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Generalization of the Flajolet Salvy identities for Euler sums

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Abstract. The exploration of linear Euler harmonic sums had its beginnings with the works of Euler in the 18^{th} century. In 1998 Flajolet and Salvy published a seminal paper in which they developed an approach to the evaluation of linear Euler harmonic sums and gave explicit formulae for several classes of Euler sums in terms of Riemann zeta and other special function values. In this paper, results given by Flajolet and Salvy are extended and generalized.

1. Introduction and notation

In a landmark paper [14] Flajolet and Salvy studied and provided explicit expressions for the four distinct classes of linear harmonic and skew Euler harmonic sums of the type

$$\mathbb{S}_{p,t}^{++}(q) := \sum_{n=1}^{\infty} \frac{H_{qn}^{(p)}}{n^t}, \qquad \mathbb{S}_{p,t}^{+-}(q) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{H_{qn}^{(p)}}{n^t}, \tag{1}$$

$$\mathbb{S}_{p,t}^{-+}(q) := \sum_{\tau=1}^{\infty} \frac{A_{qn}^{(p)}}{n^t}, \qquad \mathbb{S}_{p,t}^{--}(q) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{A_{qn}^{(p)}}{n^t}, \tag{2}$$

for the case $q = 1$. The terms $H_n^{(p)}$ and $H_{qn}^{(p)}$ are harmonic numbers of order p described by (7) and $A_n^{(p)}$ by (9). The aim of this paper is to express (1) in terms of special functions, such as the Riemann zeta function described by (6), for the case $q \in \mathbb{R}^+ \setminus \{0\}$, thereby markedly extending the applicability of (1). We emphasize that there are no closed form representations of (1) in the published literature for the general case $q \in \mathbb{R}^+ \setminus \{0\}$. Similar analysis

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will be extended to the linear skew Euler harmonic sums (2) in a forthcoming paper. The linear Euler harmonic sums of the type

$$\mathbb{S}_{p,t}^{++}(1) := \mathbb{S}_{p,t}^{++}, \tag{3}$$

with the integer $p+t$ designated as the weight, were studied in detail by Euler and Goldbach, see Flajolet and Salvy [14]. For odd weight $p+t$, Borwein et al. [6] gave the identity

$$\begin{aligned}\mathbb{S}_{p,t}^{++} = &\frac{1-(-1)^p}{2}\zeta(p)\zeta(t) + \left(1-(-1)^p\binom{p+t}{t}\binom{p+t-1}{p}\right)\frac{\zeta(p+t)}{2} \\ &+(-1)^p\sum_{j=1}^{[\frac{p}{2}]}\binom{p+t-2j-1}{t-1}\zeta(2j)\zeta(p+t-2j) \\ &+(-1)^p\sum_{j=1}^{[\frac{t}{2}]}\binom{p+t-2j-1}{p-1}\zeta(2j)\zeta(p+t-2j).\end{aligned} \tag{4}$$

Euler obtained this evaluation by computing many examples with weight $p+t \leq 13$ and then extrapolating the general formula, without a technical proof. Euler [13] stated the result

$$\mathbb{S}_{1,t}^{++} = \left(1+\frac{t}{2}\right)\zeta(t+1) - \frac{1}{2}\sum_{j=1}^{t-2}\zeta(j+1)\zeta(t-j)$$

for integer $t \geq 2$. For the alternating linear harmonic Euler sums

$$\mathbb{S}_{p,t}^{+-}(1) := \mathbb{S}_{p,t}^{+-} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n^{(p)}}{n^t},$$

Flajolet and Salvy [14] gave the expression, for odd weight $p+t$,

$$\begin{aligned}2\mathbb{S}_{p,t}^{+-} = &(1-(-1)^p)\zeta(p)\eta(t) + \eta(p+t) \\ &+2\sum_{j+2k=p}\binom{t+j-1}{t-1}(-1)^{j+1}\eta(t+j)\eta(2k) \\ &+2(-1)^p\sum_{i+2k=t}\binom{p+i-1}{p-1}\zeta(p+i)\eta(2k),\end{aligned} \tag{5}$$

where $\eta(\cdot)$ is the Dirichlet eta function. For odd weight $1+t$, Sitaramachandrarao [21] published

$$2\mathbb{S}_{1,t}^{+-} = (t+1)\eta(t+1) - \zeta(t+1) - 2\sum_{j=1}^{\frac{t}{2}-1}\zeta(2j)\zeta(t+1-2j),$$

and recently Alzer and Choi [3] obtained the nice result, for $p \in \mathbb{N} \setminus \{1\}$,

$$2\mathbb{S}_{p,1}^{+-} = p\zeta(p+1) - \sum_{j=1}^{p} \eta(j)\,\eta(p+1-j).$$

Nielsen [18] and many other researchers, see [3], [5], [6], [7], [10], [14], [22], [23], [24], [25], [26] and [32], have continued this work and it is now known that $\mathbb{S}_{p,t}^{++}$ can be explicitly evaluated, in terms of special functions such as the Riemann zeta function, in the cases when $p = t \in \mathbb{N}$, $p+t$ of odd weight, $p+t$ and of even weight in only the pair $\{(4,2),(2,4)\}$ with $p \neq t$. A valuable reciprocity (shuffle) relation

$$\mathbb{S}_{p,t}^{++} + \mathbb{S}_{t,p}^{++} = \zeta(p)\,\zeta(t) + \zeta(p+t)$$

exists to evaluate $\mathbb{S}_{p,t}^{++}$ in the case $\mathbb{S}_{t,p}^{++}$ is known (or visa-versa). In the case

$$\mathbb{S}_{4,2}^{++} = \frac{37}{12}\zeta(6) - \zeta(3)^2,$$

and it is speculated that $\mathbb{S}_{p,t}^{++}$, with even weight $(p+t) \geq 8$ cannot be evaluated in terms of special functions.

Within this research paper the standard notation

$$\mathbb{N} := \{1,2,3,\cdots\},\ \mathbb{N}_0 := \{0,1,2,3,\cdots\} = \mathbb{N} \cup \{0\},$$
$$\mathbb{Z}^- := \{-1,-2,-3,\cdots\} = \mathbb{Z}_0^- \setminus \{0\},$$

is utilized where $\mathbb{Z}$ denotes the set of integers. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of real positive numbers and $\mathbb{C}$ denotes the set of complex numbers. An important and useful special function encountered in this research is the prolific *Riemann zeta function*

$$\zeta(t) := \begin{cases} \sum_{j\geq 1} \frac{1}{j^t} = \frac{1}{1-2^{-t}} \sum_{j\geq 1} \frac{1}{(2j-1)^t};\ \Re(t) > 1 \\ \\ \frac{1}{1-2^{1-t}} \sum_{j\geq 1} \frac{(-1)^{j-1}}{j^t},\ \Re(t) > 0, t \neq 1 \end{cases}. \tag{6}$$

The *Dirichlet eta function*, $\eta(t)$, is defined by

$$\eta(t) := \sum_{n\geqslant 1} \frac{(-1)^{n-1}}{n^t} = \left(1 - 2^{1-t}\right)\zeta(t) \quad (\Re(t) > 0).$$

The *generalized harmonic numbers* $H_n^{(p)}(\upsilon)$ of order p are defined by

$$H_n^{(p)}(\upsilon) := \sum_{j=1}^{n} \frac{1}{(j+\upsilon)^p} \quad (p \in\ \mathbb{C},\ \upsilon \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1},\ n \in \mathbb{N}), \tag{7}$$

where $H_n^{(p)} := H_n^{(p)}(0)$ are the harmonic numbers of order p. We let $\Phi_{>\nu}$, $\Phi_{\geqslant\nu}$, $\Phi_{<\nu}$, and $\Phi_{\leqslant\nu}$ be the subsets of the set Φ ($\mathbb{R}$ or $\mathbb{Z}$) which are greater than, greater than or equal to, less than, and less than or equal to some $\nu \in \mathbb{R}$, respectively. The *harmonic numbers* $H_n := H_n^{(1)}$ are given by

$$H_n = \sum_{j=1}^{n} \frac{1}{j} = \gamma + \psi(n+1) \quad (n \in \mathbb{Z}_{\geqslant 0}) \quad \text{and} \quad H_0 := 0. \tag{8}$$

The *alternating* or *skew, harmonic numbers* $A_n^{(t)}$ of order t, in (2), are defined by

$$A_n^{(t)} := \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j^t} \quad (t \in \mathbb{C},\ n \in \mathbb{N}) \tag{9}$$

and $A_n := A_n^{(1)}$. The term γ represents the familiar Euler-Mascheroni constant (see, e.g., [30, Section 1.2]) and $\psi(t)$ denotes the *digamma* (or *psi*) *function* defined by

$$\psi(t) := \frac{d}{dz}(\log \Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)} \quad (t \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}),$$

where $\Gamma(t)$ is the familiar Gamma function (see, e.g., [30, Section 1.1]). There are many identities involving the psi function (see, e.g., [30, Section 1.3]),

$$\psi(t+n) = \psi(t) + \sum_{j=1}^{n} \frac{1}{t+k-1} \quad (n \in \mathbb{N}).$$

The *polygamma function* $\psi^{(k)}(t)$ is defined by

$$\begin{aligned} \psi^{(k)}(t) := & \quad \frac{d^k}{dt^k}\{\psi(t)\} = (-1)^{k+1}\, k! \sum_{\mu \geq 0}^{\infty} \frac{1}{(\mu+t)^{k+1}} \\ = & \quad (-1)^{k+1}\, k!\, \zeta(k+1, t),\ (k \in \mathbb{N};\ z \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}). \end{aligned}$$

The *generalized* (or *Hurwitz*) *zeta function* $\zeta(t, m)$, occuring when taking

$$\lim_{n->\infty} H_n^{(t)}(m) := \sum_{j=1}^{n} \frac{1}{(j+m)^t},$$

is defined by

$$\zeta(t, m) = \sum_{j \geq 0} \frac{1}{(j+m)^t} \quad (\Re(t) > 1,\ m \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}),$$

and by its meromorphic continuation to the whole complex t plane except for a simple pole at $t = 1$, with residue 1 (see, e.g., [30, Section 1.3]), can be written as

$$\psi^{(m)}(t) \quad = \quad (-1)^{m+1}\, m! \sum_{j \geq 0} \frac{1}{(j+t)^{m+1}}$$

$$= \quad (-1)^{m+1}\, m!\zeta(m+1,t).$$

Two useful identities for $\zeta(t,m)$ are

$$\zeta(t,1)=\zeta(t) \text{ and } \zeta(t,\frac{1}{2})=2^t\lambda(t).$$

The *Dirichlet lambda function* $\lambda\,(s)$ is defined as the term-wise arithmetic mean of the Dirichlet eta function and the Riemann zeta function:

$$\lambda\,(m)=\frac{\eta(m)+\zeta(m)}{2}=\lim_{n\to\infty} O_n^{(s)}=\sum_{j=1}^{\infty}\frac{1}{(2j-1)^m} \quad (\Re(m)>1).$$

The *Dirichlet beta function* $\beta(z)$ is defined by

$$\beta(z)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^z} \quad (\Re(z)>0),$$

and $\beta(2)=G$ is the *Catalan constant.* Among various properties and formulae for $\beta(z)$ is

$$\begin{aligned}\beta(z)= &\quad 4^{-z}\left\{\zeta\left(z,\,\tfrac{1}{4}\right)-\zeta\left(z,\,\tfrac{3}{4}\right)\right\}\\ = &\quad \frac{i}{2}\left\{\mathrm{Li}_z(-\mathbf{i})-\mathrm{Li}_z(\mathbf{i})\right\}.\end{aligned}$$

To bring clarity, $\mathbf{i}=\sqrt{-1}$ is used throughout this paper. The *polylogarithm function* $\mathrm{Li}_p(z)$ of order p, for each integer $p\geq 1$, is defined by (see, e.g., [30, p. 198])

$$\mathrm{Li}_p(z)=\sum_{m=1}^{\infty}\frac{z^m}{m^p} \quad (|z|\leqslant 1;\ p\in\mathbb{Z}_{\geqslant 2}) \tag{10}$$

and

$$\mathrm{Li}_1(z)=-\log\left(1-z\right), \quad (|z|\leqslant 1)\,.$$

The polylogarithm

$$\mathrm{Li}_s(-z)=-\frac{1}{\Gamma\left(s\right)}\int_0^{\infty}\frac{t^{s-1}}{z^{-1}\exp\left(t\right)+1}\,\mathrm{d}t, \tag{11}$$

in this context, is sometimes referred to as a *Fermi-Dirac integral* (see [1] or [31]). Similarly, the polylogarithm

$$\mathrm{Li}_s(z)=\frac{1}{\Gamma\left(s\right)}\int_0^{\infty}\frac{t^{s-1}}{z^{-1}\exp\left(t\right)-1}\,\mathrm{d}t \tag{12}$$

is sometimes referred to as a *Bose-Einstein integral* [31]. From Lewin ([16], p. 299), see also [11], [12, pp. 30–31], [15], [30, pp. 197–198]), Jonquiĕre's relation states

$$\begin{aligned}\mathrm{Li}_t(\frac{1}{z}) &= (-1)^{t+1}\left(\mathrm{Li}_t(z)+\frac{(2\pi\mathbf{i})^t}{t!}B\left(t,\frac{\ln z}{2\pi\mathbf{i}}\right)\right) \qquad (13)\\ &= (-1)^{t+1}\,\mathrm{Li}_t(z)+(-1)^{t+1}\,\frac{(2\pi\mathbf{i})^t}{t!}B\left(t,\frac{\ln z}{2\pi\mathbf{i}}\right)\\ &= (-1)^{t+1}\,\mathrm{Li}_t(z)+(-1)^{t+1}\,\frac{(2\pi\mathbf{i})^t}{t!}\sum_{j=0}^{t}\binom{t}{j}\left(\frac{\ln z}{2\pi\mathbf{i}}\right)^{t-j}B_j, (t\in\mathbb{Z}_{\geqslant 2}),\end{aligned}$$

where $\mathrm{Li}_t(z)$ is a polylogarithm. For a negative argument, Jonquiĕre's relation states

$$\mathrm{Li}_t(-\frac{1}{z}) = (-1)^{t+1}\left(\mathrm{Li}_t(-z)+\sum_{j=0}^{\left[\frac{t}{2}\right]}\frac{(\ln z)^{t-2j}}{(t-2j)!}\eta\left(2j\right)\ ,(t\in\mathbb{Z}_{\geqslant 2})\right). \qquad (14)$$

The relations (13) and (14) provide the analytic continuation of $\mathrm{Li}_s(z)$ in (10) outside its circle of convergence $|z|=1$. The *Bernoulli polynomials* $B_j(t)$, are defined by the generating function

$$\frac{z\ e^{tz}}{e^z-1}=\sum_{j\geq 0}\frac{z^j}{j!}B\left(j,t\right),\ \text{ where } |z|<2\pi,$$

and have the representation (see, e.g., [29, Section 1.7])

$$B\left(j,t\right)=\sum_{k=0}^{j}\left(\begin{array}{c} j\\ k\end{array}\right)B_k\ t^{j-k}=\sum_{k=0}^{j}\left(\begin{array}{c} j\\ k\end{array}\right)B_{t-k}t^k,$$

where $B_k=B\left(k,0\right)$ are the *Bernoulli numbers* represented as

$$\frac{z}{e^z-1}=\sum_{j\geq 0}\frac{z^j}{j!}B_j,\ \text{ where } |z|<2\pi,$$

so that

$$B_0=1, B_1=-\frac{1}{2}, B_{2n+1}=0, n\in\mathbb{N}. \qquad (15)$$

The *extended harmonic numbers* of order $m\in\mathbb{N}$ with index $\alpha\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant -1}$ are defined by (see [29])

$$H_\alpha^{(m)} := \begin{cases}\gamma+\psi(\alpha+1) & (m=1),\\ \zeta(m)+\frac{(-1)^{m-1}}{(m-1)!}\,\psi^{(m-1)}(\alpha+1) & (m\in\mathbb{Z}_{\geqslant 2}).\end{cases}$$

The case $m=1$ from the above definition is given in (8). It follows from (7) that

$$H_\alpha^{(m)}=H_{\alpha-1}^{(m)}+\frac{1}{\alpha^m}\quad\left(m\in\mathbb{N},\,\alpha\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}\right).$$

The multiplication formula for polygamma functions (see, e.g., [17, p. 14]) is

$$\psi^{(n)}(mz) = \delta_{n,0}\log m + \frac{1}{m^{n+1}}\sum_{j=1}^{m}\psi^{(n)}\left(z+\tfrac{j-1}{m}\right) \quad (m\in\mathbb{N},\ n\in\mathbb{Z}_{\geqslant 0}),$$

where $\delta_{n,j}$ is the Kronecker delta.

The interested reader may also consult the valuable works of [2], [8], [9], [20], [27] and [28].

2. The main results

The following theorems are the main results expressing the linear Euler harmonic sums $\mathbb{S}^{++}_{p+1,t}(q)$ and $\mathbb{S}^{+-}_{p+1,t}(q)$ in terms of special functions.

Theorem 1. *If* $p\in\mathbb{N},\ t\in\mathbb{N}\geq 2$ *and* $q\in\mathbb{R}>0$, *for odd weight* $p+1+t$, *then it follows that*

$$\begin{aligned} 2(-1)^p p!\mathbb{S}^{++}_{p+1,t}(q) &= \left(\frac{(-1)^p p!}{q^{p+1}} + q^t (t+1)_p\right)\zeta(p+t+1) \qquad (16)\\ &\quad +2(-1)^p p!\zeta(p+1)\zeta(t) \end{aligned}$$

$$-2\sum_{j=1}^{\left[\frac{t}{2}\right]}(t+1-2j)_p\, q^{t-2j}\zeta(2j)\zeta(p+t+1-2j) - \Re\left(\int_0^\infty \frac{\ln^p(y)\,\mathrm{Li}_t(y^q)}{y(1-y)}\,\mathrm{d}y\right),$$

where $\mathbb{S}^{++}_{p+1,t}(q)$ *is the linear Euler harmonic sum* (3), $\mathrm{Li}_t(\cdot)$ *is the polylogarithmic function described by* (10), $\zeta(t)$ *is the classical Riemann zeta function and* $(\cdot)_p$ *is the Pochhammer symbol. The symbol,* $[x]$ *indicates the greatest integer less than or equal to* $x\in\mathbb{R}$ *and* $\Re(z)$ *indicates the real part of* z.

Proof. First, the integral $X(p,q,t,\delta,\rho)$ is defined, for $\delta=\pm 1$ and $\rho=\pm 1$, as

$$X(p,q,t,\delta,\rho) := \int_0^\infty \frac{\ln^p(y)\,\mathrm{Li}_t(\rho y^q)}{y(1+\delta y)}\,\mathrm{d}y \qquad (17)$$

and the case of $\delta=-1$ and $\rho=1$ is considered on the real half line $x>0$ as

$$X(p,q,t,-1,1) = \int_0^1 \frac{\ln^p(y)\,\mathrm{Li}_t(y^q)}{y(1-y)}\,\mathrm{d}y + \int_1^\infty \frac{\ln^p(y)\,\mathrm{Li}_t(y^q)}{y(1-y)}\,\mathrm{d}y.$$

Using the transformation $xy=1$ in the last integral and recovering the variable y instead of x in the resultant integral, yields the following:

$$X(p,q,t,-1,1) = \int_0^1 \frac{\ln^p(y)\,\mathrm{Li}_t(y^q)}{y(1-y)}\,\mathrm{d}y - (-1)^p\int_0^1 \frac{\ln^p(y)\,\mathrm{Li}_t(\frac{1}{y^q})}{(1-y)}\,\mathrm{d}y.$$

Recalling the Jonquiěre formula (13), the last integral can be transformed as follows:

$$\begin{aligned}X(p,q,t,-1,1) &= \int_0^1 \frac{\ln^p(y)\operatorname{Li}_t(y^q)}{y(1-y)}\,dy - (-1)^p\int_0^1 \frac{\ln^p(y)\operatorname{Li}_t(\frac{1}{y^q})}{(1-y)}\,dy.\\ &= \int_0^1 \frac{\ln^p(y)\operatorname{Li}_t(y^q)}{y}\,dy + \left(1+(-1)^{p+t}\right)\int_0^1 \frac{\ln^p(y)\operatorname{Li}_t(y^q)}{(1-y)}\,dy\\ &\quad + (-1)^{p+t}\,\frac{1}{t!}\sum_{j=0}^{t}\binom{t}{j} q^{t-j}B_j\,(2\pi\mathbf{i})^j\int_0^1 \frac{\ln^{p+t-j}(y)}{(1-y)}dy.\end{aligned}$$

Standard integration procedure yields

$$\int_0^1 \frac{\ln^p(y)\operatorname{Li}_t(y^q)}{y}\,dy = \sum_{n\geq 1}\frac{(-1)^p\,p!\Gamma(p+1)}{n^t\,(nq)^{p+1}} = \frac{(-1)^p\,p!}{q^{p+1}}\zeta(p+t+1).$$

Similarly,

$$\begin{aligned}\int_0^1 \frac{\ln^p(y)\operatorname{Li}_t(y^q)}{(1-y)}\,dy &= (-1)^p\,p!\sum_{n\geq 1}\frac{1}{n^t}\left(H_{qn}^{(p+1)} - \zeta(p+1)\right)\\ &= (-1)^p\,p!\left(\mathbb{S}^{++}_{p+1,t}(q) - \zeta(t)\,\zeta(p+1)\right).\end{aligned}$$

Hence,

$$X(p,q,t,-1,1) = \left(1+(-1)^{p+t}\right)(-1)^{p+1}\,p!\left(\mathbb{S}^{++}_{p+1,t}(q) - \zeta(t)\,\zeta(p+1)\right)$$

$$+\frac{(-1)^p\,p!}{q^{p+1}}\zeta(p+t+1) + \frac{1}{t!}\sum_{j=0}^{t}(-1)^j\binom{t}{j}q^{t-j}B_j\,(2\pi\mathbf{i})^j\,(p+t-j)!\zeta(p+t+1-j).$$

To include the $\mathbb{S}^{++}_{p+1,t}(q)$ term, the weight $p+t+1$ is chosen to be odd, so that

$$\begin{aligned}X(p,q,t,-1,1) =&\frac{(-1)^p\,p!}{q^{p+1}}\zeta(p+t+1) + 2\,(-1)^{p+1}\,p!\left(\mathbb{S}^{++}_{p+1,t}(q) - \zeta(t)\,\zeta(p+1)\right)\\ &+\frac{1}{t!}\sum_{j=0}^{t}(-1)^j\binom{t}{j}q^{t-j}B_j\,(2\pi\mathbf{i})^j\,(p+t-j)!\zeta(p+t+1-j).\end{aligned}$$

Furthermore, the properties of the Bernoulli numbers, see (15) and the classical Euler, Bernoulli relation (see [30], p.166),

$$\zeta\left(2k\right)=\frac{\left(-1\right)^{k+1}B_{2k}2^{2k-1}\pi^{2k}}{\left(2k\right)!},$$

permit the following:

$$\begin{aligned}&X\left(p,q,t,-1,1\right)\\&=\frac{\left(-1\right)^{p}p!}{q^{p+1}}\zeta\left(p+t+1\right)+2\left(-1\right)^{p+1}p!\left(\mathbb{S}^{++}_{p+1,t}(0,0;q)-\zeta\left(t\right)\zeta\left(p+1\right)\right)\\&\quad+2\sum_{j=0}^{\left[\frac{t}{2}\right]}\left(t+1-2j\right)_{p}q^{t-2j}\zeta(2j)\zeta\left(p+t+1-2j\right)-q^{t-1}\left(t\right)_{p}\pi\zeta\left(p+t\right)\mathbf{i}.\end{aligned}$$

Rearranging yields

$$\begin{aligned}&2\left(-1\right)^{p}p!\mathbb{S}^{++}_{p+1,t}(q)\\&=q^{t-1}\left(t\right)_{p}\pi\zeta\left(p+t\right)\mathbf{i}-2\sum_{j=0}^{\left[\frac{t}{2}\right]}\left(t+1-2j\right)_{p}q^{t-2j}\zeta(2j)\zeta\left(p+t+1-2j\right)\\&\quad+2\left(-1\right)^{p}p!\zeta\left(t\right)\zeta\left(p+1\right)+\frac{\left(-1\right)^{p}p!}{q^{p+1}}\zeta\left(p+t+1\right)-\int_{0}^{\infty}\frac{\ln^{p}\left(y\right)\mathrm{Li}_{t}(y^{q})}{y\left(1-y\right)}\,\mathrm{d}y.\end{aligned}$$

Since the sum $\mathbb{S}^{++}_{p,t}(q)$ is a real number, we have

$$\Im\left(\int_{0}^{\infty}\frac{\ln^{p}\left(y\right)\mathrm{Li}_{t}(y^{q})}{y\left(1-y\right)}\,\mathrm{d}y\right)=q^{t-1}\left(t\right)_{p}\pi\zeta\left(p+t\right),$$

from which it follows that

$$\begin{aligned}2\left(-1\right)^{p}p!\mathbb{S}^{++}_{p+1,t}(q)&=\frac{\left(-1\right)^{p}p!}{q^{p+1}}\zeta\left(p+t+1\right)+2\left(-1\right)^{p}p!\zeta\left(p+1\right)\zeta\left(t\right)\\-2\sum_{j=0}^{\left[\frac{t}{2}\right]}\left(t+1-2j\right)_{p}q^{t-2j}\zeta(2j)\zeta\left(p+t+1-2j\right)&-\Re\left(\int_{0}^{\infty}\frac{\ln^{p}\left(y\right)\ \mathrm{Li}_{t}(y^{q})}{y\left(1-y\right)}\,\mathrm{d}y\right).\end{aligned}$$

Taking out the first term in the finite sum and noting that $\zeta(0)=-\frac{1}{2}$ yields (16), and thus the proof of the Theorem is complete. □

The next theorem establishes a recurrence relation involving the linear Euler harmonic sum $\mathbb{S}^{++}_{p+1,t}(q)$ and $\mathbb{S}^{++}_{p+1,t}(\frac{q}{2})$.

Theorem 2. *If* $p\in\mathbb{N},\ t\in\mathbb{N}\geq 2$ *and* $q\in\mathbb{R}>0$, *for odd weight* $p+1+t$, *then it follows that*

$$2\left(-1\right)^{p+1}p!\left(\mathbb{S}^{++}_{p+1,t}(q)-2^{-p}\mathbb{S}^{++}_{p+1,t}(\frac{q}{2})\right)=\frac{\left(-1\right)^{p}p!}{q^{p+1}}\zeta\left(p+t+1\right)\qquad(18)$$

$$+2\sum_{j=0}^{\left[\frac{t}{2}\right]}(t+1-2j)_p\, q^{t-2j}\zeta(2j)\eta\left(p+t+1-2j\right)$$
$$-2\left(-1\right)^p p!\eta\left(p+1\right)\zeta\left(t\right)-\Re\left(\int_0^\infty \frac{\ln^p\left(y\right)\mathrm{Li}_t(y^q)}{y\left(1+y\right)}\,\mathrm{d}y\right),$$

where $\mathbb{S}^{++}_{p+1,t}(q)$ *is the linear Euler harmonic sum* (3), $\mathrm{Li}_t(\cdot)$ *is the polylogarithmic function described by* (10), $\zeta\left(t\right)$ *is the classical Riemann zeta function,* $\eta\left(t\right)$ *is the classical Riemann eta function and* $\left(\cdot\right)_p$ *is the Pochhammer symbol.*

Proof. Following the same pattern as in Theorem 1, from (17) putting $\delta=\rho=1$ yields

$$X\left(p,q,t,1,1\right)=\int_0^\infty \frac{\ln^p\left(y\right)\mathrm{Li}_t(y^q)}{y\left(1+y\right)}\,\mathrm{d}y$$
$$=\int_0^1 \frac{\log^p\left(y\right)\mathrm{Li}_t(y^q)}{y}\,\mathrm{d}y-\int_0^1 \frac{\log^p\left(y\right)\mathrm{Li}_t(y^q)}{1+y}\,\mathrm{d}y+(-1)^p\int_0^1 \frac{\log^p\left(y\right)\mathrm{Li}_t(\frac{1}{y^q})}{1+y}\,\mathrm{d}y.$$

Using the Jonquiěre relation (13) allows the representation

$$\begin{aligned}X\left(p,q,t,1,1\right) &= \int_0^1 \frac{\log^p\left(y\right)\mathrm{Li}_t(y^q)}{y}\,\mathrm{d}y\\ &\quad-\left(1+(-1)^{p+t}\right)\int_0^1 \frac{\log^p\left(y\right)\mathrm{Li}_t(y^q)}{1+y}\,\mathrm{d}y\\ -\left(-1\right)^{p+t}\frac{\left(2\pi i\right)^t}{t!}\sum_{j=0}^{t}\binom{t}{j}B_j\frac{q^{t-j}}{\left(2\pi\mathbf{i}\right)^{t-j}}&\int_0^1 \frac{\log^{p+t-j}\left(y\right)}{1+y}\,\mathrm{d}y,\end{aligned}$$

which transforms to

$$\begin{aligned}X\left(p,q,t,1,1\right) &= \int_0^1 \frac{\log^p\left(y\right)\mathrm{Li}_t(y^q)}{y}\,\mathrm{d}y\\ &\quad-\left(1+(-1)^{p+t}\right)\int_0^1 \frac{\log^p\left(y\right)\mathrm{Li}_t(y^q)}{1+y}\,\mathrm{d}y\\ -\sum_{j=0}^{t}\frac{\left(-1\right)^j B_j}{j!}&\left(2\pi i\right)^j\left(t+1-j\right)_p\; q^{t-j}\eta\left(p+t+1-j\right).\end{aligned}\tag{19}$$

The above integrals can be expressed as

$$\int_0^1 \frac{\log^p\left(y\right)\ \mathrm{Li}_t(y^q)}{y}\,\mathrm{d}y=\frac{\left(-1\right)^{p+1}p!}{q^{p+1}}\zeta\left(p+t+1\right)$$

and

$$\int_0^1 \frac{\log^p\left(y\right)\mathrm{Li}_t(y^q)}{1+y}\,\mathrm{d}y = \frac{\left(-1\right)^p p!}{2^{p+1}}\sum_{n\geq 1}\frac{1}{n^t}\left(H^{(p+1)}_{\frac{qn-1}{2}}-H^{(p+1)}_{\frac{qn}{2}}\right)$$

$$= \frac{(-1)^p \, p!}{2^{p+1}} \sum_{n\geq 1} \frac{1}{n^t} \left(2^{p+1} H_{qn}^{(p+1)} - 2H_{\frac{qn}{2}}^{(p+1)} - 2^{p+1}\eta\,(p+1) \right).$$

For the case of odd weight $(p+1+t)$,

$$2\,(-1)^{p+1}\, p! \left(\mathbb{S}_{p+1,t}^{++}(q) - 2^{-p}\mathbb{S}_{p+1,t}^{++}(\frac{q}{2}) \right)$$

$$= \frac{(-1)^p \, p!}{q^{p+1}} \zeta\,(p+t+1) - 2\,(-1)^p \, p! \zeta\,(t)\,\eta\,(p+1) - q^{t-1}\,(t)_p\,\pi\eta\,(p+t)\,\mathbf{i}$$
$$+ 2\sum_{j=0}^{\left[\frac{t}{2}\right]} (t+1-2j)_p\, q^{t-2j} \zeta(2j)\eta\,(p+t+1-2j) - \int_0^\infty \frac{\ln^p(y)\,\mathrm{Li}_t(y^q)}{y\,(1+y)}\,\mathrm{d}y.$$

Since both terms $\mathbb{S}_{p+1,t}^{++}(q)$ and $\mathbb{S}_{p+1,t}^{++}(\frac{q}{2})$ are real numbers, the difference of the sums $\mathbb{S}_{p+1,t}^{++}(q) - 2^{-p}\mathbb{S}_{p+1,t}^{++}(\frac{q}{2})$ is also a real number and therefore the imaginary part of the integral can be expressed as

$$\Im \left(\int_0^\infty \frac{\ln^p(y)\ \mathrm{Li}_t(y^q)}{y\,(1+y)}\,\mathrm{d}y \right) = -q^{t-1}\,(t)_p\,\pi\eta\,(p+t)\,,$$

which completes the proof of Theorem 2. The identity (4) may be used in the recurrence (18). □

The next lemma indicates a procedure for the evaluation of the integrals required in Theorems 1 and 2.

Lemma 1. *If* $p \in \mathbb{N},\ t \in \mathbb{N} \geq 2$, $q \in \mathbb{R}^+, -1 \leq a < 1, \delta = \pm 1$ *and* $\rho = \pm 1$, *then*

$$X\,(p,q,t,\delta,\rho) = \lim_{a->0} \left(\frac{\partial^p}{\partial a^p} Z\,(a,q,t,\delta,\rho) \right) \tag{20}$$

$$= \int_0^\infty \frac{\ln^p(y)\,\mathrm{Li}_t(\rho y^q)}{y\,(1+\delta y)}\,\mathrm{d}y, \tag{21}$$

where

$$Z\,(a,q,t,\delta,\rho) = \int_0^\infty \frac{y^a\ \mathrm{Li}_t(\rho y^q)}{y\,(1+\delta y)}\,\mathrm{d}y.$$

Proof. Consider the integral,

$$Z\,(a,q,t,\delta,1) = \int_0^\infty \frac{y^a\ \mathrm{Li}_t(y^q)}{y\,(1+\delta y)}\,\mathrm{d}y.$$

Application of the Bose–Einstein relation (12) to the polylogarithm yields

$$Z\,(a,q,t,\delta,1) = \frac{1}{\Gamma\,(t)} \int_0^\infty x^{t-1} \left(\int_0^\infty \frac{y^a}{y\,(1+\delta y)\,(y^{-q}\exp(x)-1)}\,\mathrm{d}y \right) dx.$$

The consecutive partial derivative of $Z(a,q,t,\delta,1)$ delivers the result (20).

In the case when $\rho=-1$, application of the Fermi–Dirac relation (11) to the polylogarithm yields

$$Z(a,q,t,\delta,-1)=-\frac{1}{\Gamma(t)}\int_0^\infty x^{t-1}\left(\int_0^\infty \frac{y^a}{y(1+\delta y)(y^{-q}\exp(x)+1)}\,\mathrm{d}y\right)dx.$$

The consecutive partial derivative of $Z(a,q,t,\delta,-1)$ delivers the result (20).

The following will be useful, subsequently, in the evaluation of $\mathbb{S}^{++}_{p+1,t}(q)$ and $\mathbb{S}^{+-}_{p+1,t}(q)$ including their half argument recurrence.

If $q=3,\delta=-1$ and $p=2$, CAS Mathematica yields

$$Z(a,3,t,-1,1)=\frac{1}{\Gamma(t)}\int_0^\infty x^{t-1}\left(\int_0^\infty \frac{y^a}{y(1-y)(y^{-3}\exp(x)-1)}\,\mathrm{d}y\right)dx$$

$$=\frac{\exp(-4\mathbf{i}a\pi/3)\,\pi}{3}\left(\begin{array}{c} 3\exp(\mathbf{i}a\pi/3)\csc(a\pi)\zeta(t)-\exp(\mathbf{i}ax)\csc\left(\frac{a\pi}{3}\right)\zeta\left(t,\frac{1-a}{3}\right)\\ +\exp(\mathbf{i}(ax+2\pi/3))\csc\left(\frac{\pi+a\pi}{3}\right)\zeta\left(t,\frac{1-a}{3}\right)\\ +\exp(\mathbf{i}(ax+\pi/3))\sec\left(\frac{\pi+2a\pi}{6}\right)\zeta\left(t,\frac{2-a}{3}\right). \end{array}\right),$$

the second derivative of which is,

$$X(2,3,t,-1,1)=\lim_{a->0}\left(\frac{\partial^2}{\partial a^2}Z(a,3,t,-1,1)\right)$$

$$=\frac{8\sqrt{3}\pi^3}{243}\left(\zeta\left(t,\frac{1}{3}\right)-\zeta\left(t,\frac{2}{3}\right)\right)+3^{t-1}\pi\,(t)_2\;\zeta(t+2)\,\mathbf{i}.$$
$$+\frac{4t}{9}\left(4\times 3^t-1\right)\zeta(2)\zeta(t+1)+\frac{1}{81}(t)_3\;\zeta(t+3)$$
$$+\frac{\sqrt{3}\pi}{81}(t)_2\;\left(\zeta\left(t+2,\frac{1}{3}\right)-\zeta\left(t+2,\frac{2}{3}\right)\right).$$

If $q=\frac{1}{3},\delta=1$ and $p=2$, then Mathematica yields

$$Z\left(a,\frac{1}{3},t,1,1\right)=\frac{1}{\Gamma(t)}\int_0^\infty x^{t-1}\left(\int_0^\infty \frac{y^a}{y(1+y)\left(y^{-\frac{1}{3}}\exp(x)-1\right)}\,\mathrm{d}y\right)dx$$

$$=6^{-t}\pi\left(-2^t\csc(a\pi)\,\eta(t)+\sec\left(\frac{\pi-6a\pi}{6}\right)\zeta\left(t,\frac{1}{6}\right)+\sec\left(\frac{\pi+6a\pi}{6}\right)\zeta\left(t,\frac{1}{3}\right)\right)$$
$$+6^{-t}\pi\left(-\sec\left(\frac{\pi-6a\pi}{6}\right)\zeta\left(t,\frac{2}{3}\right)-\sec\left(\frac{\pi+6a\pi}{6}\right)\zeta\left(t,\frac{5}{6}\right)\right)$$
$$+6^{-t}\pi\exp(-3\mathbf{i}a\pi)\,3\csc(3a\pi)\left(\zeta\left(t,\frac{1-a}{2}\right)-\zeta\left(t,\frac{2-a}{2}\right)\right),$$

the second derivative of which is,

$$X\left(2,\frac{1}{3},t,1,1\right)=\lim_{a->0}\left(\frac{\partial^2}{\partial a^2}Z\left(a,\frac{1}{3},t,1,1\right)\right)$$

$$\begin{aligned}= \ & 3^{-1-t}\left(t\right)_2\ \eta\left(t+3\right)-4\cdot 3^{2-t}\ t\zeta\left(2\right)\eta\left(t+1\right)-\pi 3^{1-t}\ \left(t\right)_2\ \eta\left(t+2\right)i\\ & +40\sqrt{3}\pi^3 6^{-2-t}\left(\zeta\left(t,\frac{1}{6}\right)+\zeta\left(t,\frac{1}{3}\right)-\zeta\left(t,\frac{2}{3}\right)-\zeta\left(t,\frac{5}{6}\right)\right).\end{aligned}$$

If $q=\frac{1}{5},\delta=-1,\ \rho=1$ and $p=1$, Mathematica yields the simplification,

$$X\left(1,\frac{1}{5},t,-1,-1\right)=\lim_{a->0}\left(\frac{\partial}{\partial a}Z\left(a,\frac{1}{5},t,-1,-1\right)\right)$$

$$\begin{aligned}= & \frac{27}{5^t}\eta\left(t\right)\zeta\left(2\right)+\frac{t\left(1+t\right)}{2\times 5^t}\eta\left(t+2\right)\\ & +3\times 2^{3-t}\times 5^{t-1/2}\zeta\left(2\right)\phi\left(\zeta\left(t,\frac{1}{10}\right)-\zeta\left(t,\frac{2}{5}\right)-\zeta\left(t,\frac{3}{5}\right)+\zeta\left(t,\frac{9}{10}\right)\right)\\ & +3\times 2^{3-t}\times 5^{t-1/2}\zeta\left(2\right)\phi^{\star}\left(\zeta\left(t,\frac{1}{5}\right)-\zeta\left(t,\frac{3}{10}\right)-\zeta\left(t,\frac{7}{10}\right)+\zeta\left(t,\frac{4}{5}\right)\right),\end{aligned}$$

where ϕ and $\phi^{\star}$ are the two zeros of the Fibonacci recurrence,

$$\phi=\frac{1+\sqrt{5}}{2},\ \ \phi^{\star}=\frac{1-\sqrt{5}}{2}.$$

□

Two examples are offered.

Example 1. Consider the following.
1) If $p=2,q=3$ and even $t\in\mathbb{N}\geq 2$, then from Theorem 1 and Lemma 1 it follows that

$$\begin{aligned}4\mathbb{S}^{++}_{3,t}(3)=4\sum_{n=1}^{\infty}\frac{H^{(3)}_{3n}}{n^t}= & \frac{2}{3^3}\zeta\left(t+3\right)+4\zeta\left(2\right)\zeta\left(t\right)-\frac{4t}{9}\left(4\times 3^t-1\right)\zeta\left(2\right)\zeta\left(t+1\right)\\ & -2\sum_{j=0}^{\left[\frac{t}{2}\right]}3^{t-2j}\left(t+1-2j\right)_2\zeta(2j)\zeta\left(t+3-2j\right)-\frac{1}{81}\left(t\right)_3\ \zeta\left(t+3\right)\\ & -\frac{8\sqrt{3}\pi^3}{243}\left(\zeta\left(t,\frac{1}{3}\right)-\zeta\left(t,\frac{2}{3}\right)\right)-\frac{\sqrt{3}\pi}{81}\left(t\right)_2\ \left(\zeta\left(t+2,\frac{1}{3}\right)-\zeta\left(t+2,\frac{2}{3}\right)\right).\end{aligned}$$

2) If $p=2,q=\frac{1}{3}$ and even $t\in\mathbb{N}\geq 2$, from Theorem 2 and Lemma 1 it follows that

$$4\left(\mathbb{S}^{++}_{3,t}(\frac{1}{3})-\frac{1}{4}\mathbb{S}^{++}_{3,t}(\frac{1}{6})\right)=2\times 3^3\zeta\left(t+3\right)-3^{-1-t}\left(t\right)_2\ \eta\left(t+3\right)$$

$$+ 2\sum_{j=0}^{\left[\frac{t}{2}\right]}\left(\frac{1}{3}\right)^{t-2j}(t+1-2j)_2\,\zeta(2j)\eta\left(t+3-2j\right)+4t3^{2-t}\,\zeta\left(2\right)\eta\left(t+1\right)$$

$$-\,40\sqrt{3}\pi^3 6^{-2-t}\left(\zeta\left(t,\frac{1}{6}\right)+\zeta\left(t,\frac{1}{3}\right)+\zeta\left(t,\frac{2}{3}\right)-\zeta\left(t,\frac{5}{6}\right)\right).$$

Next the alternating linear harmonic Euler sums $\mathbb{S}^{+-}_{p+1,t}(q)$ are investigated in terms of special functions.

Theorem 3. *If $p\in\mathbb{N},\ t\in\mathbb{N}\geq 1$ and $q\in\mathbb{R}>0$, then, for odd weight $p+1+t$, it follows that*

$$2\left(-1\right)^p p!\mathbb{S}^{+-}_{p+1,t}(q)=\frac{\left(-1\right)^p p!}{q^{p+1}}\eta\left(p+t+1\right)+2\left(-1\right)^p p!\eta(t)\zeta\left(p+1\right) \tag{22}$$

$$-2\sum_{j=0}^{\left[\frac{t}{2}\right]}\left(t+1-2j\right)_p q^{t-2j}\eta(2j)\zeta\left(p+t+1-2j\right)+\int_0^\infty\frac{\ln^p\left(y\right)\mathrm{Li}_t(-y^q)}{y\left(1-y\right)}\,\mathrm{d}y,$$

where $\mathbb{S}^{+-}_{p+1,t}(q)$ is the linear Euler harmonic sum (3), $\mathrm{Li}_t(\cdot)$ is the polylogarithmic function described by (10), $\zeta\left(t\right)$ is the classical Riemann zeta function and $\left(\cdot\right)_p$ is the Pochhammer symbol.

Proof. Using a similar approach to that in Theorem 1, the following integral is investigated on the real half line $x>0$:

$$X\left(p,q,t,-1,-1\right)=\int_0^1\frac{\ln^p\left(y\right)\mathrm{Li}_t(-y^q)}{y\left(1-y\right)}\,\mathrm{d}y+\int_1^\infty\frac{\ln^p\left(y\right)\mathrm{Li}_t(-y^q)}{y\left(1-y\right)}\,\mathrm{d}y.$$

Using the transformation $xy=1$ in the last integral and recovering the variable y instead of x in the resultant integral, yields the following:

$$X\left(p,q,t,-1,-1\right)=\int_0^1\frac{\ln^p\left(y\right)\mathrm{Li}_t(-y^q)}{y\left(1-y\right)}\,\mathrm{d}y-\left(-1\right)^p\int_0^1\frac{\ln^p\left(y\right)\mathrm{Li}_t(-\frac{1}{y^q})}{\left(1-y\right)}\,\mathrm{d}y.$$

Recalling the Jonquiĕre formula (14), the last integral is expressed as

$$X\left(p,q,t,-1,-1\right)=\int_0^1\frac{\ln^p\left(y\right)\mathrm{Li}_t(-y^q)}{y\left(1-y\right)}\,\mathrm{d}y-\left(-1\right)^p\int_0^1\frac{\ln^p\left(y\right)\mathrm{Li}_t(-\frac{1}{y^q})}{\left(1-y\right)}\,\mathrm{d}y.$$

$$=\int_0^1\frac{\ln^p\left(y\right)\mathrm{Li}_t(-y^q)}{y}\,\mathrm{d}y+\left(1+(-1)^{p+t}\right)\int_0^1\frac{\ln^p\left(y\right)\mathrm{Li}_t(-y^q)}{\left(1-y\right)}\,\mathrm{d}y$$

$$+\,2\left(-1\right)^{p+t}\sum_{j=0}^{\left[\frac{t}{2}\right]}\frac{q^{t-j}}{\left(t-2j\right)!}\eta\left(2j\right)\int_0^1\frac{\ln^{p+t-2j}\left(y\right)}{\left(1-y\right)}\mathrm{d}y.$$

The following integrals can be obtained routinely,

$$\int_0^1 \frac{\ln^p(y)\,\mathrm{Li}_t(-y^q)}{y}\,\mathrm{d}y = \frac{(-1)^p\,p!}{q^{p+1}}\eta(p+t+1),$$

$$\int_0^1 \frac{\ln^{p+t-2j}(y)}{1-y}\,\mathrm{d}y = (-1)^{p+t}(p+t-2j)!\zeta(p+t+1-2j)$$

and

$$\begin{aligned}\int_0^1 \frac{\ln^p(y)\,\mathrm{Li}_t(-y^q)}{(1-y)}\,\mathrm{d}y &= (-1)^p\,p!\sum_{n\geq 1}\frac{(-1)^n}{n^t}\zeta(p+1,nq)\\ &= (-1)^p\,p!\left(\mathbb{S}^{+-}_{p+1,t}(q)-\eta(t)\zeta(p+1)\right).\end{aligned}$$

Hence, it follows that

$$X(p,q,t,-1,-1) = \left(1+(-1)^{p+t}\right)(-1)^p\,p!\left(\mathbb{S}^{+-}_{p+1,t}(q)-\eta(t)\zeta(p+1)\right)$$

$$+\frac{(-1)^p\,p!}{q^{p+1}}\eta(p+t+1)+2\sum_{j=0}^{[\frac{t}{2}]}\frac{q^{t-j}}{(t-2j)!}\eta(2j)(p+t-2j)!\zeta(p+t+1-2j).$$

To include the $\mathbb{S}^{+-}_{p+1,t}(q)$ term, the weight $p+t+1$ is chosen to be odd, so that

$$X(p,q,t,-1,-1) = 2(-1)^p\,p!\left(\mathbb{S}^{+-}_{p+1,t}(q)-\eta(t)\zeta(p+1)\right)$$

$$+\frac{(-1)^p\,p!}{q^{p+1}}\eta(p+t+1)+\ 2\sum_{j=0}^{[\frac{t}{2}]}(t+1-2j)_p\,q^{t-j}\eta(2j)\zeta(p+t+1-2j).$$

Rearranging yields

$$2(-1)^p\,p!\mathbb{S}^{+-}_{p+1,t}(q) = \frac{(-1)^p\,p!}{q^{p+1}}\eta(p+t+1)+2(-1)^p\,p!\eta(t)\zeta(p+1)$$

$$-2\sum_{j=0}^{[\frac{t}{2}]}(t+1-2j)_p\,q^{t-2j}\eta(2j)\zeta(p+t+1-2j)+\int_0^\infty \frac{\ln^p(y)\,\mathrm{Li}_t(-y^q)}{y(1-y)}\,\mathrm{d}y.$$

This completes the proof of (3) and the identity (22) is achieved. □

As shown in Theorem 2, the next theorem establishes a recurrence relation involving the linear Euler harmonic sum $\mathbb{S}^{+-}_{p+1,t}(q)$ and $\mathbb{S}^{+-}_{p+1,t}(\frac{q}{2})$.

Theorem 4. *If $p \in \mathbb{N}$, $t \in \mathbb{N} \geq 1$ and $q \in \mathbb{R} > 0$, then the following identity is valid for odd weight $p+1+t$:*

$$2(-1)^p p!\left(\mathbb{S}^{+-}_{p+1,t}(q) - 2^{-p}\mathbb{S}^{+-}_{p+1,t}(\frac{q}{2})\right) = \frac{(-1)^{p+1} p!}{q^{p+1}}\eta(p+t+1) \tag{23}$$
$$-2\sum_{j=0}^{[\frac{t}{2}]}(t+1-2j)_p\, q^{t-2j}\eta(2j)\eta(p+t+1-2j)$$
$$+2(-1)^p p!\eta(p+1)\eta(t) - \int_0^\infty \frac{\ln^p(y)\operatorname{Li}_t(-y^q)}{y(1+y)}\,\mathrm{d}y,$$

where $\mathbb{S}^{+-}_{p+1,t}(q)$ is the linear Euler harmonic sum (3), $\operatorname{Li}_t(\cdot)$ is the polylogarithmic function described by (10), $\zeta(t)$ is the classical Riemann zeta function, $\eta(t)$ is the classical Riemann eta function and $(\cdot)_p$ is the Pochhammer symbol.

Proof. Following the same pattern as in Theorem 2, and from (17), putting $\delta = 1$ and $\rho = -1$, it follows that

$$X(p,q,t,1,-1) = \int_0^\infty \frac{\ln^p(y)\operatorname{Li}_t(-y^q)}{y(1+y)}\,\mathrm{d}y = \int_0^1 \frac{\log^p(y)\operatorname{Li}_t(-y^q)}{y}\,\mathrm{d}y$$
$$-\int_0^1 \frac{\log^p(y)\operatorname{Li}_t(-y^q)}{1+y}\,\mathrm{d}y + (-1)^p\int_0^1 \frac{\log^p(y)\operatorname{Li}_t(-\frac{1}{y^q})}{1+y}\,\mathrm{d}y.$$

Using the Jonquiĕre relation (14) allows the representation

$$\begin{aligned} X(p,q,t,1,-1) &= \int_0^1 \frac{\log^p(y)\operatorname{Li}_t(-y^q)}{y}\,\mathrm{d}y \\ &\quad -(-1)^{p+t}\sum_{j=0}^{[\frac{t}{2}]}\frac{q^{t-j}}{(t-2j)!}\eta(2j)\int_0^1 \frac{\log^{p+t-2j}(y)}{1+y}\,\mathrm{d}y \\ &\quad -\left(1+(-1)^{p+t}\right)\int_0^1 \frac{\log^p(y)\operatorname{Li}_t(-y^q)}{1+y}\,\mathrm{d}y. \end{aligned}$$

The following integrals can then be obtained:

$$\begin{aligned} \int_0^1 \frac{\log^p(y)\operatorname{Li}_t(-y^q)}{y}\,\mathrm{d}y &= \frac{(-1)^{p+1} p!}{q^{p+1}}\eta(p+t+1), \\ \int_0^1 \frac{\log^{p+t-2j}(y)}{1+y}\mathrm{d}y &= -(p+t-2j)!\eta(p+t+1-2j) \end{aligned}$$

and

$$\int_0^1 \frac{\ln^p(y)\operatorname{Li}_t(-y^q)}{1+y}dx = (-1)^p p!\sum_{n\geq 1}\frac{(-1)^n}{n^t}\sum_{j\geq 0}\frac{(-1)^j}{(qn+j+1)^{p+1}}$$

$$\begin{aligned}
&= \frac{(-1)^p\, p!}{2^{p+1}} \sum_{n\geq 1} \frac{(-1)^n}{n^t} \left(\zeta\left(p+1, \frac{qn+1}{2}\right) - \zeta\left(p+1, \frac{qn+2}{2}\right)\right) \\
&= \frac{(-1)^p\, p!}{2^{p+1}} \sum_{n\geq 1} \frac{(-1)^n}{n^t} \left(\psi^{(p)}\left(\frac{qn+2}{2}\right) - \psi^{(p)}\left(\frac{qn+1}{2}\right)\right) \\
&= \frac{(-1)^p\, p!}{2^{p+1}} \sum_{n\geq 1} \frac{(-1)^n}{n^t} \left(H^{(p+1)}_{\frac{qn}{2}} - H^{(p+1)}_{\frac{qn-1}{2}} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{(-1)^p\, p!}{2^{p+1}} \sum_{n\geq 1} \frac{(-1)^{n+1}}{n^t} \left(2^{p+1} H^{(p+1)}_{qn} - 2H^{(p+1)}_{\frac{qn}{2}} - 2^{p+1}\eta\left(p+1\right)\right) \\
&= (-1)^p\, p! \sum_{n\geq 1} \frac{(-1)^{n+1}}{n^t} \left(H^{(p+1)}_{qn} - 2^{-p} H^{(p+1)}_{\frac{qn}{2}} \right) - (-1)^p\, p!\eta\left(t\right)\eta\left(p+1\right).
\end{aligned}$$

For odd weight $p+1+t$, we have

$$\begin{aligned}
\int_0^1 \frac{\log^p\left(y\right)\mathrm{Li}_t(-y^q)}{y}\,\mathrm{d}y \;=\; & \frac{(-1)^{p+1}\, p!}{q^{p+1}} \eta\left(p+t+1\right) + 2\left(-1\right)^p p!\eta\left(t\right)\eta\left(p+1\right) \\
& -2\sum_{j=0}^{\left[\frac{t}{2}\right]} \left(t+1-2j\right)_p q^{t-2j} \eta(2j)\eta\left(p+t+1-2j\right) \\
& -2\left(-1\right)^p p! \sum_{n\geq 1} \frac{(-1)^{n+1}}{n^t} \left(H^{(p+1)}_{qn} - 2^{-p} H^{(p+1)}_{\frac{qn}{2}} \right),
\end{aligned}$$

and rearranging results in (23) completes the proof of Theorem 4. □

Some examples follow.

Example 2. Consider the following.

1) If $p=1, q=\frac{1}{5}$ and $t\in\mathbb{N}\geq 2$ is odd, then, from Theorem 3 and Lemma 1, it follows that

$$2\mathbb{S}^{+-}_{2,t}(\frac{1}{5}) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H^{(2)}_{\frac{n}{5}}}{n^t} = \left(25 - \frac{t\left(1+t\right)}{2\times 5^t}\right)\eta\left(t+2\right)$$

$$+\left(2 - \frac{27}{5^t}\right)\zeta\left(2\right)\eta\left(t\right) + 2\sum_{j=0}^{\left[\frac{t}{2}\right]} 5^{2j-t}\left(t+1-2j\right)_1 \eta(2j)\zeta\left(t+2-2j\right)$$

$$-3\times 2^{3-t}\times 5^{t-1/2}\zeta\left(2\right)\phi\left(\zeta\left(t,\frac{1}{10}\right) - \zeta\left(t,\frac{2}{5}\right) - \zeta\left(t,\frac{3}{5}\right) + \zeta\left(t,\frac{9}{10}\right)\right)$$

$$-3\times 2^{3-t}\times 5^{t-1/2}\zeta\left(2\right)\phi^{\star}\left(\zeta\left(t,\frac{1}{5}\right) - \zeta\left(t,\frac{3}{10}\right) - \zeta\left(t,\frac{7}{10}\right) + \zeta\left(t,\frac{4}{5}\right)\right).$$

2) If $p=1,q=4$ and $t\in\mathbb{N}\geq 2$ is odd, then, from Theorem 4 and Lemma 1, it follows that

$$2\left(\mathbb{S}^{+-}_{2,t}(4)-\frac{1}{2}\mathbb{S}^{+-}_{2,t}(2)\right)=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H^{(2)}_{4n}}{n^t}-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H^{(2)}_{2n}}{n^t}$$

$$=2^{t-3}\pi t\beta(t+1)+\frac{(t-1)(t+2)}{32}\eta(t+2)+2\eta(2)\eta(t)-\frac{15}{16}\zeta(2)\eta(t)$$

$$+2\sum_{j=0}^{\left[\frac{t}{2}\right]}(t+1-2j)_1 4^{t-2j}\eta(2j)\eta(t+2-2j)$$

$$-3\times 2^{-5/2-t}\zeta(2)\left(\zeta\left(t,\frac{1}{8}\right)-\zeta\left(t,\frac{3}{8}\right)-\zeta\left(t,\frac{5}{8}\right)+\zeta\left(t,\frac{7}{8}\right)\right)$$

$$-2^{-9/2-t}\pi t\left(\zeta\left(t+1,\frac{1}{8}\right)+\zeta\left(t+1,\frac{3}{8}\right)-\zeta\left(t+1,\frac{5}{8}\right)-\zeta\left(t+1,\frac{7}{8}\right)\right).$$

3) The lengthy calculations involved for Theorem 3 do not give a nice compact representation for the Euler sum. Consider $p=3,q=2$ and odd t, $t\in\mathbb{N}\geq 2$. Then, from Theorem 3 and Lemma 1, it follows that

$$\mathbb{S}^{+-}_{4,t}(2)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H^{(4)}_{2n}}{n^t}=\frac{1}{12}\left(\frac{3}{8}-\frac{1}{64}t(1+t)(2+t)(3+t)\right)\eta(t+4)$$

$$+\frac{1}{6}\sum_{j=0}^{\left[\frac{t}{2}\right]}5^{t-2j}(t+1-2j)_3\eta(2j)\zeta(t+4-2j)-\frac{7}{128}\eta(t)\zeta(4)$$

$$-2^{t-5}\pi^3 t\beta(t+1)-\frac{1}{3}2^{t-3}\pi t(t+1)(t+2)\beta(t+3)$$

$$-\frac{1}{64}t(1+t)\eta(t+2)\zeta(2).$$

Remark 1. From the relationship

$$\mathbb{S}^{++}_{p+1,t}(2q)=2^{t-1}\left(\mathbb{S}^{++}_{p+1,t}(q)-\mathbb{S}^{+-}_{p+1,t}(q)\right),$$

for $q=1$, the identities (4) and (5) can be used to evaluate $\mathbb{S}^{++}_{p+1,t}(1)$ and $\mathbb{S}^{+-}_{p+1,t}(1)$, respectively, and this yields a general identity for $\mathbb{S}^{++}_{p+1,t}(2)$. Alternatively, from (16), for odd weight $p+1+t$, the following is obtained:

$$2(-1)^p p!\mathbb{S}^{++}_{p+1,t}(2)=\frac{(-1)^p p!}{2^{p+1}}\zeta(p+t+1)+2(-1)^p p!\zeta(p+1)\zeta(t)$$

$$-2\sum_{j=0}^{\left[\frac{t}{2}\right]}(t+1-2j)_p 2^{t-2j}\zeta(2j)\zeta(p+t+1-2j)-\Re\left(\int_0^{\infty}\frac{\ln^p(y)\,\mathrm{Li}_t(y^2)}{y(1-y)}\,\mathrm{d}y\right).$$

For the evaluation of the integral, consider

$$Z(a,2,t,-1,1)=\Re\left(\frac{1}{\Gamma(t)}\int_0^\infty x^{t-1}\left(\int_0^\infty \frac{y^a}{y(1-y)(y^{-2}\exp(x)-1)}\,\mathrm{d}y\right)dx\right).$$
$$=\frac{\pi}{2}\zeta(t)\left(\cot\left(\frac{a\pi}{2}\right)-\tan\left(\frac{a\pi}{2}\right)\right)$$
$$+\frac{\pi}{2}\tan\left(\frac{a\pi}{2}\right)\zeta\left(t,\frac{1-a}{2}\right)-\frac{\pi}{2}\cot\left(\frac{a\pi}{2}\right)\zeta\left(t,1-\frac{a}{2}\right).$$

The consecutive derivative operator, delivers the needed result

$$\begin{aligned} X(p,2,t,-1,1) &= \lim_{a->0}\left(\frac{\partial^p}{\partial a^p}Z(a,2,t,-1,1)\right) \\ &= \int_0^\infty \frac{\ln^p(y)\,\mathrm{Li}_t(y^2)}{y(1-y)}\,\mathrm{d}y. \end{aligned}$$

If $p=4$, the following identity is obtained:

$$X(4,2,t,-1,1)=\Re\left(\int_0^\infty \frac{\ln^4(y)\,\mathrm{Li}_t(y^2)}{y(1-y)}\,\mathrm{d}y\right)=3t\left(15\times 2^t-7\right)\zeta(4)\zeta(t{+}1)$$
$$+\frac{1}{2}(t)_3\left(3\times 2^{t+2}-7\right)\zeta(2)\zeta(t+3)-\frac{1}{160}(t)_5\zeta(t+5),$$

and therefore, for even t, $t\geq 2$, it follows that,

$$2\times 4!\mathbb{S}^{++}_{5,t}(2)=\frac{4!}{2^5}\zeta(t+5)+2\times 4!\zeta(5)\zeta(t)$$
$$-2\sum_{j=0}^{\left[\frac{t}{2}\right]}(t+1-2j)_4\,2^{t-2j}\zeta(2j)\zeta(t{+}5{-}2j)-3t\left(15\times 2^t-7\right)\zeta(4)\zeta(t{+}1)$$
$$-\frac{1}{2}(t)_3\left(3\times 2^{t+2}-7\right)\zeta(2)\zeta(t+3)+\frac{1}{160}(t)_5\zeta(t+5).$$

If $q=1$, then (4) and (16) are equivalent. From (16), it follows that

$$2(-1)^p p!\mathbb{S}^{++}_{p+1,t}(1)=\left((-1)^p p!+(t+1)_p\right)\zeta(p+t+1)+2(-1)^p p!\zeta(p{+}1)\zeta(t)$$
$$-2\sum_{j=1}^{\left[\frac{t}{2}\right]}(t+1-2j)_p\,\zeta(2j)\zeta(p+t+1-2j)-\Re\left(\int_0^\infty \frac{\ln^p(y)\,\mathrm{Li}_t(y)}{y(1-y)}\,\mathrm{d}y\right).$$

From (20), it follows that

$$\Re\left(\int_0^\infty \frac{\ln^p(y)\,\mathrm{Li}_t(y)}{y(1-y)}\,\mathrm{d}y\right)=\lim_{a->0}\left(\frac{\partial^p}{\partial a^p}(\pi\cot(a\pi))(\zeta(t)-\zeta(t,1-a))\right).$$

Consider the case $p=5$ and $t>1$, an odd integer. Then it follows that

$$-2\times 5!\mathbb{S}^{++}_{6,,t}(1)=(-5!+(t+1)_5)\zeta(t+6)-2\times 5!\zeta(6)\zeta(t)$$

$$-2\sum_{j=1}^{\left[\frac{t}{2}\right]}(t+1-2j)_5\,\zeta(2j)\zeta\left(t+6-2j\right)-\frac{4}{3}\left(t\right)_2\pi^4\zeta(t+2)$$
$$-\frac{5}{3}\left(t\right)_4\pi^2\zeta(t+4)+\frac{1}{6}\left(t\right)_6\pi^4\zeta(t+6).$$

Similarly, if $q=1$, then (5) and (22) are equivalent. From (22), it follows that

$$2\left(-1\right)^p p!\mathbb{S}^{+-}_{p+1,t}(1)=\left(-1\right)^p p!\eta\left(p+t+1\right)+2\left(-1\right)^p p!\eta(t)\zeta\left(p+1\right) \tag{24}$$
$$-2\sum_{j=0}^{\left[\frac{t}{2}\right]}(t+1-2j)_p\,\eta(2j)\zeta\left(p+t+1-2j\right)+\int_0^\infty\frac{\ln^p\left(y\right)\mathrm{Li}_t(-y)}{y\left(1-y\right)}\,\mathrm{d}y.$$

From (20), it follows that

$$\int_0^\infty\frac{\ln^p\left(y\right)\mathrm{Li}_t(-y)}{y\left(1-y\right)}\,\mathrm{d}y=\lim_{a->0}\left(\frac{\partial^p}{\partial a^p}\left(\pi\csc\left(a\pi\right)\right)\begin{pmatrix}2^{-t}\left(\zeta(t\ \frac{1-a}{2})-\zeta(t,1-\frac{a}{2})\right)\\-\exp\left(-\mathbf{i}a\pi\right)\eta(t)\end{pmatrix}\right)$$

and therefore from (24) the resulting identity is

$$2\left(-1\right)^p p!\mathbb{S}^{+-}_{p+1,t}(1)=\left(-1\right)^p p!\eta\left(p+t+1\right)+2\left(-1\right)^p p!\eta(t)\zeta\left(p+1\right)$$
$$-2\sum_{j=0}^{\left[\frac{t}{2}\right]}(t+1-2j)_p\,\eta(2j)\zeta\left(p+t+1-2j\right)$$
$$+\lim_{a->0}\left(\frac{\partial^p}{\partial a^p}\left(\pi\csc\left(a\pi\right)\right)\begin{pmatrix}2^{-t}\left(\zeta(t,\ \frac{1-a}{2})-\zeta(t,1-\frac{a}{2})\right)\\-\exp\left(-\mathbf{i}a\pi\right)\eta(t)\end{pmatrix}\right),$$

which is in a form slightly different from (5). If $p=4$ and t is an even integer, it follows that

$$48\mathbb{S}^{+-}_{5,t}\left(1\right)=24\eta\left(t+5\right)+48\eta(t)\zeta\left(5\right)-2\sum_{j=0}^{\left[\frac{t}{2}\right]}(t+1-2j)_4\,\eta(2j)\zeta\left(t+5-2j\right)$$
$$+\frac{7}{15}\left(t\right)_1\pi^4\eta(t+1)+\frac{2}{3}\left(t\right)_3\pi^2\eta(t+3)+\frac{3}{15}\left(t\right)_5\eta(t+5).$$

Remark 2. It is observed that a similar integral as that described by (21) was encountered in the course of a work on statistical plasma physics, in the so-called Sommerfeld temperature-expansion of the electronic entropy. Such a formalism plays a major role in the calculation of the equation of state of dense plasmas (see [4]). In particular, an integral representation for $\zeta\left(4\right)$, was obtained in [19].

3. Concluding remarks

The series (4) and (5) were developed by Borwein et al. [6], [7] and Flajolet et al. [14] both being generalizations of earlier work initiated by Euler in the 18^{th} century. In this paper, both (4) and (5) are generalized by extending the harmonic numbers from integer argument to a real positive argument. There exist, scattered throughout the literature individual results, such as $\mathbb{S}^{++}_{p+1,t}(\frac{1}{2})$ and $\mathbb{S}^{+-}_{p+1,t}(\frac{1}{2})$, however there is no systematic study of the expressions $\mathbb{S}^{++}_{p+1,t}(q)$ and $\mathbb{S}^{+-}_{p+1,t}(q)$ for $q \in \mathbb{R}^{+}$.

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