hold. Properties (1), (2) and (3) hold because the mapping $\tau : M \times N \mapsto M \otimes_R N$, $(m,n) \mapsto m \otimes n$, is an R-tensorial mapping.

Now for all $m \in M$ and $n \in N$ we have

$$\begin{aligned} m \otimes n + 0 \otimes 0 &= m \otimes n + 0 \otimes (0n) = m \otimes n + (0 \cdot 0) \otimes n \\ &= m \otimes n + 0 \otimes n = (m+0) \otimes n = m \otimes n. \end{aligned}$$

Therefore $0 \otimes 0$ is the zero element of the abelian group $M \otimes_R N$ and $0 \otimes 0 = 0 \otimes n$ for all $n \in N$ and similarly $0 \otimes 0 = m \otimes 0$ for all $m \in M$. Hence property (4) holds.

Finally, for all $m \in M$ and $n \in N$ we have

$$m \otimes n + (-m) \otimes n = (m + (-m)) \otimes n = 0 \otimes n = 0 \otimes 0.$$

Thus $-(m \otimes n) = (-m) \otimes n$ and similarly $-(m \otimes n) = m \otimes (-n)$. Therefore property (5) also holds. □

We now know that we can represent every element of a tensor product as a sum of elementary tensors (note that this representation is not unique). Next, it is easy to see how to represent the unique group homomorphism $\overline{\beta}$ used in the construction of a tensor product.

Lemma 5.6. *Let R be a ring, M_R and ${}_RN$ partial R-modules, A an abelian group and $\beta : M \times N \to A$ an R-tensorial mapping. Then*

$$\overline{\beta} : M \otimes_R N \to A, \quad \sum_{k=1}^{k^*} m_k \otimes n_k \mapsto \sum_{k=1}^{k^*} \beta(m_k, n_k),$$

is a group homomorphism.

The previous lemma is useful in situations where we need to define a mapping $\overline{\beta}$ from the tensor product to some abelian group: it suffices to define an R-tensorial mapping β on a set $M \times N$.

Next we show that under certain assumptions we can view the tensor product as a global module.

Proposition 5.7. *Let R, S be rings, ${}_RM_S$ a partial left polite (R,S)-bimodule, ${}_RM$ a global left R-module and ${}_SN$ a partial left S-module. Then ${}_R(M \otimes_S N)$ is a global left R-module with respect to the action*

$$r\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right) := \sum_{k=1}^{k^*} r m_k \otimes n_k,$$

where $\sum_{k=1}^{k^*} m_k \otimes n_k \in M \otimes_S N$.

Proof. Let R, S be rings, ${}_RM_S$ a partial left polite (R,S)-bimodule, ${}_RM$ a global left R-module and ${}_SN$ a partial left S-module. Let us consider the abelian group $(M \otimes_S N; +)$. For every $r \in R$ define a mapping

$$\beta_r : M \times N \to M \otimes_S N, \quad \beta_r(m,n) = rm \otimes n.$$

Notice that we can define such a mapping for every $r \in R$ because ${}_RM$ is a global module. Fix $r \in R$. It is easy to see that β_r is additive in both arguments. If $m \in M$, $n \in N$ and $s \in S$ are such that the products ms and sn exist, then there exists $r(ms)$ thanks to globality of ${}_RM$ and there exists $(rm)s$ and $(rm)s = r(ms)$ thanks to left politeness of ${}_RM_S$. Therefore

$$\beta_r(ms,n) = r(ms) \otimes n = (rm)s \otimes n = rm \otimes sn = \beta_r(m, sn),$$

which means that β_r is an S-tensorial mapping. By Lemma 5.6, there exists a well-defined group homomorphism

$$\overline{\beta_r} : M \otimes_S N \to M \otimes_S N, \quad \sum_{k=1}^{k^*} m_k \otimes n_k \mapsto \sum_{k=1}^{k^*} rm_k \otimes n_k.$$

Define a mapping

$$R \times (M \otimes_S N) \to M \otimes_S N, \quad \left(r, \sum_{k=1}^{k^*} m_k \otimes n_k\right) \mapsto \overline{\beta_r}\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right) = \sum_{k=1}^{k^*} rm_k \otimes n_k.$$

This mapping is well defined and exactly coincides with the R-action in the formulation of this proposition.

It is straightforward to show that $M \otimes_S N$ satisfies all the conditions of a global module under the given R-action. □

Similarly, the following proposition holds.

Proposition 5.8. *Let R, S be rings, ${}_SN_R$ a partial right polite (S,R)-bimodule, N_R a global right R-module and M_S a partial right S-module. Then $(M \otimes_S N)_R$ is a global right R-module with respect to the action*

$$\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right) r := \sum_{k=1}^{k^*} m_k \otimes n_k r,$$

where $\sum_{k=1}^{k^*} m_k \otimes n_k \in M \otimes_S N$.

We note that some of the actions used in the previous two results need to be global since, as we saw in the proof of Proposition 5.7, we want to define the mapping β_r for all $r \in R$ and this could not be done if the actions were not global.

The following theorem claims that under certain assumptions the tensor product of partial modules is associative up to an isomorphism.

Theorem 5.9. *Let R and S be rings, M_R a partial right R-module, ${}_RN_S$ a global (R,S)-bimodule and ${}_SP$ a partial left S-module. Then there exists a group isomorphism*

$$\alpha : (M \otimes_R N) \otimes_S P \to M \otimes_R (N \otimes_S P), \quad (m \otimes n) \otimes p \mapsto m \otimes (n \otimes p).$$

Proof. Let R and S be rings, M_R a partial right R-module, ${}_RN_S$ a global (R,S)-bimodule and ${}_SP$ a partial left S-module. We know that the tensor product $M \otimes_R (N \otimes_S P)$ is an abelian group. Fix $p \in P$ and define a mapping

$$\gamma_p : M \times N \to M \otimes_R (N \otimes_S P), \quad (m,n) \mapsto m \otimes (n \otimes p).$$

It is easy to see that γ_p is additive in both arguments. If $m \in M$, $n \in N$ and $r \in R$ are such that the product mr is defined, then

$$\gamma_p(mr,n) = mr \otimes (n \otimes p) = m \otimes r(n \otimes p) = m \otimes (rn \otimes p) = \gamma_p(m, rn).$$

Notice that the products rn and $r(n \otimes p)$ exist because ${}_RN$ and ${}_R(N \otimes_S P)$ are global modules. Therefore γ_p is an R-tensorial mapping. Thanks to Lemma 5.6, the mapping

$$\overline{\gamma_p} : M \otimes_R N \to M \otimes_R (N \otimes_S P), \quad \sum_{k=1}^{k^*} m_k \otimes n_k \mapsto \sum_{k=1}^{k^*} \gamma_p(m_k, n_k) = \sum_{k=1}^{k^*} m_k \otimes (n_k \otimes p)$$

is a group homomorphism. Next define a mapping

$$\delta : (M \otimes_R N) \times P \to M \otimes_R (N \otimes_S P),$$
$$\left(\sum_{k=1}^{k^*} m_k \otimes n_k, p\right) \mapsto \overline{\gamma_p}\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right) = \sum_{k=1}^{k^*} m_k \otimes (n_k \otimes p).$$

This mapping is additive in the first argument since $\overline{\gamma_p}$ is a group homomorphism. If $p, p' \in P$ and $\sum_{k=1}^{k^*} m_k \otimes n_k \in M \otimes_R N$, then

$$\delta\left(\sum_{k=1}^{k^*} m_k \otimes n_k, p + p'\right) = \overline{\gamma_{p+p'}}\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right) = \sum_{k=1}^{k^*} m_k \otimes (n_k \otimes (p + p'))$$
$$= \sum_{k=1}^{k^*} m_k \otimes (n_k \otimes p + n_k \otimes p') = \sum_{k=1}^{k^*} (m_k \otimes (n_k \otimes p) + m_k \otimes (n_k \otimes p'))$$
$$= \sum_{k=1}^{k^*} m_k \otimes (n_k \otimes p) + \sum_{k=1}^{k^*} m_k \otimes (n_k \otimes p')$$

$$= \delta\left(\sum_{k=1}^{k^*} m_k \otimes n_k, p\right) + \delta\left(\sum_{k=1}^{k^*} m_k \otimes n_k, p'\right).$$

Also, if $s \in S$ is such that the product sp exists, then by Proposition 5.8

$$\begin{aligned}\delta\left(\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right)s, p\right) &= \overline{\gamma_p}\left(\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right)s\right) = \overline{\gamma_p}\left(\sum_{k=1}^{k^*} m_k \otimes n_k s\right)\\ &= \sum_{k=1}^{k^*} m_k \otimes (n_k s \otimes p) = \sum_{k=1}^{k^*} m_k \otimes (n_k \otimes sp)\\ &= \overline{\gamma_{sp}}\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right) = \delta\left(\sum_{k=1}^{k^*} m_k \otimes n_k, sp\right).\end{aligned}$$

Therefore δ is S-tensorial. Now, by Lemma 5.6, the mapping

$$\alpha := \overline{\delta} : (M \otimes_R N) \otimes_S P \to M \otimes_R (N \otimes_S P),$$
$$\sum_{h=1}^{h^*}(m_h \otimes n_h) \otimes p_h \mapsto \sum_{h=1}^{h^*} m_h \otimes (n_h \otimes p_h)$$

is a group homomorphism. Analogously there also exists a group homomorphism

$$\beta : M \otimes_R (N \otimes_S P) \to (M \otimes_R N) \otimes_S P,$$
$$\sum_{h=1}^{h^*} m_h \otimes (n_h \otimes p_h) \mapsto \sum_{h=1}^{h^*}(m_h \otimes n_h) \otimes p_h.$$

Notice that

$$(\alpha \circ \beta)\left(\sum_{h=1}^{h^*} m_h \otimes (n_h \otimes p_h)\right) = \alpha\left(\sum_{h=1}^{h^*}(m_h \otimes n_h) \otimes p_h\right) = \sum_{h=1}^{h^*} m_h \otimes (n_h \otimes p_h),$$
$$(\beta \circ \alpha)\left(\sum_{h=1}^{h^*}(m_h \otimes n_h) \otimes p_h\right) = \beta\left(\sum_{h=1}^{h^*} m_h \otimes (n_h \otimes p_h)\right) = \sum_{h=1}^{h^*}(m_h \otimes n_h) \otimes p_h.$$

Therefore $\alpha \circ \beta = \mathrm{id}_{M \otimes_R (N \otimes_S P)}$ and $\beta \circ \alpha = \mathrm{id}_{(M \otimes_R N) \otimes_S P}$. In conclusion, $M \otimes_R (N \otimes_S P)$ and $(M \otimes_R N) \otimes_S P$ are isomorphic abelian groups. □

6. Tensor product of homomorphisms of partial modules

Now let us define a tensor product of homomorphisms of partial modules. Let R be a ring, M_R, M'_R partial right and ${}_RN$, ${}_RN'$ partial left R-modules and $f : M \to M'$, $g : N \to N'$ homomorphisms of partial R-modules. Define a mapping

$$(f; g) : M \times N \to M' \otimes_R N', \quad (m, n) \mapsto f(m) \otimes g(n).$$

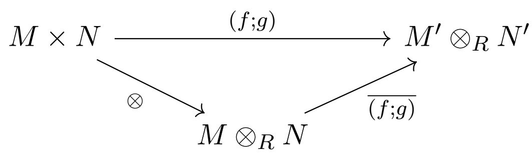
The mapping $(f;g)$ is additive in both arguments. If $m \in M$, $n \in N$ and $r \in R$ are such that the products mr and rn exist, then there exist products $f(m)r$, $rg(n)$ and

$$\begin{aligned}(f;g)(mr,n) &= f(mr) \otimes g(n) = f(m)r \otimes g(n) = f(m) \otimes rg(n) \\ &= f(m) \otimes g(rn) = (f;g)(m,rn).\end{aligned}$$

Thus $(f;g)$ is R-tensorial. By Lemma 5.6, there exists a group homomorphism $\overline{(f;g)} : M \otimes_R N \to M' \otimes_R N'$ such that $\overline{(f;g)} \circ \otimes = (f;g)$, i.e.,

$$\overline{(f;g)}(m \otimes n) = (\overline{(f;g)} \circ \otimes)(m,n) = (f;g)(m,n) = f(m) \otimes g(n),$$

which means that the diagram



commutes.

Definition 6.1. The mapping $\overline{(f;g)}$ is called a *tensor product of the homomorphisms f and g*. Denote $f \otimes g := \overline{(f;g)}$.

It is straightforward to check that a tensor product of homomorphisms has the following properties.

Proposition 6.2. *Let R be a ring, M_R, M'_R, M''_R partial right and ${}_RN$, ${}_RN'$, ${}_RN''$ partial left R-modules and $f, f^* : M_R \to M'_R$, $f' : M'_R \to M''_R$, $g, g^* : {}_RN \to {}_RN'$, $g' : {}_RN' \to {}_RN''$ homomorphisms of partial R-modules. Then*

(1) $\mathrm{id}_M \otimes \mathrm{id}_N = \mathrm{id}_{M \otimes_R N}$*;*
(2) $(f + f^*) \otimes g = f \otimes g + f^* \otimes g$*;*
(3) $f \otimes (g + g^*) = f \otimes g + f \otimes g^*$*;*
(4) $f \otimes \mathbf{0} = \mathbf{0} \otimes g = \mathbf{0}$*;*
(5) *if f and g are surjective, then $f \otimes g$ is also surjective;*
(6) $(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)$*;*
(7) *if f and g are retractions, coretractions or isomorphisms, then so is $f \otimes g$. In the third case one has $(f \otimes g)^{-1} = f^{-1} \otimes g^{-1}$.*

7. Exactness of a tensor functor

In this section we show that a tensor functor of partial modules is right exact under certain assumptions. For that, we first define short exact sequences.

Definition 7.1. Let R be a ring. Let M_R^γ be a partial right R-module for every $\gamma \in \{1, \dots, n\}$ and $f_\gamma : M_R^\gamma \to M_R^{\gamma+1}$ $(\gamma \neq n)$ a homomorphism of partial right R-modules. Then the sequence

$$M_R^1 \xrightarrow{f_1} \dots \xrightarrow{f_{\gamma-2}} M_R^{\gamma-1} \xrightarrow{f_{\gamma-1}} M_R^\gamma \xrightarrow{f_\gamma} M_R^{\gamma+1} \xrightarrow{f_{\gamma+1}} \dots \xrightarrow{f_{n-1}} M_R^n$$

is called *exact* (in PMod_R) if for all $\gamma \in \{1, \dots, n-1\}$ we have $\operatorname{im} f_\gamma = \ker f_{\gamma+1}$ (as sets).

Definition 7.2. Let R be a ring, L_R, M_R, N_R partial right R-modules and $f : L_R \to M_R$, $g : M_R \to N_R$ homomorphisms of partial modules. An exact sequence

$$\{0\} \xrightarrow{\mathbf{0}} L_R \xrightarrow{f} M_R \xrightarrow{g} N_R \xrightarrow{\mathbf{0}} \{0\}$$

is called a *short exact sequence.*

It is easy to see that for every short exact sequence as above, the homomorphism f is injective and the homomorphism g is surjective. Next we define tensor functors for partial modules.

Proposition 7.3. *Let M_R be a partial right R-module. Then there exists a functor $M \otimes_R \_\_ : {}_R\mathsf{PMod} \to \mathsf{Ab}$, which maps every partial left R-module ${}_RN$ to the abelian group $M \otimes_R N$ and every homomorphism $g : {}_RN \to {}_RN'$ of partial left R-modules to the homomorphism $\mathrm{id}_M \otimes g$ of abelian groups. We call this functor a tensor functor.*

Similarly we also get a tensor functor $\_\_ \otimes_R N : \mathsf{PMod}_R \to \mathsf{Ab}$. The following theorem shows how a tensor functor behaves on a short exact sequence.

Theorem 7.4. *Let R be a ring and ${}_RN$ a partial left R-module. If the sequence*

$$\{0\} \xrightarrow{\mathbf{0}} M_R \xrightarrow{f} K_R \xrightarrow{g} L_R \xrightarrow{\mathbf{0}} \{0\},$$

where g is a full homomorphism, is exact in PMod_R, then the sequenece

$$M \otimes_R N \xrightarrow{f \otimes \mathrm{id}_N} K \otimes_R N \xrightarrow{g \otimes \mathrm{id}_N} L \otimes_R N \xrightarrow{\mathbf{0}} \{0\}$$

is exact in Ab.

Proof. Let us consider a short exact sequence

$$\{0\} \xrightarrow{\mathbf{0}} M_R \xrightarrow{f} K_R \xrightarrow{g} L_R \xrightarrow{\mathbf{0}} \{0\}, \tag{1}$$

where g is a full homomorphism, in PMod_R and a sequence

$$M \otimes_R N \xrightarrow{f \otimes \mathrm{id}_N} K \otimes_R N \xrightarrow{g \otimes \mathrm{id}_N} L \otimes_R N \xrightarrow{\mathbf{0}} \{0\} \tag{2}$$

in Ab. Since the sequence (1) is exact, the mapping g is surjective. By Proposition 6.2, the mapping $g \otimes \mathrm{id}_N$ is also surjective. We have $g \circ f = \mathbf{0}$ because $\operatorname{im} f = \ker g$. Therefore, by Proposition 6.2,

$$(g \otimes \mathrm{id}_N) \circ (f \otimes \mathrm{id}_N) = (g \circ f) \otimes (\mathrm{id}_N \circ \mathrm{id}_N) = \mathbf{0} \otimes \mathrm{id}_N = \mathbf{0}.$$

Hence $\operatorname{im}(f \otimes \mathrm{id}_N) \subseteq \ker(g \otimes \mathrm{id}_N)$. Next we will show the reverse inclusion. There exists a surjective group homomorphism φ such that the diagram

$$\begin{array}{ccc} K \otimes_R N & \xrightarrow{\quad g\otimes \mathrm{id}_N \quad} & L \otimes_R N \\ & {\scriptstyle\pi}\searrow \qquad \nearrow{\scriptstyle\varphi} & \\ & (K \otimes_R N)/(\operatorname{im}(f \otimes \mathrm{id}_N)) & \end{array}$$

commutes. Therefore

$$\begin{aligned} \varphi\left(\sum_{h=1}^{h^*}[k_h \otimes n_h]\right) &= \varphi\left(\sum_{h=1}^{h^*}\pi(k_h \otimes n_h)\right) = \sum_{h=1}^{h^*}(g \otimes \mathrm{id}_N)(k_h \otimes n_h) \\ &= \sum_{h=1}^{h^*} g(k_h) \otimes n_h. \end{aligned}$$

Define a mapping

$$\beta : L \times N \to (K \otimes_R N)/(\operatorname{im}(f \otimes \mathrm{id}_N)), \quad (l, n) \mapsto [k \otimes n],$$

where k is such that $g(k) = l$. Notice that such k exists because g is surjective. We show that β is well-defined. Let $k, k' \in K$ be such that $g(k) = g(k') = l$ for some $l \in L$. Then $k' - k \in \ker g = \operatorname{im} f$ and therefore $(k' - k) \otimes n \in \operatorname{im}(f \otimes \mathrm{id}_N)$ and $[(k' - k) \otimes n] = [0 \otimes 0]$. Now

$$\begin{aligned} [k \otimes n] &= [k \otimes n] + [0 \otimes 0] = [k \otimes n] + [(k' - k) \otimes n] = [(k + k' - k) \otimes n] \\ &= [k' \otimes n]. \end{aligned}$$

Hence the mapping β is well defined. Next we show that β is R-tensorial. Let $l, l' \in L$ and $k, k' \in K$ be such that $g(k) = l$ and $g(k') = l'$. Then

$$g(k + k') = g(k) + g(k') = l + l'$$

and thus for every $n, n' \in N$ we have

$$\beta(l + l', n) = [(k + k') \otimes n] = [k \otimes n] + [k' \otimes n] = \beta(l, n) + \beta(l', n),$$
$$\beta(l, n + n') = [k \otimes (n + n')] = [k \otimes n] + [k \otimes n'] = \beta(l, n) + \beta(l, n').$$

If $r \in R$ is such that the product $lr = g(k)r$ exists, then, since g is a full homomorphism, there exists $k'' \in K$ such that the product $k''r$ exists and $g(k'') = g(k) = l$. Therefore the product $g(k'')r$ is defined and

$$lr = g(k)r = g(k'')r = g(k''r).$$

If also the product rn exists, then

$$\beta(lr, n) = [k''r \otimes n] = [k'' \otimes rn] = \beta(l, rn).$$

Hence the mapping β is R-tensorial. The mapping

$$\overline{\beta} : L \otimes_R N \to (K \otimes_R N)/\operatorname{im}(f \otimes \mathrm{id}_N), \quad \sum_{h=1}^{h^*} l_h \otimes n_h \mapsto \sum_{h=1}^{h^*} [k_h \otimes n_h],$$

where, for every h, we have $l_h = g(k_h)$, is a group homomorphism by Lemma 5.6. Now

$$(\overline{\beta} \circ \varphi)\left(\sum_{h=1}^{h^*}[k_h \otimes n_h]\right) = \overline{\beta}\left((\varphi\circ\pi)\left(\sum_{h=1}^{h^*} k_h \otimes n_h\right)\right) = \overline{\beta}\left((g\otimes\mathrm{id}_N)\left(\sum_{h=1}^{h^*} k_h \otimes n_h\right)\right)$$
$$= \overline{\beta}\left(\sum_{h=1}^{h^*} g(k_h) \otimes n_h\right) = \sum_{h=1}^{h^*}[k_h \otimes n_h],$$

where $\sum_{h=1}^{h^*}[k_h \otimes n_h] \in (K \otimes_R N)/\operatorname{im}(f \otimes \mathrm{id}_N)$. So $\overline{\beta} \circ \varphi = \mathrm{id}_{(K\otimes_R N)/\operatorname{im}(f\otimes \mathrm{id}_N)}$. On the other hand,

$$(\varphi \circ \overline{\beta})\left(\sum_{h=1}^{h^*} l_h \otimes n_h\right) = \varphi\left(\sum_{h=1}^{h^*}[k_h \otimes n_h]\right) = \varphi\left(\pi\left(\sum_{h=1}^{h^*} k_h \otimes n_h\right)\right)$$
$$= (g \otimes \mathrm{id}_N)\left(\sum_{h=1}^{h^*} k_h \otimes n_h\right) = \sum_{h=1}^{h^*} g(k_h) \otimes n_h = \sum_{h=1}^{h^*} l_h \otimes n_h$$

for every $\sum_{h=1}^{h^*} l_h \otimes n_h \in L \otimes_R N$. Therefore $\varphi \circ \overline{\beta} = \mathrm{id}_{L\otimes_R N}$ also. So φ is a group isomorphism and $\ker\varphi = \{0\}$. We have

$$\{0\} = \ker\varphi = \ker(g \otimes \mathrm{id}_N)/\operatorname{im}(f \otimes \mathrm{id}_N).$$

We get $\ker(g \otimes \mathrm{id}_N) = \operatorname{im}(f \otimes \mathrm{id}_N)$. Thus the sequence (2) is exact. □

Example 7.5. Let M_R be a partial right R-module, which satisfies conditions (M4′), (M5′) and $N \subseteq M$ a subset such that N_R is a global module under the operations of the partial module M_R. Then $\pi : M \to M/N$, $m \mapsto [m]$ is a full homomorphism. A mapping $\iota : N_R \to M_R$, $n \mapsto n$ is a homomorphism of partial modules and $\operatorname{im}\iota = N = \ker\pi$. Therefore the sequence

$$N \otimes_R K \xrightarrow{\iota\otimes\mathrm{id}_K} M \otimes_R K \xrightarrow{\pi\otimes\mathrm{id}_K} (M/N) \otimes_R K \xrightarrow{\mathbf{0}} \{0\},$$

where ${}_RK$ is a partial R-module, is exact.

Finally, we define exact functors and notice a useful corollary of Theorem 7.4.

Definition 7.6. Let R be a ring and $\mathcal{A} \subseteq \mathsf{PMod}_R$ a subcategory. A functor $F : \mathcal{A} \to \mathsf{Ab}$ is called *right exact* if for every short exact sequence

$$\{0\} \xrightarrow{\mathbf{0}} M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \xrightarrow{\mathbf{0}} \{0\}$$

the sequence

$$F(M_1) \xrightarrow{F(f)} F(M_2) \xrightarrow{F(g)} F(M_3) \xrightarrow{\mathbf{0}} \{0\}$$

is exact. Dually we can define *left exact* functors.

Corollary 7.7. *Let R be a ring and ${}_RN$ a partial left R-module. Then* $\_\_ \otimes_R N : \mathsf{PMod}^{\mathsf{s}}_R \to \mathsf{Ab}$ *is a right exact tensor functor.*

8. Hom-tensor adjunction

In this section we prove the main theorem of this paper, i.e., we show that under certain assumptions a tensor functor of partial modules is a left adjoint functor of a hom-functor of partial modules. First we introduce tensor functors, whose target category is a category of modules.

Proposition 8.1. *Let ${}_SM_R$ be a left polite partial (S, R)-bimodule such that ${}_SM$ is a global left S-module. Then there exists a covariant tensor functor* $M \otimes_R \_\_ : {}_R\mathsf{PMod} \to {}_S\mathsf{Mod}$.

Proof. By Proposition 7.3, we have a tensor functor $M \otimes_R \_\_ : {}_R\mathsf{PMod} \to \mathsf{Ab}$. The tensor product ${}_S(M \otimes_R N)$ is a global left S-module by Proposition 5.7 and therefore $M \otimes_R \_\_ : \mathrm{Ob}({}_R\mathsf{PMod}) \to \mathrm{Ob}({}_S\mathsf{Mod})$ is a mapping. For every $g \in \mathrm{Hom}_{{}_R\mathsf{PMod}}(N, N')$ the mapping

$$(M \otimes_R \_\_)(g) = \mathrm{id}_M \otimes g : M \otimes_R N \to M \otimes_R N'$$

is a morphism in the category ${}_S\mathsf{Mod}$. Hence, we may consider a tensor functor $M \otimes_R \_\_ : {}_R\mathsf{PMod} \to {}_S\mathsf{Mod}$. □

Similarly we can consider a tensor functor $\_\_ \otimes_S M : \mathsf{PMod}_S \to \mathsf{Mod}_R$, where M_R is a global module and a partial module ${}_SM_R$ is right polite.

Recall that ${}_S\mathsf{PMod}'$ is the category of partial left S-modules, which satisfy condition (M4′). Also recall from the proof of Corollary 2.12 that if ${}_SM \in \mathrm{Ob}({}_S\mathsf{PMod})$ and ${}_SN \in \mathrm{Ob}({}_S\mathsf{PMod}')$, then we can consider the abelian group $\mathrm{Hom}({}_SM, {}_SN)$. So for a partial bimodule ${}_SM_R$ we have a hom-functor $\mathrm{Hom}({}_SM_R, \_\_) : {}_S\mathsf{PMod}' \to \mathsf{Ab}$. Next we see that under certain extra assumptions we can treat the abelian group $\mathrm{Hom}({}_SM_R, {}_SN)$ as a partial left R-module.

Proposition 8.2. *Let R and S be rings and ${}_SM_R$ a partial right polite (S, R)-bimodule. Then there exists a covariant hom-functor* $\mathrm{Hom}({}_SM_R, \_\_) : {}_S\mathsf{PMod}' \to {}_R\mathsf{PMod}'$.

Proof. Let ${}_SM_R \in \mathrm{Ob}({}_S\mathsf{PMod}_R)$ be a partial right polite bimodule and ${}_SN \in \mathrm{Ob}({}_S\mathsf{PMod}')$. Our goal is to turn the abelian group $\mathrm{Hom}({}_SM_R, {}_SN)$ into a partial left R-module. Let $f \in \mathrm{Hom}({}_SM_R, {}_SN)$ and $r \in R$. Define the product rf if and only if the product mr exists for all $m \in M$ or if $f = \mathbf{0}$. In the first case define

$$(rf)(m) = f(mr)$$

for every $m \in M$ and it the second case define $r\mathbf{0} = \mathbf{0}$, i.e., $(r\mathbf{0})(m) = 0$ for every $m \in M$. Clearly $r\mathbf{0} = \mathbf{0} \in \mathrm{Hom}({}_SM_R, {}_SN)$. Assume now that $r \in R$ is such that the product mr exists for every $m \in M$. Then there exist the products m_1r, m_2r and $(m_1 + m_2)r$, where $m_1, m_2 \in M$, and

$$\begin{aligned}(rf)(m_1 + m_2) &= f((m_1 + m_2)r) = f(m_1r + m_2r) = f(m_1r) + f(m_2r)\\ &= (rf)(m_1) + (rf)(m_2).\end{aligned}$$

If $s \in S$ is such that the product sm, $m \in M$, exists, then there also exists the product $(sm)r$ thanks to the previous assumption about r. Since ${}_SM_R$ is right polite, the product $s(mr)$ also exists and

$$(rf)(sm) = f((sm)r) = f(s(mr)) = sf(mr) = s(rf)(m).$$

Therefore $rf \in \mathrm{Hom}({}_SM_R, {}_SN)$.

Finally we need to check if the conditions of a partial module hold for the R-action defined above. Notice that condition (M2) holds. For an arbitrary $f \in \mathrm{Hom}({}_SM_R, {}_SN)$ the product $0f$ is defined, because for every $m \in M$ the product $m0$ exists and therefore

$$(0f)(m) = f(m0) = f(0) = 0.$$

Hence condition (M3) holds. Now let $r, r' \in R$, $f, g \in \mathrm{Hom}({}_SM_R, {}_SN)$. If the products rf, rg and $r(f + g)$ exist, then mr exists for every $m \in M$ and

$$\begin{aligned}(r(f + g))(m) &= (f + g)(mr) = f(mr) + g(mr) = (rf)(m) + (rg)(m)\\ &= (rf + rg)(m).\end{aligned}$$

Thus condition (M4) holds. If the products rf, $r'f$ and $(r + r')f$ exist, then the products mr, mr' and $m(r + r')$ exist for every $m \in M$ and

$$\begin{aligned}((r + r')f)(m) &= f(m(r + r')) = f(mr + mr') = f(mr) + f(mr')\\ &= (rf)(m) + (r'f)(m) = (rf + r'f)(m).\end{aligned}$$

Therefore condition (M5) holds. If the products $r'f$, $r(r'f)$ exist, then also the products mr' and mr exist for every $m \in M$. Fix $m \in M$. Then the product $(mr)r'$ is defined. Hence the product $m(rr')$ exists and the product $(r'r)f$ is defined. Now

$$((rr')f)(m) = f(m(rr')) = f((mr)r') = (r'f)(mr) = (r(r'f))(m).$$

Therefore condition (M6) holds. We have shown that we can treat the set $\mathrm{Hom}({}_SM_R, {}_SN)$ as a partial left R-module.

Next, let $g : {}_SN \to {}_SN'$ be a homomorphism of partial left S-modules satisfying condition (M4′). We show that

$$g \circ \_ \colon \operatorname{Hom}({}_SM_R, {}_SN) \to \operatorname{Hom}({}_SM_R, {}_SN')$$

is a homomorphism of partial left R-modules. Take $f, f' \in \operatorname{Hom}({}_SM_R, {}_SN)$ and $m \in M$. Then

$$\begin{aligned}(g \circ \_)(f + f')(m) &= (g \circ (f + f'))(m) = g((f + f')(m)) = g(f(m) + f'(m)) \\ &= g(f(m)) + g(f'(m)) = (g \circ f)(m) + (g \circ f')(m) \\ &= (g \circ \_)(f)(m) + (g \circ \_)(f')(m) \\ &= ((g \circ \_)(f) + (g \circ \_)(f'))(m),\end{aligned}$$

thus $g \circ \_$ is additive. Let now $r \in R$ and $f \in \operatorname{Hom}({}_SM_R, {}_SN)$ be such that the product rf exists. If $f = \mathbf{0}$, then

$$(g \circ \_)(rf) = g \circ (r\mathbf{0}) = g \circ \mathbf{0} = \mathbf{0} = r\mathbf{0} = r(g \circ \mathbf{0}) = r(g \circ f) = (r(g \circ \_))(f).$$

Otherwise there exists the product mr for every $m \in M$ and

$$\begin{aligned}(g \circ \_)(rf)(m) &= (g \circ (rf))(m) = g((rf)(m)) = g(f(mr)) = (g \circ f)(mr) \\ &= (r(g \circ f))(m) = (r(g \circ \_))(f)(m).\end{aligned}$$

Hence $g \circ \_$ is a homomorphism of partial left R-modules. In conclusion we may consider the hom-functor $\operatorname{Hom}({}_SM_R, \_) : {}_S\mathsf{PMod}' \to {}_R\mathsf{PMod}'$. □

Next we construct a mapping needed for the proof of the main theorem.

Proposition 8.3. *Let R and S be rings. For every ${}_SM_R \in \operatorname{Ob}({}_S\mathsf{Mod}_R)$, ${}_RN \in \operatorname{Ob}({}_R\mathsf{PMod})$ and ${}_SP \in \operatorname{Ob}({}_S\mathsf{PMod}')$ the mapping*

$$\chi : \operatorname{Hom}({}_RN, {}_R(\operatorname{Hom}({}_SM_R, {}_SP))) \to \operatorname{Hom}({}_S(M \otimes_R N), {}_SP),$$

$$\chi(f)\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right) = \sum_{k=1}^{k^*} f(n_k)(m_k),$$

is bijective.

Proof. Let ${}_SM_R \in \operatorname{Ob}({}_S\mathsf{Mod}_R)$, ${}_RN \in \operatorname{Ob}({}_R\mathsf{PMod})$ and ${}_SP \in \operatorname{Ob}({}_S\mathsf{PMod}')$. First, we show that $\chi(f)$ is well defined for every homomorphism $f \in \operatorname{Hom}({}_RN, {}_R(\operatorname{Hom}({}_SM_R, {}_SP)))$. Let a mapping $\beta : M \times N \to P$ be such that

$$\beta(m, n) \coloneqq f(n)(m),$$

where $m \in M$ and $n \in N$. It is straightforward to check that β is additive in both arguments. Let $r \in R$, $m \in M$ and $n \in N$ be such that the product rn exists. Notice that the product mr also exists, since M_R is a global module. Then there also exists $rf(n)$, furthermore, $rf(n) = f(rn)$, and

$$\beta(mr, n) = f(n)(mr) = (rf(n))(m) = f(rn)(m) = \beta(m, rn).$$

Hence β is R-tensorial. By Lemma 5.6, the mapping $\overline{\beta} : M \otimes_R N \to P$, defined by the equality

$$\overline{\beta}\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right) = \sum_{k=1}^{k^*} f(n_k)(m_k),$$

is a group homomorphism. Notice that $\chi(f) = \overline{\beta}$, so $\chi(f)$ is well-defined.

Next we show that χ is well defined. For that we need to show that $\chi(f)$ preserves the partial left S-action. Take $s \in S$ ja $\nu = \sum_{k=1}^{k^*} m_k \otimes n_k \in M \otimes_R N$. Then

$$\begin{aligned}\chi(f)(s\nu) &= \chi(f)\left(s\sum_{k=1}^{k^*} m_k \otimes n_k\right) = \chi(f)\left(\sum_{k=1}^{k^*}(sm_k) \otimes n_k\right)\\ &= \sum_{k=1}^{k^*} f(n_k)(sm_k) = \sum_{k=1}^{k^*} s(f(n_k)(m_k))\\ &\overset{(\bullet)}{=} s\sum_{k=1}^{k^*} f(n_k)(m_k) = s\chi(f)\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right)\\ &= s\chi(f)(\nu).\end{aligned}$$

Here the equality ($\bullet$) holds, because the partial left S-module satisfies condition (M4$'$) and thus the product $s\sum_{k=1}^{k^*} f(n_k)(m_k)$ exists. Therefore χ is well defined.

Next we show that χ is bijective. Let $f_1, f_2 \in \mathrm{Hom}({}_RN, {}_R(\mathrm{Hom}({}_SM_R, {}_SP)))$ be such that $\chi(f_1) = \chi(f_2)$. Then, among other things, for every $m \in M$ and $n \in N$ we have

$$f_1(n)(m) = \chi(f_1)(m \otimes n) = \chi(f_2)(m \otimes n) = f_2(n)(m).$$

Since $f_1(n)(m) = f_2(n)(m)$ for every $m \in M$, we have $f_1(n) = f_2(n)$ and since the latter equality holds for every $n \in N$, $f_1 = f_2$. Therefore χ is injective.

Finally, let us show that χ is surjective. Let $g \in \mathrm{Hom}({}_S(M \otimes_R N), {}_SP)$ and take $f : {}_RN \to {}_R(\mathrm{Hom}({}_SM_R, {}_SP))$ such that

$$f(n)(m) = g(m \otimes n)$$

for every $m \in M$, $n \in N$. The mapping g is a homomorphism of partial S-modules, hence, for every $n \in N$, $m, m' \in M$ and $s \in S$,

$$\begin{aligned}f(n)(m + m') &= g((m + m') \otimes n) = g(m \otimes n) + g(m' \otimes n)\\ &= f(n)(m) + f(n)(m'),\\ f(n)(sm) &= g((sm) \otimes n) = g(s(m \otimes n)).\end{aligned}$$

Notice that the product $sg(m \otimes n)$ exists, because $s(m \otimes n)$ exists, and

$$g(s(m \otimes n)) = sg(m \otimes n) = sf(n)(m).$$

Thus $f(n) : {}_SM_R \to {}_SP$ is a homomorphism of partial left S-modules. Now, if $n, n' \in N$ and $m \in M$, then

$$f(n+n')(m) = g(m \otimes (n+n')) = g(m \otimes n) + g(m \otimes n') = f(n)(m) + f(n')(m).$$

Assume $r \in R$ is such that the product rn exists. Then

$$f(rn)(m) = g(m \otimes rn) = g(mr \otimes n) = f(n)(mr) = (rf(n))(m).$$

Notice that the mapping $rf(n)$ is defined, because M_R is a global module and therefore mr exists for every $m \in M$. Hence $f \in \mathrm{Hom}({}_RN, {}_R(\mathrm{Hom}({}_SM_R, {}_SP)))$. We also notice that for all $\sum_{k=1}^{k^*} m_k \otimes n_k \in M \otimes_R N$ we have

$$\chi(f)\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right) = \sum_{k=1}^{k^*} f(n_k)(m_k) = \sum_{k=1}^{k^*} g(m_k \otimes n_k) = g\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right).$$

This means that $\chi(f) = g$ and therefore χ is surjective. □

We are now ready to prove the main theorem of this paper.

Theorem 8.4. *Let R and S be rings and ${}_SM_R$ a global (S, R)-bimodule. The tensor functor $M \otimes_R \_\_ : {}_R\mathsf{PMod} \to {}_S\mathsf{PMod}'$ is a left adjoint functor of the hom-functor $\mathrm{Hom}({}_SM_R, \_\_) : {}_S\mathsf{PMod}' \to {}_R\mathsf{PMod}$.*

Proof. Let ${}_SM_R$ be a global bimodule. Due to Proposition 5.7, ${}_S(M \otimes_R N)$ is a global left S-module for every ${}_RN \in \mathrm{Ob}({}_R\mathsf{PMod})$ and therefore it satisfies condition (M4′). This allows us to consider the functor $M \otimes_R \_\_ : {}_R\mathsf{PMod} \to {}_S\mathsf{PMod}'$. We denote a family of homomorphisms

$$\xi := (\xi_{N,P})_{\substack{N \in \mathrm{Ob}({}_R\mathsf{PMod}) \\ P \in \mathrm{Ob}({}_S\mathsf{PMod}')}} : \mathrm{Hom}({}_R\_\_, {}_R(\mathrm{Hom}({}_SM_R, \_\_))) \Rightarrow \mathrm{Hom}({}_S(M \otimes_R \_\_), {}_S\_\_),$$

where for every ${}_RN \in \mathrm{Ob}({}_R\mathsf{PMod})$ and ${}_SP \in \mathrm{Ob}({}_S\mathsf{PMod}')$ we have $\xi_{N,P} := \chi$ defined in Proposition 8.3. We know that every component of ξ is a bijection. We fix ${}_RN, {}_RN' \in \mathrm{Ob}({}_R\mathsf{PMod})$, ${}_SP \in \mathrm{Ob}({}_S\mathsf{PMod}')$, $f \in \mathrm{Hom}({}_RN, {}_RN')$ and show that the diagram

$$\begin{array}{cccc}
{}_RN & \mathrm{Hom}({}_RN, {}_R(\mathrm{Hom}({}_SM_R, {}_SP))) & \xrightarrow{\xi_{N,P}} & \mathrm{Hom}({}_S(M \otimes_R N), {}_SP) \\
\Big\downarrow f & \Big\uparrow __\circ f & & \Big\uparrow __\circ(\mathrm{id}_M \otimes f) \\
{}_RN' & \mathrm{Hom}({}_RN', {}_R(\mathrm{Hom}({}_SM_R, {}_SP))) & \xrightarrow{\xi_{N',P}} & \mathrm{Hom}({}_S(M \otimes_R N'), {}_SP)
\end{array}$$

commutes. Let $\varphi \in \operatorname{Hom}({}_RN', {}_R(\operatorname{Hom}({}_SM_R, {}_SP)))$ and $\sum_{k=1}^{k^*} m_k \otimes n_k \in M \otimes_R N$. Now

$$\begin{aligned}
(\xi_{N,P} \circ (__ \circ f))(\varphi)\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right) &= (\xi_{N,P}((__ \circ f)(\varphi)))\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right) \\
&= (\xi_{N,P}(\varphi \circ f))\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right) = (\chi(\varphi \circ f))\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right) \\
&= \sum_{k=1}^{k^*} (\varphi \circ f)(n_k)(m_k) = \sum_{k=1}^{k^*} \varphi(f(n_k))(m_k)
\end{aligned}$$

and

$$\begin{aligned}
&((__ \circ (\mathrm{id}_M \otimes f)) \circ \xi_{N',P})(\varphi)\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right) \\
&= (__ \circ (\mathrm{id}_M \otimes f))(\xi_{N',P}(\varphi))\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right) \\
&= (\xi_{N',P}(\varphi) \circ (\mathrm{id}_M \otimes f))\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right) \\
&= \chi'(\varphi)\left((\mathrm{id}_M \otimes f)\left(\sum_{k=1}^{k^*} m_k \otimes n_k\right)\right) \\
&= \chi'(\varphi)\left(\sum_{k=1}^{k^*} \mathrm{id}_M(m_k) \otimes f(n_k)\right) \\
&= \chi'(\varphi)\left(\sum_{k=1}^{k^*} m_k \otimes f(n_k)\right) = \sum_{k=1}^{k^*} \varphi(f(n_k))(m_k)
\end{aligned}$$

where χ' is the bijection from Proposition 8.3 for the partial modules ${}_RN', {}_SP$ and the global module ${}_SM_R$. Therefore, ξ is natural in the first variable.

Similarly, ξ is also natural in the second variable. In conclusion, ξ is a natural isomorphism, which proves the adjunction $M \otimes_R \_\_ \dashv \operatorname{Hom}({}_SM_R, \_\_)$. □

Every ring can be viewed as a (bi)module over itself. Therefore, we can make the following corollary.

Corollary 8.5. *Let R be a ring. The tensor functor ${}_RR \otimes_R \_\_ \colon {}_R\mathsf{PMod} \to {}_R\mathsf{PMod}'$ is a left adjoint functor of the hom-functor $\operatorname{Hom}({}_RR_R, \_\_) \colon {}_R\mathsf{PMod}' \to {}_R\mathsf{PMod}$.*

Every (partial) module can be viewed as an abelian group. Therefore, we can also make the following corollary.

Corollary 8.6. *Let R be a ring and M_R a global right R-module. The tensor functor $M \otimes_R \_\_ : {}_R\mathsf{PMod} \to \mathsf{Ab}$ is a left adjoint functor of the hom-functor $M \otimes_R \_\_ : {}_R\mathsf{PMod} \to \mathsf{Ab}$.*

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References

[1] E. Batista, *Partial actions: what they are and why we care*, Bull. Belg. Math. Soc. Simon Stevin **24** (2017), 35–71. DOI

[2] F. Borceaux, *Handbook of Categorical Algebra 1: Basic Category Theory*, Cambridge University Press, 1994.

[3] M. Dokuchaev, *Partial actions: a survey*, Contemp. Math. **537** (2011), 173–184. DOI

[4] M. Dokuchaev, *Recent developments around partial actions*, São Paulo J. Math. Sci. **13** (2019), 195–247. DOI

[5] G. Grätzer, *Universal Algebra*, Springer, New York, 2008.

[6] J. Hu and J. Vercruysse, *Geometrically partial actions*, Trans. Amer. Math. Soc. **373** (2020), 4085–4143. DOI

[7] H. Saarse and K. Väljako, *Tensor product of partial acts*. Acta Comment. Univ. Tartu. Math. **26** (2022), 265–280. DOI

[8] P. Sarraco and J. Vercruysse, *Geometric partial comodules over flat coalgebras in Abelian categories are globalizable*, J. Pure Appl. Algebra **228** (2024), 107502. DOI

[9] P. Sarraco and J. Vercruysse, *Globalization for geometric partial comodules*, J. Algebra **602** (2022), 37–59. DOI

[10] P. Sarraco and J. Vercruysse, *On the globalization of geometric partial (co)modules in the categories of topological spaces and algebras*, Semigroup Forum **105** (2022), 534–550. DOI

[11] H. Whitney, *Tensor products of Abelian groups*, Duke Math. J.**4** (1938), 495–528. DOI

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