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On the Diophantine equation $a^x + 8^y = z^2$

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Abstract. In this article, we studied the Diophantine equation $a^x + 8^y = z^2$, where a is a fixed positive integer with $a \equiv 3 \pmod 4$ and x, y, z are non-negative integers. The results show all non-negative integer solutions of this Diophantine equation.

1. Introduction

A Diophantine equation is one for which only integer solutions are of interest. It is a popular topic in number theory and has many vital applications in algebra, analytical geometry, and trigonometry. The Diophantine equations have been studied since antiquity and are mathematically both challenging and attractive because of the great diversity of methods that solve them.

Many authors have studied the Diophantine equation $a^x + b^y = z^2$, where a, b are fixed positive integers and x, y, z are non-negative integers (see, for instance [1, 3, 5, 4, 2]).

In 2021, Pakapongpun and Chattae [3] showed that the Diophantine equation $a^x + (a+2)^y = z^2$, where a is a positive integer with $a \equiv 3 \pmod{20}$, has the unique non-negative integer solution $(x, y, z) = (1, 0, \sqrt{a+1})$, where $a + 1$ is a perfect square number.

In 2022, Viriyapong and Viriyapong [5] studied the Diophantine equation $n^x + 19^y = z^2$, where n is a positive integer with $a \equiv 2 \pmod{57}$. They proved that this equation has a unique non-negative integer solution $(x, y, z) = (3, 0, 3)$, where $n = 2$.

In 2024, Srisawat and Sriprad [4] showed that the Diophantine equation $a^x + b^y = z^2$, where a, b are positive odd integers with $a \equiv 1 \pmod 3$ and $b \equiv 1 \pmod 3$, has no non-negative integer solution.

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2. Preliminaries

Theorem 1 (Catalan's Conjecture [2])**.** *The Diophantine equation* $a^x - b^y = 1$*, where* a, b, x, y *are integers with* $\min\{a, b, x, y\} > 1$*, has the unique solution* $(a, b, x, y) = (3, 2, 2, 3)$*.*

Lemma 1. *The Diophantine equation* $1 + 8^y = z^2$*, where* y, z *are non-negative integers, has the unique non-negative integer solution* $(y, z) = (1, 3)$*.*

Proof. Assume that there exist non-negative integers y, z such that $1 + 8^y = z^2$. If $y = 0$, then $z^2 = 2$, which is impossible. Thus $y \geq 1$. We obtain $z^2 = 8^y + 1 \geq 8 + 1 = 9$. Then $z \geq 3$. According to Theorem 1, we have $y = 1$. It implies that $z^2 = 9$, so $z = 3$. Thus, $(y, z) = (1, 3)$ is a non-negative integer solution of equation $1 + 8^y = z^2$. $\square$

Lemma 2. *The Diophantine equation* $a^x + 1 = z^2$*, where* a *is a fixed positive integer with* $a \equiv 3 \pmod 4$ *and* x, z *are non-negative integers, has the unique non-negative integer solution* $(x, z) = (1, \sqrt{a+1})$*, where* $a + 1$ *is a perfect square.*

Proof. Let a be a fixed positive integer with $a \equiv 3 \pmod 4$. Assume that there exist non-negative integers x, z such that $a^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$, which is impossible. Thus $x \geq 1$. Since $a \geq 3$, we obtain $z^2 = a^x + 1 \geq 3 + 1 = 4$. Then $z \geq 2$. According to Theorem 1, we have $x = 1$. It implies that $z^2 = a + 1$, so $z = \sqrt{a+1}$. Thus, $(x, z) = (1, \sqrt{a+1})$, where $a + 1$ is a perfect square, is a non-negative integer solution of equation $a^x + 1 = z^2$. $\square$

3. Main results

Theorem 2. *The Diophantine equation* $a^x + 8^y = z^2$*, where* a *is a fixed positive integer with* $a \equiv 3 \pmod 4$ *and* x, y, z *are non-negative integers, has the non-negative integer solutions*

(1) $(x, y, z) = (0, 1, 3)$ *for all* a*,*
(2) $(x, y, z) = (1, 0, \sqrt{a+1})$*, if* $a + 1$ *is a perfect square,*
(3) $(x, y, z) = (2, \frac{1}{3}(2 + \log_2(a+1)), a + 2)$*, if* $a = 2^{3s+1} - 1$ *for some* $s \in \mathbb{Z}^+$*.*

Proof. Let a be a fixed positive integer with $a \equiv 3 \pmod 4$. Assume that there exist non-negative integers x, y, z such that $a^x + 8^y = z^2$. We divide the proof into four cases as follows:

Case 1. $x = 0$ and $y = 0$.
We have $z^2 = 2$, which is a contradiction.

Case 2. $x = 0$ and $y > 0$.
We have $1 + 8^y = z^2$. By Lemma 1, we obtain $(x, y, z) = (0, 1, 3)$.

Case 3. $x > 0$ and $y = 0$.
We have $a^x+1 = z^2$. By Lemma 2, we obtain $(x, y, z) = (1, 0, \sqrt{a+1})$, where $a + 1$ is a perfect square.

Case 4. $x > 0$ and $y > 0$.
Since a^x and 8^y are an odd and an even integer, respectively, $z^2 = a^x + 8^y$ is an odd integer. It implies that z is an odd integer. We obtain $z^2 \equiv 1 \pmod 4$. Now, since $8^y \equiv 0 \pmod 4$ and $z^2 = a^x + 8^y$, we have $z^2 \equiv a^x \pmod 4$. Thus $a^x \equiv 1 \pmod 4$. We get that x is even. Let $x = 2k, k \in \mathbb{Z}^+$. We obtain $a^{2k} + 8^y = z^2$. Then $z^2 - a^{2k} = 8^y$. Thus $(z - a^k)(z + a^k) = 2^{3y}$. There exist integer $u \geq 0$ such that $z - a^k = 2^u$ and $z + a^k = 2^{3y-u}$, where $3y - u > u, 3y - 2u > 0$. We obtain $2a^k = 2^{3y-u} - 2^u = 2^u(2^{3y-2u} - 1)$. We get that $u = 1$. It implies that $2^{3y-2} - a^k = 1$.

If $k = 1$, we have $x = 2$ and $z = a + 2$, also $a = 2^{3y-2} - 1$. The latter is equivalent to $y = \frac{1}{3}(2 + \log_2(a + 1))$, and it is not difficult to check that y is an integer if and only if a is of form $2^{3s+1} - 1$ for some $s \in \mathbb{Z}^+ \cup \{0\}$. It can be checked from the original equation that $(x, y, z) = (2, \frac{1}{3}(2 + \log_2(a + 1)), a + 2)$ is indeed a solution to the original equation $a^x + 8^y = z^2$ if $a = 2^{3s+1} - 1$ for some $s \in \mathbb{Z}^+ \cup \{0\}$. Since, moreover, we have assumed $a \equiv 3 \pmod 4$ for the original equation, solutions are given when $s \in \mathbb{Z}^+$.

If $k > 1$, we obtain $2^{3y-2} = a^k + 1 > 3 + 1 = 4 = 2^2$. Then $3y - 2 > 2$. By Theorem 1, the equation $2^{3y-2} - a^k = 1$ has no solution.

By Cases 1–4, the non-negative integer solutions of the Diophantine equation $a^x + 8^y = z^2$, where a is a fixed positive integer with $a \equiv 3 \pmod 4$, are given by

(1) $(x, y, z) = (0, 1, 3)$ for all a,
(2) $(x, y, z) = (1, 0, \sqrt{a+1})$, where $a + 1$ is a perfect square,
(3) $(x, y, z) = (2, \frac{1}{3}(2+\log_2(a+1)), a+2)$, where $a = 2^{3s+1} - 1$ for some $s \in \mathbb{Z}^+$.

□

Corollary. The following are obvious conclusions from Theorem 2.

(1) The Diophantine equation $3^x+8^y = z^2$, where x, y, z are non-negative integers, has exactly two non-negative integer solutions $(x, y, z) = (0, 1, 3), (1, 0, 2)$.
(2) The Diophantine equation $7^x+8^y = z^2$, where x, y, z are non-negative integers, has a unique non-negative integer solution $(x, y, z) = (0, 1, 3)$.
(3) The Diophantine equation $15^x + 8^y = z^2$, where x, y, z are non-negative integers, has exactly three non-negative integer solutions $(x, y, z) = (0, 1, 3), (1, 0, 4), (2, 2, 17)$.

(4) The Diophantine equation $27^x + 8^y = z^2$, where x, y, z are non-negative integers, has a unique non-negative integer solution $(x, y, z) = (0, 1, 3)$.
(5) The Diophantine equation $35^x + 8^y = z^2$, where x, y, z are non-negative integers, has exactly two non-negative integer solutions $(x, y, z) = (0, 1, 3), (1, 0, 6)$.
(6) The Diophantine equation $127^x + 8^y = z^2$, where x, y, z are non-negative integers, has exactly two non-negative integer solutions $(x, y, z) = (0, 1, 3), (2, 3, 129)$.

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