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A note on Delta-points in Lipschitz-free spaces

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Abstract. A norm one element x of a Banach space is a Daugavet-point (respectively, a Δ-point) if every slice of the unit ball (respectively, every slice of the unit ball containing x) contains an element that is almost at distance 2 from x. It is known that any finitely supported element μ in the unit sphere of a Lipschitz-free space $\mathcal{F}(M)$ is a Δ-point if and only if for every $\varepsilon > 0$ and a slice S of $B_{\mathcal{F}(M)}$ with $\mu \in S_{\mathcal{F}(M)}$ there exist $u, v \in M$ with $u \neq v$ such that $m_{uv} \in S$ and $d(u,v) < \varepsilon$. The aim of this note is to show that this characterization can also be applied to certain convex series of molecules.

1. Introduction

Daugavet and Δ-points were first introduced in [2], as pointwise versions of the Daugavet property and the diametral local diameter 2 property (DLD2P), respectively. Recall that a Banach space X has the *Daugavet property* (respectively, the *DLD2P*), if for every $x \in S_X$ and for every slice of the unit ball (respectively, every slice of the unit ball containing x) there exists an element in the slice that is almost at distance 2 from x. By a *slice of the unit ball* we mean any set of the form

$$S(x^*, \alpha) := \{y \in B_X \colon x^*(y) > 1 - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$. For a Banach space X and element $x \in S_X$ we say that

(1) x is a *Daugavet-point* if for every slice S of B_X and for every $\varepsilon > 0$ there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$;
(2) x is a Δ-*point* if for every slice S of B_X with $x \in S$ and for every $\varepsilon > 0$ there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$.

From this definition it is clear that a Banach space X has the Daugavet property (respectively the DLD2P) if and only if every element of the unit

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sphere is a Daugavet point (respectively a Δ-point). The study of Daugavet and Δ-points has received a lot of attention in the recent years (see, e.g, [2], [3], [6]). In Lipschitz-free spaces, these points were first considered in [10]. In particular, it was shown that a molecule m_{xy} is a Δ-point if and only if for every $\varepsilon > 0$ and a slice S of $B_{\mathcal{F}(M)}$ with $m_{xy} \in S$ there exist $u, v \in M$ with $u \neq v$ such that $m_{uv} \in S$ and $d(u, v) < \varepsilon$ (see [10, Theorem 4.7]). Furthermore, one direction can be applied to any element of the unit sphere of a Lipschitz-free space, i.e., if $\mu \in S_{\mathcal{F}(M)}$ is such that for every $\varepsilon > 0$ and a slice S of $B_{\mathcal{F}(M)}$ with $\mu \in S$ there exist $u, v \in M$ with $u \neq v$ such that $m_{uv} \in S$ and $d(u, v) < \varepsilon$, then μ is a Δ-point. However, it is unknown whether the converse also holds for arbitrary elements of the unit sphere. Several papers have addressed this question, and thus far we know that the characterization holds in the following cases:

- for finitely supported elements (see [11, Theorem 4.4]);
- for elements of a Lipschitz-free space over a proper metric space (see [12, Theorem 3.2]);
- for elements of a Lipschitz-free space over a subset of an $\mathbb{R}$-tree (see [1, Theorem 4.5]);
- for $\mu \in S_{\mathcal{F}(M)}$ such that

$$\lim_{\delta \to 0} \sup_{x,y \in \mathrm{supp}(\mu) \cup \{0\}} \alpha \left(\{ p \in M \,:\, d(p, x) + d(p, y) < (1 + \delta) d(x, y) \} \right) = 0$$

(see [1, Proposition 4.3]).

The potential characterization is strongly related to the open question that is well-known among the experts in the field: whether a Lipschitz-free space over a uniformly discrete metric space can contain Δ-points. Recall that a metric space M is *uniformly discrete* if there exists $\varepsilon > 0$ such that $d(u, v) > \varepsilon$ for every $u, v \in M$ with $u \neq v$. It is clear that if one were to prove the aforementioned characterization, then the answer would be negative. Furthermore, it is known that every element in a Lipschitz-free space over a uniformly discrete metric space is a convex series of molecules (see [4]), thus it would be sufficient to prove the characterization for all convex series of molecules.

The aim of this note is to generalize the technique from the proof of [11, Theorem 4.4] to prove the characterization for certain convex series of molecules (see Theorem 1.1). Whilst we do not manage to prove the characterization for all convex series of molecules, this is a step forward for solving the question. Furthermore, diving into the details of why our result does not work for all convex series of molecules could also prove useful if one were to look for a counterexample to the question mentioned in the previous paragraph.

Let us now also introduce Lipschitz-free spaces. Let M be a metric space with metric d and a fixed point 0. We denote by $\mathrm{Lip}_0(M)$ the Banach space of all Lipschitz functions $f\colon M \to \mathbb{R}$ with $f(0) = 0$ equipped with the obvious linear structure and the norm

$$\|f\| := \sup\Big\{\frac{|f(x) - f(y)|}{d(x,y)} : x, y \in M, x \neq y\Big\}.$$

Let $\delta\colon M \to \mathrm{Lip}_0(M)^*$ be the canonical isometric embedding of M into $\mathrm{Lip}_0(M)^*$, which is given by $x \mapsto \delta_x$ where $\delta_x(f) = f(x)$. The norm closed linear span of $\delta(M)$ in $\mathrm{Lip}_0(M)^*$ is called the *Lipschitz-free space over* M and is denoted by $\mathcal{F}(M)$ (see [8] and [13] for the background). The Lipschitz-free space is a predual of the space of Lipschitz functions, i.e., $\mathcal{F}(M)^* = \mathrm{Lip}_0(M)$. An element in $\mathcal{F}(M)$ of the form

$$m_{xy} := \frac{\delta_x - \delta_y}{d(x,y)}$$

for $x, y \in M$ with $x \neq y$ is called a *molecule*. It is well known that $\|m_{xy}\| = 1$ for every $x, y \in M$ with $x \neq y$ and that the closed convex hull of all molecules forms the unit ball of the Lipschitz-free space.

We are now ready to present our main result.

Theorem 1.1. *Let $I \subset \mathbb{N}$, let $\lambda_i > 0$, $i \in I$, with $\sum_{i\in I} \lambda_i = 1$, and $m_{x_iy_i} \in S_{\mathcal{F}(M)}$, $i \in I$ be such that $\mu := \sum_{i\in I} \lambda_i m_{x_iy_i} \in S_{\mathcal{F}(M)}$. Assume that $\max_{i,j\in I} d(x_i, y_j) < \infty$ and there exists $\delta > 0$ such that for every pairwise distinct $k_1, \dots, k_m \in I$ and $k_{m+1} := k_1$ we have*

$$\sum_{j=1}^{m} d(x_{k_j}, y_{k_{j+1}}) > \sum_{j=1}^{m} d(x_{k_j}, y_{k_j}) \Rightarrow (1-\delta)\sum_{j=1}^{m} d(x_{k_j}, y_{k_{j+1}}) > \sum_{j=1}^{m} d(x_{k_j}, y_{k_j}).$$

Then μ is a Δ-point if and only if for every $\varepsilon > 0$ and every slice S of $B_{\mathcal{F}(M)}$ with $\mu \in S$ there exist $u, v \in M$ with $u \neq v$ such that $m_{uv} \in S$ and $d(u, v) < \varepsilon$.

The theorem can be applied, for example, if $I = \mathbb{N}$ and $d(x_i, y_j) = 1$ for every $i, j \in \mathbb{N}$, a case which is not covered by previous results. Thus this result provides us with new information in our attempt to characterize an arbitrary element of a Lipschitz-free space. Additionally, it also provides an alternative proof for showing that the Lipschitz-free space over $\mathbb{N}$ with the discrete metric does not contain Δ-points.

2. Proof of the main theorem

Throughout the section, let M be a metric space with a fixed point 0. Let $I \subseteq \mathbb{N}$, $\lambda_i > 0$, $i \in I$ with $\sum_{i\in I} \lambda_i = 1$, and $m_{x_iy_i} \in S_{\mathcal{F}(M)}$, $i \in I$ be such that $\mu := \sum_{i\in I} \lambda_i m_{x_iy_i} \in S_{\mathcal{F}(M)}$. By [5, Theorem 2.4], for every sequence

$k_1, \ldots, k_{m+1} \in I$ with $k_1 = k_{m+1}$, we have

$$\sum_{j=1}^{m} d(x_{k_j}, y_{k_{j+1}}) \geq \sum_{j=1}^{m} d(x_{k_j}, y_{k_j}). \tag{1}$$

Furthermore, by, e.g., [11, Section 4] the equality in (1) yields that for every $j \in \{1, \ldots, m\}$ with $x_{k_j} \neq y_{k_{j+1}}$ there exist $\nu \in S_{\mathcal{F}(M)}$ and $\lambda \in (0,1]$ such that $\mu = \lambda m_{x_{k_j} y_{k_{j+1}}} + (1-\lambda)\nu$, or equivalently, there exists $\lambda \in (0,1]$ such that $\left\| \mu - \lambda m_{x_{k_j} y_{k_{j+1}}} \right\| = 1 - \lambda$.

Next we will define numbers a_{ij} that measure how close we are to attaining the equality in (1). For $i, j \in I$, let

$$a_{ij} = \sup \left\{ \frac{\sum_{j=1}^{m} d(x_{k_j}, y_{k_j})}{\sum_{j=1}^{m} d(x_{k_j}, y_{k_{j+1}})} : k_1, \ldots, k_{m+1} \in I, k_1 = k_{m+1} = i, k_2 = j \right\}.$$

By (1) we have $a_{ij} \leq 1$. It is straightforward to check that we get the same result if we assume in the definition of a_{ij} that $k_1, \ldots, k_m$ are pairwise distinct. Thus, if I is finite, then for $i, j \in I$ we have $a_{ij} = 1$ if and only if

$$\sum_{j=1}^{m} d(x_{k_j}, y_{k_j}) = \sum_{j=1}^{m} d(x_{k_j}, y_{k_{j+1}})$$

for some $k_1, \ldots, k_{m+1} \in I$, with $k_1 = k_{m+1} = i$ and $k_2 = j$.

Now we proceed with the main proof, which will be divided into several parts.

Lemma 2.1. *Let $\mu := \sum_{i \in I} \lambda_i m_{x_i y_i} \in S_{\mathcal{F}(M)}$ and a_{ij}, $i, j \in I$, be as specified in the beginning of the section. There exists $f \in S_{\mathrm{Lip}_0(M)}$ with $\langle f, \mu \rangle = 1$ such that for all $i, j \in I$ we have*

$$f(x_i) - f(y_j) \leq a_{ij} d(x_i, y_j).$$

Proof. The proof follows the same line as the $(iv) \Rightarrow (i)$ part of the proof of [5, Theorem 2.4], with small modifications. For $i, j \in I$, let

$$\beta_{ij} = a_{ij} d(x_i, y_j) - d(x_i, y_i).$$

We wish to apply [5, Lemma 2.3]. Clearly $a_{ii} = 1$ and thus $\beta_{ii} = 0$ for every $i \in I$. For any finite sequence $k_1, \ldots, k_{m+1}$ with $k_1 = k_{m+1}$ we get

$$\sum_{j=1}^{m} a_{k_j k_{j+1}} d(x_{k_j}, y_{k_{j+1}}) \geq \min_{j \in \{1, \ldots, m\}} a_{k_j k_{j+1}} \sum_{j=1}^{m} d(x_{k_j}, y_{k_{j+1}}) \geq \sum_{j=1}^{m} d(x_{k_j}, y_{k_j})$$

and therefore

$$\sum_{j=1}^{m} \beta_{k_j k_{j+1}} \geq 0.$$

By [5, Lemma 2.3] we obtain real numbers α_i, $i \in I$, such that $\alpha_i \leq \alpha_j + \beta_{ij}$ for every $i, j \in I$. Now we define a function f on the set $\{x_i, y_i : i \in I\}$ by $f(y_i) = \alpha_i$ and $f(x_i) = \alpha_i + d(x_i, y_i)$. Then for every $i, j \in I$ we have

$$\begin{aligned} f(x_i) - f(y_j) &= \alpha_i - \alpha_j + d(x_i, y_i) \leq \beta_{ij} + d(x_i, y_i) = a_{ij} d(x_i, y_j), \\ f(y_i) - f(x_j) &= \alpha_i - \alpha_j - d(x_j, y_j) \leq \beta_{ij} - d(x_j, y_j) \\ &\leq d(x_i, y_j) - d(x_i, y_i) - d(x_j, y_j) \leq d(y_i, x_j), \\ f(x_i) - f(x_j) &= \alpha_i - \alpha_j + d(x_i, y_i) - d(x_j, y_j) \leq \beta_{ij} + d(x_i, y_i) - d(x_j, y_j) \\ &\leq d(x_i, y_j) - d(x_j, y_j) \leq d(x_i, x_j), \\ f(y_i) - f(y_j) &= \alpha_i - \alpha_j \leq \beta_{ij} \leq d(x_i, y_j) - d(x_i, y_i) \leq d(y_i, y_j). \end{aligned}$$

This shows that there are no conflicting assignments of values of f and that the Lipschitz constant of f is 1. Clearly $f(m_{x_i y_i}) = 1$ for every $i \in I$ and thus $\langle f, \mu \rangle = 1$. We extend f to the entire M by the McShane–Whitney Theorem and add a constant so that $f(0) = 0$. Then $f \in S_{\mathcal{F}(M)}$ is the desired function. □

Before introducing the next lemma, let us recall the Lipschitz function

$$f_{xy}(p) = \frac{d(x, y)}{2} \cdot \frac{d(y, p) - d(x, p)}{d(x, p) + d(y, p)},$$

where $x, y \in M$ with $x \neq y$. This function played an important role in the proof of [10, Theorem 4.7]. The key property of f_{xy}, which is stated in [7, Lemma 3.6], is the following: for every $u, v \in M$ and $\alpha \in (0, 1)$ with $u \neq v$ and $m_{uv} \in S(f_{xy}, \alpha)$ we have

$$(1 - \alpha) \max \{d(x, v) + d(y, v), d(x, u) + d(y, u)\} < d(x, y).$$

Our aim with the next lemma is to construct a function with a similar property for any convex series of molecules in Lipschitz-free spaces. The proof of the lemma is analogous to the proof of [11, Lemma 4.3], but has small modifications.

Lemma 2.2. *Let* $\mu := \sum_{i \in I} \lambda_i m_{x_i y_i} \in S_{\mathcal{F}(M)}$ *and* a_{ij}, $i, j \in I$, *be as specified in the beginning of the section. There exist* $f_\mu \in S_{\mathrm{Lip}_0(M)}$ *such that the following holds:*

(a) $\langle f_\mu, \mu \rangle = 1$;
(b) For every $u, v \in M$ *and* $\alpha \in (0, 1)$ *with* $u \neq v$ *and* $m_{uv} \in S(f_\mu, \alpha)$ *there exist* $i, j \in I$ *with* $x_i \neq y_j$ *and* $a_{ij} > 1 - \alpha$ *such that*

$$(1 - \alpha) \max \{d(x_i, v) + d(y_j, v), d(x_i, u) + d(y_j, u)\} < d(x_i, y_j).$$

Proof. Without loss of generality we may assume that $0 \in \{x_i, y_i : i \in I\}$. By Lemma 2.1 there exists $g \in S_{\mathrm{Lip}_0(M)}$ with $\langle g, \mu \rangle = 1$ such that for all $i, j \in I$ we have

$$g(x_i) - g(y_j) \leq a_{ij} d(x_i, y_j).$$

For every $i \in I$ let $h_i \colon M \to \mathbb{R}$ be given by

$$h_i(p) = \sup\left\{\frac{g(x_i) - g(y_j)}{d(x_i,p) + d(y_j,p)} d(x_i,p) \colon j \in I, x_i \neq y_j\right\},$$

for every $p \in M$. Since $g(x_i) - g(y_i) = d(x_i, y_i)$, h_i is nonnegative for every $i \in I$. For all $i, j \in I$ we have $d(x_i, y_j) \geq g(x_i) - g(y_j)$ and

$$\begin{aligned}\frac{d(x_i,y_j)}{2} - f_{x_i y_j}(p) &= \frac{d(x_i,y_j)}{2}\frac{d(x_i,p) + d(y_j,p) - (d(y_j,p) - d(x_i,p))}{d(x_i,p) + d(y_j,p)} \\ &= \frac{d(x_i,y_j)}{g(x_i) - g(y_j)}\frac{g(x_i) - g(y_j)}{d(x_i,p) + d(y_j,p)} d(x_i,p), \qquad (2)\end{aligned}$$

and so from [13, Proposition 1.32] and [7, Lemma 3.6] we get that h_i is a Lipschitz function with Lipschitz constant at most 1. Define $f_\mu \colon M \to \mathbb{R}$ by

$$f_\mu(p) = \sup_{i \in I}\left\{g(x_i) - h_i(p)\right\}.$$

By [13, Proposition 1.32] f_μ is a Lipschitz function with Lipschitz constant at most 1. For every $i \in I$ we have

$$f_\mu(x_i) \geq g(x_i) - h_i(x_i) = g(x_i).$$

For fixed $j \in I$ and $\delta > 0$ let $i \in I$ be such that $f_\mu(y_j) < g(x_i) - h_i(y_j) + \delta$. If $x_i = y_j$, then $f_\mu(y_j) < g(y_j) - h_i(y_j) + \delta = g(y_j) + \delta$, otherwise

$$\begin{aligned}f_\mu(y_j) &< g(x_i) - h_i(y_j) + \delta \\ &\leq g(x_i) - \frac{g(x_i) - g(y_j)}{d(x_i,y_j) + d(y_j,y_j)} d(x_i,y_j) + \delta = g(y_j) + \delta.\end{aligned}$$

As $\delta > 0$ was arbitrary, we must have $f_\mu(y_j) \leq g(y_j)$ for every $j \in I$. This gives us $f_\mu(x_i) - f_\mu(y_j) \geq g(x_i) - g(y_j)$ for every $i, j \in I$. Furthermore,

$$d(x_i,y_i) \geq f_\mu(x_i) - f_\mu(y_i) \geq g(x_i) - g(y_i) = d(x_i,y_i)$$

for every $i \in I$ and thus $f_\mu(p) = g(p)$ for every $p \in \{x_i, y_i \colon i \in I\}$. Hence $f_\mu \in S_{\mathrm{Lip}_0(M)}$ and $\langle f_\mu, \mu\rangle = \langle g, \mu\rangle = 1$.

We will now show that condition (b) holds. Fix $u, v \in M$ with $u \neq v$ and $\alpha \in (0,1)$ such that $m_{uv} \in S(f_\mu, \alpha)$. Let $\delta > 0$ be such that $m_{uv} \in S(f_\mu, \alpha - 2\delta)$ and let $i \in I$ be such that $f_\mu(u) \leq g(x_i) - h_i(u) + \delta d(u,v)$. Then $f_\mu(v) \geq g(x_i) - h_i(v)$ giving us

$$(1 - \alpha + 2\delta) d(u,v) < f_\mu(u) - f_\mu(v) \leq h_i(v) - h_i(u) + \delta d(u,v),$$

i.e, $(1 - \alpha + \delta) d(u,v) < h_i(v) - h_i(u)$. There exists $j \in I$ with $x_i \neq y_j$ such that

$$h_i(v) < \frac{g(x_i) - g(y_j)}{d(x_i,v) + d(y_j,v)} d(x_i,v) + \delta d(u,v).$$

By (2) we get that

$$(1 - \alpha) d(u,v) < h_i(v) - h_i(u) - \delta d(u,v)$$

$$\begin{aligned}
&< \frac{g(x_i)-g(y_j)}{d(x_i,v)+d(y_j,v)}d(x_i,v) - \frac{g(x_i)-g(y_j)}{d(x_i,u)+d(y_j,u)}d(x_i,u)\\
&= \frac{g(x_i)-g(y_j)}{d(x_i,y_j)}\left(f_{x_iy_j}(u)-f_{x_iy_j}(v)\right)\\
&\le a_{ij}\left(f_{x_iy_j}(u)-f_{x_iy_j}(v)\right)\\
&\le \min\left\{a_{ij}d(u,v), f_{x_iy_j}(u)-f_{x_iy_j}(v)\right\}.
\end{aligned}$$

Thus $a_{ij} > 1-\alpha$ and from [7, Lemma 3.6] we get

$$d(x_i,y_j) > (1-\alpha)\max\{d(x_i,u)+d(y_j,u), d(x_i,v)+d(y_j,v)\}.$$

Hence the condition (b) holds. □

Now we are ready to prove our main theorem. The proof is similar to the proof of [11, Theorem 4.4].

Proof of Theorem 1.1. One implication is a direct consequence of [10, Theorem 2.6].

Assume that μ is a Δ-point. Fix $\varepsilon > 0$ and a slice $S := S(f,\alpha)$ with $\mu \in S$. By [9, Lemma 2.1] we can assume that $\alpha < \delta$ and

$$\left(\frac{1}{(1-\alpha)^2}-1\right)\max_{i,j\in I} d(x_i,y_j) < \varepsilon.$$

Let $\gamma > 0$ be such that $\mu \in S(f,\alpha-\gamma)$ and let $J \subset I$, finite, be such that

$$\left\|\mu - \frac{\sum_{i\in J}\lambda_i m_{x_iy_i}}{\sum_{i\in J}\lambda_i}\right\| < \gamma.$$

Let $\nu = \sum_{i\in J}\lambda_i m_{x_iy_i}/\sum_{i\in J}\lambda_i$ and let

$$b_{ij} = \sup\left\{\frac{\sum_{j=1}^m d(x_{k_j},y_{k_j})}{\sum_{j=1}^m d(x_{k_j},y_{k_{j+1}})} : k_1,\dots,k_{m+1}\in J, k_1=k_{m+1}=i, k_2=j\right\}.$$

Then $\nu \in S_{\mathcal{F}(M)}$, and thus by Lemma 2.2 there exist $f_\nu \in S_{\mathrm{Lip}_0(M)}$ such that $\langle f_\nu,\nu\rangle = 1$ and for every $m_{uv} \in S(f_\nu,\alpha)$ there exist $i,j\in J$ with $x_i \neq y_j$ and $b_{ij} > 1-\alpha$ such that

$$(1-\alpha)\max\{d(x_i,v)+d(y_j,v), d(x_i,u)+d(y_j,u)\} < d(x_i,y_j).$$

Furthermore, notice that if $b_{ij} > 1-\alpha$, then there exist $k_1,\dots,k_{m+1}\in J$ with $k_1 = k_{m+1} = i$ and $k_2 = j$ such that

$$\frac{\sum_{j=1}^m d(x_{k_j},y_{k_j})}{\sum_{j=1}^m d(x_{k_j},y_{k_{j+1}})} > 1-\alpha > 1-\delta.$$

We may additionally assume that $k_1, \dots, k_m$ are pairwise distinct and thus by our assumption

$$\sum_{j=1}^{m} d(x_{k_j}, y_{k_j}) = \sum_{j=1}^{m} d(x_{k_j}, y_{k_{j+1}}).$$

Therefore, by the arguments presented in the beginning of the section for every $i, j \in J$ with $b_{ij} > 1-\alpha$ there exists $l_{ij} \in (0,1]$ such that $\left\|\mu - l_{ij} m_{x_i y_j}\right\| \leq 1 - l_{ij}$.

Our aim is to show that there exist $u, v \in M$ with $u \neq v$ such that $m_{uv} \in S$ and $d(u,v) < \varepsilon$. Set $g = f + f_\mu$. Then

$$\langle g, \mu \rangle = \langle f_\nu, \mu \rangle + \langle f, \mu \rangle > \langle f_\nu, \nu \rangle - \gamma + \langle f, \mu \rangle > 2 - \alpha,$$

i.e., $\mu \in S\left(g/\|g\|, 1 - (2-\alpha)/\|g\|\right)$. Since μ is a Δ-point, by [10, Remark 2.4] there exist $u, v \in M$ with $u \neq v$ such that $\langle g, m_{uv} \rangle > 2 - \alpha$ and

$$\|\mu - m_{uv}\| \geq 2 - \alpha \min\left\{l_{ij} \colon i, j \in J, b_{ij} > 1 - \alpha\right\}.$$

It is easy to see that $\langle f_\nu, m_{uv} \rangle > 1 - \alpha$ and $\langle f, m_{uv} \rangle > 1 - \alpha$. This gives us $m_{uv} \in S$. Furthermore, as $\langle f_\mu, m_{uv} \rangle > 1 - \alpha$, there exist $i, j \in J$ with $x_i \neq y_j$ and $b_{ij} > 1 - \alpha$ such that

$$(1-\alpha) \max\left\{d(x_i, v) + d(y_j, v), d(x_i, u) + d(y_j, u)\right\} < d(x_i, y_j). \tag{3}$$

Then $\|\mu - m_{uv}\| \geq 2 - \alpha l_{ij}$ and from $\left\|\mu - l_{ij} m_{x_i y_j}\right\| \leq 1 - l_{ij}$ we get

$$\begin{aligned} 2 - \alpha l_{ij} &\leq \|\mu - m_{uv}\| \\ &\leq l_{ij} \|m_{x_i y_j} - m_{uv}\| + \left\|\mu - l_{ij} m_{x_i y_j}\right\| + (1 - l_{ij}) \|m_{uv}\| \\ &\leq l_{ij} \|m_{x_i y_j} - m_{uv}\| + 2 - 2 l_{ij}, \end{aligned}$$

i.e., $\|m_{x_i y_j} - m_{uv}\| \geq 2 - \alpha$. From [11, Lemma 1.2] we deduce that

$$\begin{aligned} d(x_i, u) + d(y_j, v) &\geq d(x_i, y_j) + d(u, v) - \alpha \max\left\{d(x_i, y_j), d(u, v)\right\} \\ &> (1 - \alpha)\left(d(x_i, y_j) + d(u, v)\right). \end{aligned}$$

Since $b_{ij} > 1 - \alpha$ we have $b_{ij} = 1$ and thus $\left\langle f_\nu, m_{x_i y_j} \right\rangle = 1$. Hence

$$\|m_{x_i y_j} + m_{uv}\| \geq \left\langle f_\nu, m_{x_i y_j} \right\rangle + \langle f_\nu, m_{uv} \rangle > 2 - \alpha.$$

We apply [11, Lemma 1.2] once more to get

$$d(x_i, v) + d(y_j, u) > (1 - \alpha)\left(d(x_i, y_j) + d(u, v)\right).$$

Combining these inequalities, we have

$$\min\left\{d(x_i, v) + d(y_j, u), d(x_i, u) + d(y_j, v)\right\} > (1 - \alpha)\left(d(x_i, y_j) + d(u, v)\right). \tag{4}$$

By (3) and (4) we have

$$d(u, v) < \frac{d(x_i, v) + d(y_j, u) + d(x_i, u) + d(y_j, v)}{2(1 - \alpha)} - d(x_i, y_j)$$

$$< \frac{2d(x_i, y_j)}{2(1-\alpha)^2} - d(x_i, y_j)$$
$$\leq \left(\frac{1}{(1-\alpha)^2} - 1 \right) \max_{i',j' \in I} d(x_{i'}, y_{j'}) < \varepsilon.$$

Consequently we have found $u, v \in M$ with $u \neq v$ such that $m_{uv} \in S$ and $d(u, v) < \varepsilon$. □

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