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Cohomological descent of derived category and Fourier–Mukai to singular rational cohomology

Hafiz Syed Husain

Abstract. This paper presents some nontrivial computational results on derived category and Fourier–Mukai technique in algebraic geometry. In particular, it aims at presenting calculations involving spherical twists as a certain class of Fourier–Mukai functors and its cohomological descent on the singular rational cohomology of smooth projective variety. The purpose of this investigation is to present a new perspective, based upon Fourier–Mukai technique, on solving classical problems involving characteristic classes: in particular, the Chern and the Euler characteristics.

1. Introduction

The homological algebra of varieties involves two main ideas in characterizing our understanding of varieties: the bounded derived category of complexes of coherent sheaves on the variety X, denoted by $D^b(X)$, and the derived functor called the Fourier–Mukai [3]. Besides being a host of the Fourier–Mukai functor, $D^b(X)$ under certain condition serves as a strong invariant in determining X up to isomorphism, and sometimes, up to birationality ([4], [14]). On the other hand, ever since Mukai first defined and later used Fourier–Mukai functor in solving moduli problems [20], it has become increasingly applicable in homological algebra in general and algebraic geometry in particular. The significance of Fourier–Mukai may be estimated by noting the fact that almost all interesting derived functors of algebraic geometry are special cases of Fourier–Mukai [5].

Beyond algebra and geometry, Fourier–Mukai technique involving $D^b(X)$ is also useful in mirror symmetry of mathematical physics. For instance, many dualities in string theory from mirror symmetry, arise as a result of

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some equivalence between $D^b(X)$ and $D^b(Y)$ where X and Y are mirror Calabi–Yau. In such a case, one of X and Y is the moduli space of stable (or semistable) holomorphic vector bundles on the other [16]. Since all equivalences $D^b(X) \simeq D^b(Y)$ for any smooth complex projective varieties X and Y are special cases of Fourier–Mukai [17], this means that many results of mirror symmetry in string theory are nothing but the data $(X, Y, D^b(X), D^b(Y), \Phi_{\mathcal{K}})$, where $\Phi_{\mathcal{K}}$ is the Fourier–Mukai transform giving the required equivalence $D^b(X) \simeq D^b(Y)$ with the unique kernel $\mathcal{K} \in D^b(X \times Y)$, which is the universal quotient on the product $X \times Y$, in case Y is the fine moduli space parameterizing stable (resp. semistable) holomorphic vector bundles on X ([1, 5, 12, 20]). This data determines the homological algebra of varieties X and Y. The consequence of this significance of mathematical explanation of dualities in mirror symmetry led to Maxim Kontsevich's conjecture about the mathematical proof of mirror symmetry stated in terms of the equivalence between $D^b(X)$ and the Fukaya category of Y [15]. Besides this, there have been some recent developments in formalizing the geometric interpretation of fractional derivative in terms of modified projective limit involving principal parts of a vector bundle which may be considered as the non-derived Fourier–Mukai [11].

In the present investigation, we discuss the cohomological descent of Fourier–Mukai from $D^b(X)$ to $\mathrm{H}^*(X, \mathbb{Q})$ via the Grothendieck ring $K(X)$ and the rational Chow ring $\mathrm{CH}(X) \otimes \mathbb{Q}$ which is made possible by certain maps defined in terms of Chern characteristic and Mukai vector ([13, 17, 20]). With focus on a particular class of Fourier–Mukai transform, called the spherical twist, we present some nontrivial calculations involving cohomological version of Fourier–Mukai with the intention that these calculations will not only help us understand how Fourier–Mukai technique is useful in both derived algebraic geometry of varieties and their rational cohomology, but more importantly, they will present new perspective on solving some older problems involving characteristic classes: especially Euler and Chern characteristic. In particular, we present explicit proof of how cohomological descent of Fourier–Mukai corresponding to spherical twist helps us give a formula which connects Chern characteristic with Euler characteristic (cf. Proposition 2). This can be seen as a Fourier–Mukai generalization of classical Grothendieck–Riemann–Roch formula in [7] p. 436.

2. Preliminaries

Let X and Y be some smooth projective varieties over some field K of dimension n and m respectively. Let $D^b(X)$ (resp. $D^b(Y)$) denote the bounded derived category of complexes of coherent sheaves on X (resp. Y).

Definition 1. For $\mathcal{E}^\bullet \in D^b(X)$ with a fixed $\mathcal{K}^\bullet \in D^b(X \times Y)$, we define *Fourier–Mukai functor* as the derived functor

$$\Phi_{\mathcal{K}^\bullet} : D^b(X) \longrightarrow D^b(Y) :=_{\mathrm{df}} \Phi_{\mathcal{K}^\bullet}(\mathcal{E}^\bullet) = R\pi_{Y*}(\pi_X^*(\mathcal{E}^\bullet) \overset{L}{\otimes} \mathcal{K}^\bullet),$$

where $R\pi_{Y*}$ is the right derived push-forward of the projection $\pi_Y : X \times Y \to Y$, and π_X^* denotes the pullback of the projection $\pi_X : X \times Y \to X$ and $\overset{L}{\otimes}$ is the derived tensor product. We call $\mathcal{K}^\bullet$ the *kernel* of the Fourier–Mukai functor. If $\Phi_{\mathcal{K}^\bullet}$ is an equivalence of the derived categories, then $\Phi_{\mathcal{K}^\bullet}$ is called the *Fourier–Mukai transform* [3].

With this notation, a sheaf $\mathcal{E}$ is just a complex concentrated at degree zero. From this point onwards, $\mathcal{E}$ will denote both, a complex and a sheaf, and the context will make it clear what is meant. Also, we will write $\otimes = \overset{L}{\otimes}$. In [22] spherical objects were defined associated with a Fourier–Mukai transform, which they called *spherical twist*. This was an autoequivalence of $D^b(X)$. This helped them describe braid group actions. We thus have the following.

Definition 2. An object $\mathcal{S} \in D^b(X)$ is called *spherical* if the following two conditions are satisfied: (i) $\mathcal{S} \otimes \omega_X \simeq \mathcal{S}$ and (ii) $\mathrm{Ext}^*(\mathcal{S}, \mathcal{S}) \simeq \mathrm{H}^*(\mathbb{S}^n, \mathbb{C})$, where ω_X is the canonical bundle of X and $\mathbb{S}^n$ is the n-dimensional sphere such that $n = \dim(\mathrm{X})$.

Let $\mathcal{K}_\mathcal{S} \in D^b(X \times X)$ be what completes the following composition of morphisms, say f,

$$\pi_X^*(\mathcal{S}^\vee) \otimes \pi_Y^*(\mathcal{S}) \longrightarrow Ri_* i^*(\pi_X^*(\mathcal{S})^\vee \otimes \pi_Y^*) \simeq Ri_*((\mathcal{S})^\vee \otimes \mathcal{S} \overset{tr}{\to} \mathcal{O}_X), \quad (1)$$

to the distinguish triangle,

$$\pi_X^*(\mathcal{S}^\vee) \otimes \pi_Y^*(\mathcal{S}) \longrightarrow \mathcal{O}_\triangle \longrightarrow \mathcal{K}_\mathcal{S} \simeq Cone(f) \longrightarrow (\pi_X^*(\mathcal{S}^\vee) \otimes \pi_Y^*(\mathcal{S}))[1], \ (2)$$

where tr is the trace map ([13], p. 77), and $\mathrm{Cone}(f)$ is the mapping cone of the composition (1) (cf. [10], III. 3.2 for mapping cone). Then from [22], we get the following definition.

Definition 3. The *spherical twist* $T_\mathcal{S}$ associated with spherical object $\mathcal{S}$ is the Fourier–Mukai transform $T_\mathcal{S} :=_{df} \Phi_{\mathcal{K}_\mathcal{S}} : D^b(X) \to D^b(X)$, i.e., Fourier–Mukai transform as an *autoequivalence* with the kernel $\mathcal{K}_\mathcal{S}$ as in (2).

There are many interesting examples of spherical twists on $D^b(X)$, in particular when X is a true Calabi–Yau. For instance, when X is a Calabi–Yau, any line bundle $\mathcal{L} \in \mathrm{Pic}(X)$, $\mathcal{L} \otimes (-) : D^b(X) \to D^b(X)$ determines a spherical twist, see [13] and [1] for some non-trivial examples.

There was a well-recognized drawback of this definition of spherical twist since it depends upon the kernel as the $\mathrm{Cone}(f)$, where f is the composition in (1), which is not functorial and thus not uniquely defined. However, this drawback has recently been remedied by the technique of DG-enhancement, and so, we will always assume that such an enhancement is already available when working with mapping cones (cf. [1, 2, 21] for details).

Let $K(X), K(Y), K(X \times Y)$, and $\mathrm{H}^*(X, \mathbb{Q}), \mathrm{H}^*(Y, \mathbb{Q}), \mathrm{H}^*(X \times Y, \mathbb{Q})$, respectively, denote the Grothendieck ring and the ring of singular rational cohomology of X, Y and $X \times Y$. Let us also fix $K = \mathbb{C}$ from this point onwards. Then, building on the work by [16], [17] defined and used what later came to be known as the cohomological version of Fourier–Mukai functor (cf. [5] for a brief but comprehensive introductory survey).

Definition 4. Given a Fourier–Mukai functor $\Phi_{\mathcal{K}} : D^b(X) \to D^b(Y)$, the map

$$\Phi^H_{v(\mathcal{K})} : \mathrm{H}^*(X, \mathbb{Q}) \to \mathrm{H}^*(Y, \mathbb{Q}) :=_{\mathrm{df}} \Phi^H_{v(\mathcal{K})}(\mathcal{E}) = \pi_{Y*}(\pi_X^*(\mathcal{E}) \cdot v([\mathcal{K}])), \qquad (3)$$

is called the *cohomological Fourier–Mukai*, where $v : K(X \times Y) \to \mathrm{H}^*(X \times Y, \mathbb{Q})$ given by $v([\mathcal{K}]) = \mathrm{ch}(\mathcal{K}) \cdot \sqrt{(Td(X \times Y)}$ denotes the Mukai vector, ch(-) the Chern characteristic, [-] is the descent from $D^b(-)$ to the K-ring, π_X^* and π_{Y*} being the induced pullback and pushforward at the level of cohomology and Td(-) denotes the Todd class (cf. [5, 13, 17, 20]).

Admitting an abuse of language, we will write $[\mathcal{E}] = \mathcal{E}$ for any complex $\mathcal{E} \in \mathrm{Kom}(\mathrm{Coh}(X))$, where the latter denotes the category of complexes of coherent sheaves on a smooth projective variety X.

A remark on the meaning of 'cohomological descent.' Though we will restrict the use of the phrase 'cohomological Fourier–Mukai' to the Definition 4 only, but we will use the phrase 'cohomological descent' in the sense explained by (9) below, which will include both: the cohomological Fourier–Mukai and the image of any object in $D^b(X)$ involving the composition of the maps ch(-) and $[-]$, which is a class in $\mathrm{H}^*(X, \mathbb{Q})$. Note that the latter includes associating the class of Fourier–Mukai as well. Our Proposition 2 should be seen as this latter case.[1]

3. Cohomological descent of derived category and Fourier–Mukai

Many interesting examples of actions on rational cohomology arise from Fourier–Mukai functor. Some of the simpler ones include the action of $\Phi^H_{v(\mathcal{O}_\triangle)} : \mathrm{H}^*(X, \mathbb{Q}) \to \mathrm{H}^*(X, \mathbb{Q})$, given by

$$\Phi^H_{v(\mathcal{O}_\triangle[n])}(\beta) = \pi_{Y*}(\pi_X^*(\beta) \cdot v(\mathcal{O}_\triangle[n])),$$

[1] This paper is a part of the author's PhD research work.

where $\mathcal{O}_{\triangle}[n] \simeq Ri_*(\mathcal{O}_X)[n]$ is the shifted diagonal sheaf of the diagonal embedding $i : X \to X \times X$. The corresponding action is given by

$$\Phi^H_{v(\mathcal{O}_{\triangle}[n])}(\beta) = (-1)^n \cdot \beta.$$

This follows from the Mukai vector calculation as

$$\begin{aligned} v(\mathcal{O}_{\triangle}[n]) &= \mathrm{ch}(\mathcal{O}_{\triangle}[n]) \cdot \sqrt{\mathrm{Td}(X \times X)} \\ &= (-1)^n \mathrm{ch}(\mathcal{O}_{\triangle}) \cdot \sqrt{\mathrm{Td}(X \times X)}. \end{aligned} \tag{4}$$

Now,

$$\begin{aligned} \mathrm{ch}(\mathcal{O}_{\triangle}) \cdot \mathrm{Td}(X \times X) &= i_*(\mathrm{ch}(\mathcal{O}_X) \cdot i^*(\sqrt{\mathrm{Td}(X \times X)}) \\ &= i_*(1 \cdot i^*(\sqrt{\mathrm{Td}(X \times X)}) \\ &= i_*(1) \cdot (\sqrt{\mathrm{Td}(X \times X)}), \\ \mathrm{ch}(\mathcal{O}_{\triangle}) \cdot \sqrt{\mathrm{Td}(X \times X)} &= i_*(1). \end{aligned} \tag{5}$$

From (4) and (5), we obtain

$$\begin{aligned} \Phi^H_{v(\mathcal{O}_{\triangle}[n])}(\beta) &= \pi_{Y*}(\pi_X^*(\beta) \cdot (-1)^n i_*(1)) \\ &= \pi_{Y*}(i_*(i^*(\pi_X^*(\beta)) \cdot (-1)^n) \\ &= (-1)^n \beta. \end{aligned}$$

Similarly, if $\mathcal{L} \in \mathrm{Pic}(X)$ is any line bundle on X, then $\mathcal{L} \otimes (-) : D^b(X) \to D^b(X)$ determines an autoequivalence with kernel $Ri_*(\mathcal{L})$. The corresponding Fourier–Mukai cohomological action is given as

$$\Phi^H_{v(i_*(\mathcal{L}))} = \pi_{Y*}(\pi_X^*(\beta) \cdot v(i_*(\mathcal{L})). \tag{6}$$

One may calculate Mukai vector $v(i_*(\mathcal{L}))$ in the same way as above, and obtain

$$v(i_*(\mathcal{L})) = i_*(\mathrm{ch}(\mathcal{L})).$$

Putting in (6), we similarly obtain

$$\Phi^H_{v(i_*(\mathcal{L}))} = \beta \cdot \mathrm{ch}(\mathcal{L}). \tag{7}$$

Equality (7) shows that cohomological Fourier–Mukai action does not preserve degree, especially when $c_1(\mathcal{L}) \neq 0$, where $c_1(\mathcal{L})$ denotes the first Chern class of $\mathcal{L}$. In classical algebraic geometry, Chern classes were associated with locally free sheaves or vector bundles ([7], Appendix A). However, since for a smooth projective variety X, Kom(Coh(X)) has enough locally free sheaves, implies that $\forall \mathcal{E} \in D^b(X), \exists\, \mathcal{E}^\bullet \in$ Kom(Coh(X)) as a complex of locally free sheaves, such that $\mathcal{E}$ is quasi-isomorphic to $\mathcal{E}^\bullet$ in Kom(Coh(X)), and thus $\mathcal{E} \simeq \mathcal{E}^\bullet$ in $D^b(X)$. This implies that many interesting homological computations of invariants involving both $\mathcal{E}$ and $\mathcal{E}^\bullet$ coincide. Secondly, we may define such concepts as, Chern characteristic classes, Euler Characteristic, Todd class, etc. for general complexes $\mathcal{E} \in D^b(X)$. Thus, in classical algebraic geometry, if

$$\mathrm{ch}(-) = \mathrm{ch}_0(-) + \mathrm{ch}_1(-) \cdots \mathrm{ch}_n(-) \in A(X) \otimes \mathbb{Q} = \bigoplus_{j=0}^{j=n} \mathrm{CH}^j(X) \otimes \mathbb{Q} \quad (8)$$

denotes the Chern characteristic in terms of Chern characters ch_j such that, $\mathrm{ch}_0(-) = c_0(-), \mathrm{ch}_1(-) = c_1(-), \mathrm{ch}_2 = (\frac{1}{2}(c_1(-)^2 + c_2(-))$, etc., with $c_j(-) \in \mathrm{CH}^j(X) \otimes \mathbb{Q}$ denoting the j^{th}-Chern class of the locally free sheaf, $A(X) \otimes \mathbb{Q} \simeq \bigoplus_{j=0}^{j=n} \mathrm{CH}^j(X) \otimes \mathbb{Q}$ denoting the Chow ring with intersection pairing on j-codimensional subvarieties of X considered as rationally equivalent cycles, then all these notions are carried to the rational cohomology with intersection product as

$$D^b(X) \xrightarrow{[-]} K(X) \otimes \mathbb{Q} \simeq \mathrm{CH}^*(X) \otimes \mathbb{Q} \xrightarrow{\mathrm{ch}(-)} \mathrm{H}^{2\cdot *}(X, \mathbb{Q}) = \bigoplus_{j=0}^{j=n} \mathrm{H}^{2j}(X, \mathbb{Q}) \quad (9)$$

(cf. [3] 1.2, [6] 14.3, [13] 5.2 for the detailed exposition on the definition of the maps '[-]' and 'ch(-)'). We will refer to (9) as the descent from the level of $D^b(X)$ to the rational cohomology $\mathrm{H}^{2\cdot *}(X, \mathbb{Q})$ via the isomorphism $K(X) \otimes \mathbb{Q} \simeq \mathrm{CH}^*(X) \otimes \mathbb{Q}$. Besides, where we want to emphasize, we will omit tensoring with $\mathbb{Q}$ and keep denoting the rational Grothendieck ring and the rational Chow ring with $K(X)$ and $\mathrm{CH}^*(X)$ respectively and assume the isomorphism in (9). We elaborate this transition as follows.

For instance, the diagonal embedding $\triangle : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ determines a line bundle in $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$. From the canonical structure sequence of the diagonal,

$$0 \longrightarrow \mathcal{I}_\triangle \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow \mathcal{O}_\triangle \longrightarrow 0, \quad (10)$$

we obtain the Chern characteristic polynomial

$$c_t(\mathcal{I}_\triangle) \cdot c_t(\mathcal{O}_\triangle) = c_t(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}) \Rightarrow c_1(\mathcal{O}_\triangle) + c_1(\mathcal{I}_\triangle) = 0.$$

This gives $c_1(\mathcal{O}_\triangle) = [\triangle]$ and $c_1(\mathcal{I}_\triangle) = -[\triangle]$ in $\mathrm{H}^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Q})$. After identifying $\mathbb{P}^1 \times \mathbb{P}^1$ with the image of Segre embedding $i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, we get $\mathrm{Im}(i) = Q$ as the quadric surface in $\mathbb{P}^3$. Then, we equivalently get $c_1(\mathcal{O}_\triangle) = (1,1), c_1(\mathcal{I}_\triangle) = (-1,-1) \in \mathrm{H}^2(Q, \mathbb{Q})$. But when $n \geq 2, \triangle : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ does not determine $\mathcal{O}_\triangle$ as a vector bundle. However, we do have a locally free resolution of $\mathcal{O}_\triangle$, obtained from Koszul complex as

$$\begin{aligned} 0 \longrightarrow \bigwedge^n (\mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes \Omega_{\mathbb{P}^n}(1)) \longrightarrow \bigwedge^{n-1} (\mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes \Omega_{\mathbb{P}^n}(1)) \longrightarrow \cdots \\ \cdots \longrightarrow (\mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes \Omega_{\mathbb{P}^n}(1)) \longrightarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \longrightarrow \mathcal{O}_\triangle \longrightarrow 0, \quad (11) \end{aligned}$$

giving us the isomorphism $\mathcal{E}^\bullet \simeq \mathcal{O}_\triangle$ in $D^b(X)$ ([3] 2.2.1). It is a straightforward exercise to show that for $n = 1$, the truncation of (11) reduces to

(10), giving us the same results on Chern polynomial and the Chern classes. We present our elaboration for $n = 2$ to show how the Chern character, Chern polynomial, etc. makes sense for the diagonal sheaf for any n using the structure of derived category. The case for $n \geq 3$ is similarly assumed, though the computation for the Chern characters become too tedious which can be seen from the statement of Proposition 1. Descending from the level of $D^b(X)$ to the level of $\mathrm{H}^{2\cdot *}(X, \mathbb{Q})$ through $A(X) \otimes \mathbb{Q} \simeq K(X) \otimes \mathbb{Q}$ ([6] 14.3), we have

$$\mathrm{ch}(\mathcal{E}^0) - \mathrm{ch}(\mathcal{E}^1) + \mathrm{ch}(\mathcal{E}^2) = \mathrm{ch}(\mathcal{O}_\triangle), \tag{12}$$

which may be computed by the formula as follows.

Proposition 1. *Let $\triangle : \mathbb{P}^2 \to \mathbb{P}^2 \times \mathbb{P}^2$ be the diagonal embedding with $\pi_1, \pi_2 : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ being the projections onto the first and second factors respectively. Let $\overline{\xi}_1 = \pi_1^*([\mathbb{P}^1]), \overline{\xi}_2 = \pi_2^*([\mathbb{P}^1]) \in CH^1(\mathbb{P}^2 \times \mathbb{P}^2)$ be the 3-fold hypersurfaces obtained as the pullbacks of the class of line $[\mathbb{P}^1]$ in $CH^1(\mathbb{P}^2)$. Assume that $\mathcal{O}_\triangle \in D^b(\mathbb{P}^2 \times \mathbb{P}^2)$ with quasi-isomorphism given by $\mathcal{E}^\bullet$ as in* (11). *Then the Chern characteristic of the diagonal sheaf $ch(\mathcal{O}_\triangle) \in H^{2\cdot *}(\mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Q})$ in* (12) *is determined as*

$$\begin{aligned} ch(\mathcal{O}_\triangle) = (1, \overline{\xi_2} - 6\overline{\xi_1}, &\frac{1}{2}((\overline{\xi_2} - 6\overline{\xi_1})^2 - 2(3\overline{\xi_1}^2 - 5\overline{\xi_1\xi_2} + (c+4c')\overline{\xi_2}^2)), \\ &\frac{1}{6}((\overline{\xi_2} - 6\overline{\xi_1})^3 - 3(\overline{\xi_2} - 6\overline{\xi_1})(3\overline{\xi_1}^2 - 5\overline{\xi_1\xi_2} + (c+4c')\overline{\xi_2}^2)), \\ &\frac{1}{24}((\overline{\xi_2} - 6\overline{\xi_1})^4 - 4(\overline{\xi_2} - 6\overline{\xi_1})^2(3\overline{\xi_1}^2 - 5\overline{\xi_1\xi_2} + (c+4c')\overline{\xi_2}^2) \\ &+ 2(3\overline{\xi_1}^2 - 5\overline{\xi_1\xi_2} + (c+4c')\overline{\xi_2}^2)^2). \end{aligned} \tag{13}$$

Proof. Obviously $\mathrm{ch}(\mathcal{E}^0) = 1$. To calculate $\mathrm{ch}(\mathcal{E}^1)$, we first calculate the Chern polynomial $c_t(\mathcal{E}^1)$, which is

$$c_t(\mathcal{O}_{\mathbb{P}^2}(-1) \boxtimes \Omega_{\mathbb{P}^2}(1)) = \Pi_{i,j}(1 + (c_i(\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(-1))) + c_j(\pi_2^*(\Omega_{\mathbb{P}^2}(1)))t)). \tag{14}$$

In the absence of splitting filtration of rank-2 locally free sheaf $\Omega_{\mathbb{P}^2}$, we calculate the Chern classes $c_j(\pi_2(\Omega_{\mathbb{P}^2}(1))$ in (14) by using the twisted-1 Euler sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^2}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow 0,$$

where twisting is made using some ample divisor $H_1 \in \mathrm{Pic}(\mathbb{P}^2) \simeq \mathrm{Cl}(\mathbb{P}^2)$. Since

$$c_t(\Omega_{\mathbb{P}^2}(1)) \cdot c_t(\mathcal{O}_{\mathbb{P}^2}(1)) = c_t(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}),$$

we get

$$c_1(\Omega_{\mathbb{P}^2}(1)) = -c_1(\mathcal{O}_{\mathbb{P}^2}(1)) = -H_1,$$
$$c_2(\Omega_{\mathbb{P}^2}(1)) = -H_1^2 = -(\deg(H_1)^2) = -d_{H_1}^2,$$

where d_{H_1} denotes the degree of H_1. On the other hand, since there exists no second Chern class for $\mathcal{O}_{\mathbb{P}^2}(-1)$, we just have its first Chern class

$c_1(\mathcal{O}_{\mathbb{P}^2}(-1)) = -H_1$. Since Chern classes commute with pullback after a suitable shift of degree of grading on CH^* and $\mathrm{Cl}(\mathbb{P}^2 \times \mathbb{P}^2) \simeq \mathrm{Cl}(\mathbb{P}^2) \times \mathrm{Cl}(\mathbb{P}^2)$ ([7], II.6), then using formula 5.17 in [6],

$$c_j(\mathcal{O}_{\mathbb{P}^2}(-1) \boxtimes \Omega_{\mathbb{P}^2}(1)) = \Sigma_{i=0}^{j} \binom{r-j+i}{i} c_1^i(\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(-1))) \cdot c_{j-i}(\pi_2^*(\Omega_{\mathbb{P}^2}(1))) \tag{15}$$

we get from (14)

$$\begin{aligned} c_t(\mathcal{O}_{\mathbb{P}^2}(-1) \boxtimes \Omega_{\mathbb{P}^2}(1)) &= 1 + (2\pi_1^*(-H_1) + \pi_2^*(-H_1))t + (\pi_1^*(-H_1)^2 \\ &\quad + \pi_1^*(-H_1) \cdot \pi_2^*(-H_1) + \pi_2^*(-d_{H_1}^2))t^2 \\ &= 1 + (2(-1,0) + (0,-1))t + ((-1,0)^2 \\ &\quad + (-1,0) \cdot (0,-1) - \pi_2^*(d_{H_1}^2))t^2, \end{aligned} \tag{16}$$

with $\pi_1^*, \pi_2^* : \mathrm{CH}(X)^{n-j} \to \mathrm{CH}^{n-j-r}(X \times X)$, where r denotes the relative dimension, i.e., $r = \dim(X \times X) - \dim(X), n = \dim(X)$ and j is the dimension of the cycle as a subvariety ([8] 1.7). This gives us $\pi_2^*(d_{H_1})^2 = \alpha \in \mathrm{H}^4(\mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Q})$. Since

$$\mathrm{CH}^*(\mathbb{P}^2 \times \mathbb{P}^2) \simeq \mathrm{CH}^*(\mathbb{P}^2) \otimes \mathrm{CH}^*(\mathbb{P}^2) \simeq \mathbb{Z}[\xi_1, \xi_2]/(\xi_1^3, \xi_2^3) \tag{17}$$

with

$$\mathrm{CH}^j(\mathbb{P}^2 \times \mathbb{P}^2) \simeq \mathrm{CH}^{4-j}(\mathbb{P}^2 \times \mathbb{P}^2) \simeq (\mathbb{Z}[\xi_1, \xi_2]/(\xi_1^3, \xi_2^3))_j, \tag{18}$$

where $\overline{\xi_1}, \overline{\xi_2}$ correspond to the pullbacks $\pi_1^*([\mathbb{P}^1]), \pi_2^*([\mathbb{P}^1]) \in \mathrm{CH}^1(\mathbb{P}^2 \times \mathbb{P}^2)$ of the class of line in $\mathbb{P}^2$ and $(\mathbb{Z}[\xi_1, \xi_2]/(\xi_1^3, \xi_2^3))_j$ denotes the j^{th}-degree part of the grading on the Chow ring CH^* ([6], 2.10), we obtain α as a quadratic form $a\xi_1^2 + b\xi_1\xi_1 + c\xi_2^2$ with $a = b = 0$. Then we can easily read Chern classes $c_j(\mathcal{O}_{\mathbb{P}^2}(-1) \boxtimes \Omega_{\mathbb{P}^2}(1))$ as polynomials $P_j(\overline{\xi_1}, \overline{\xi_2}) \in (\mathbb{Z}[\xi_1, \xi_2]/(\xi_1^3, \xi_2^3))_j$ which correspond to j-codimensional subvarieties in CH^*. These are respectively the homogeneous polynomials: $P_0(\overline{\xi_1}, \overline{\xi_2}) = 1$, $P_1(\overline{\xi_1}, \overline{\xi_2}) = -2\overline{\xi_1} - \overline{\xi_2}$ and $P_2(\overline{\xi_1}, \overline{\xi_2}) = -\overline{\xi_1}^2 - \overline{\xi_1 \xi_2} - c\overline{\xi_2}^2$.

Now we finally compute the Chern character $\mathrm{ch}(\mathcal{E}^2)$ by determining the Chern classes

$$c_j(\bigwedge^2(\mathcal{O}_{\mathbb{P}^2}(-1) \boxtimes \Omega_{\mathbb{P}^2}(1))).$$

From λ-ring construction on K-groups, we can write $\mathcal{E}^2$ as (see [9], V)

$$\begin{aligned} \bigwedge^2(\mathcal{O}_{\mathbb{P}^2}(-1) \boxtimes \Omega_{\mathbb{P}^2}(1)) &\simeq \pi_1^*(\mathcal{O}_{\mathbb{P}^2}(-1))^{\otimes 2} \otimes \pi_2^*(\Omega_{\mathbb{P}^2}(1)) \otimes \pi_2^*(\omega_{\mathbb{P}^2}(1)) \\ &\simeq \pi_1^*(\mathcal{O}_{\mathbb{P}^2}(-2)) \otimes \pi_2^*(\Omega_{\mathbb{P}^2}(-2)). \end{aligned}$$

Then, from (15), we get

$$c_j(\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(-2)) \otimes \pi_2^*(\Omega_{\mathbb{P}^2}(-2)))$$

$$= \Sigma_{i=0}^{j}\binom{r-j+i}{i} c_1^i(\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(-2))) \cdot c_{j-i}(\pi_2^*(\Omega_{\mathbb{P}^2}(-2))).$$

We again twist the Euler sequence, as above, but this time by (-2) and get $c_1(\Omega_{\mathbb{P}^2}(-2)) = 2H_1$ and $c_2(\Omega_{\mathbb{P}^2}(-2)) = 4H_1^2 = 4d_{H_1}^2$, where d_{H_1} is the degree of H_1 as the ample divisor. Then putting in the formula above we obtain the total Chern class

$$c(\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(-2)) \otimes \pi_2^*(\Omega_{\mathbb{P}^2}(-2))) = 1 + (2(-2,0) + (0,2))_1 + ((-2,0)^2 + (-2,0)(0,2) + 4\pi_2^*(d_{H_1}^2))_2,$$

where $(\text{-})_j$ on the R.H.S corresponds to the j^{th}-Chern class. Then, from (17) and (18), we get the total Chern class in terms of the homogeneous polynomials $P_j(\overline{\xi_1}, \overline{\xi_2})$ corresponding to the CH^j with $\overline{\xi_1}, \overline{\xi_2}$ same as above,

$$\mathrm{ch}(\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(-2)) \otimes \pi_2^*(\Omega_{\mathbb{P}^2}(-2))) = 1 + (2\overline{\xi_2} - 4\overline{\xi_1})_1 + (4\overline{\xi_1}^2 - 4\overline{\xi_1\xi_2} + 4c'\overline{\xi_2}^2)_2. \tag{19}$$

From (14)–(19) we obtain the total Chern class of $\mathcal{O}_\triangle$ as

$$\begin{aligned} c(\mathcal{O}_\triangle) &= c(\mathcal{E}^0) - c(\mathcal{E}^1) + c(\mathcal{E}^2) \\ &= 1 + (\overline{\xi_2} - 6\overline{\xi_1})_1 + (3\overline{\xi_1}^2 - 5\overline{\xi_1\xi_2} + (c + 4c')\overline{\xi_2}^2)_2. \end{aligned}$$

From (8) and (12) we get the result. □

In the absence of the locally free resolution of the type (11) for $\mathcal{O}_\triangle$ for some general smooth projective variety X, it is still an open problem to work out the calculations or formula for the $\mathrm{ch}(\mathcal{O}_\triangle)$ even at the level of derived category, let alone find a general formula for some general complex of coherent sheaves on some X. Though the situation does admit a generalization up to certain extent. For example, if Y is any smooth projective variety of $\dim(Y) = n$, embedded inside some $\mathbb{P}^N$ for some N, then the diagonal of its projective bundle admits a locally free resolution of the type (11), where the role of projective space is replaced by the projective bundle of the normal bundle of Y, i.e. $\mathbb{P}(\mathcal{N}_Y)$ of dimension $(r-1)$ where r denotes the rank$(\mathcal{N}_Y)$= codim $(Y, \mathbb{P}^N)$ ([19]). Now (11) gives rise to Beilinson's spectral sequence which, after applying the Fourier–Mukai functor to (11), orthogonally decomposes $D^b(\mathbb{P}^n)$ into n-distinct components given by

$$D^b(\mathbb{P}^n) =< \mathcal{O}_{\mathbb{P}^n}(a), \mathcal{O}_{\mathbb{P}^n}(a+1), \ldots, \mathcal{O}_{\mathbb{P}^n}(a+(n-1), \mathcal{O}_{\mathbb{P}^n}(a+n) > .$$

Generalizing this situation to the projective bundle case for any smooth projective variety Y, we get the orthogonal decomposition of $D^b(\mathbb{P}(\mathcal{N}_Y))$ as

$$\begin{aligned} &< \pi^*(D^b(Y)) \times \mathcal{O}_{\mathbb{P}^n}(a), \pi^*(D^b(Y)) \times \mathcal{O}_{\mathbb{P}^n}(a+1), \\ &\ldots, \pi^*(D^b(Y)) \times \pi^*(D^b(Y))\mathcal{O}_{\mathbb{P}^n}(a+n) > . \end{aligned}$$

Though these presentations help in simplifying the structure of $D^b(\mathbb{P}^n)$ and $D^b(\mathbb{P}(\mathcal{N}_Y))$, in neither case do we get the presentation of $\mathcal{O}_\triangle$ where $\triangle$ is the diagonal embedding of Y in $Y \times Y$. In fact, in some cases, especially when X is Calabi-Yau, there is no such resolution known for the diagonal sheaf of X in $X \times X$ ([13], [18]).

On the other hand, we know that ch(-): $K(X) \to \mathrm{H}^*(X, \mathbb{Q})$ is not surjective in general. Thus, we do not have an object $\mathcal{E} \in D^b(X)$ for every class $\beta \in \mathrm{H}^*(X, \mathbb{Q})$ when we make the descent (i.e. (9)) from $D^b(X)$. So, any formula obtained at the level of $D^b(X)$ does not usually find a general extension to $\mathrm{H}^*(X, \mathbb{Q})$.

Now, turning to the Fourier–Mukai functor, in particular the spherical twist (cf. Def. 3), there is one nontrivial result that we can claim at the level of $D^b(X)$ which descends to $\mathrm{H}^*(X, \mathbb{Q})$ via $K(X)$ (cf. Proposition 2 below) such that this result can be seen as a Fourier-Mukai version of Grothendieck–Riemann–Roch theorem in ([7] 5.3 p. 436).

Proposition 2. *Let $\mathcal{S}$ be a spherical object in $D^b(X)$ and let $\chi(-)$ denote the Euler characteristic. Then the Chern characteristic of the spherical twist $T_{\mathcal{S}}$ is given by the formula*

$$ch(T_{\mathcal{S}}(\mathcal{E})) = ch(\mathcal{E}) - \chi(\mathcal{S}^\vee \otimes \mathcal{E}) \cdot ch(\mathcal{S}). \tag{20}$$

In order to prove the statement, we need the following result used in the work by Seidel and Thomas [22]. In the absence of its explicit proof, we give our proof here.

Lemma 1. *Let $\mathcal{S} \in D^b(X)$ be a spherical object, then the associated twist $T_{\mathcal{S}} : D^b(X) \to D^b(X), \forall \mathcal{E} \in D^b(X)$ is given as $T_{\mathcal{S}}(\mathcal{E})$ which completes the morphism*

$$\bigoplus_j Ext^j(\mathcal{S}, \mathcal{E})[-j] \otimes \mathcal{S} \longrightarrow \mathcal{E}$$

to the distinguished triangle

$$\bigoplus_j Ext^j(\mathcal{S}, \mathcal{E})[-j] \otimes \mathcal{S} \longrightarrow \mathcal{E} \longrightarrow T_{\mathcal{S}}(\mathcal{E}) \longrightarrow \bigoplus_j Ext^j(\mathcal{S}, \mathcal{E})[-j+1] \otimes \mathcal{S},$$

or equivalently,

$$T_{\mathcal{S}}(\mathcal{E}) :=_{df} Cone(\bigoplus_j Ext^j(\mathcal{S}, \mathcal{E})[-j] \otimes \mathcal{S} \longrightarrow \mathcal{E}).$$

Proof. Let $\pi_1, \pi_2 : X \times X \to X$ be the projections onto the first and second factors, respectively. Then, from Definitions 1, 3, and from distinguished triangle (2), we obtain $T_{\mathcal{S}}(\mathcal{E})$ as

$$R\pi_{2*}(\pi_1^*(\mathcal{E}) \otimes \pi_1^*(\mathcal{S}^\vee) \otimes \pi_2^*(\mathcal{S})) \longrightarrow R\pi_{2*}(\pi_1^*(\mathcal{E}) \otimes \mathcal{O}_\triangle) \longrightarrow$$

$$T_{\mathcal{S}}(\mathcal{E}) = R\pi_{2*}(\pi_1^*(\mathcal{E}) \otimes \mathcal{K}_{\mathcal{S}}) \longrightarrow R\pi_{2*}(\pi_1^*(\mathcal{E}) \otimes (\pi_1^*(\mathcal{S}^\vee) \otimes \pi_2^*(\mathcal{S}))[1]),$$

which gives the following after applying projection formula onto first and second terms of the above triangle:

$$R\pi_{2*}\pi_1^*(\mathcal{E}\otimes\mathcal{S}^\vee)\otimes\mathcal{S}\longrightarrow\mathcal{E}\longrightarrow T_{\mathcal{S}}(\mathcal{E})=R\pi_{2*}(\pi_1^*(\mathcal{E})\otimes\mathcal{K}_{\mathcal{S}})$$
$$\longrightarrow(R\pi_{2*}(\pi_1^*(\mathcal{E})\otimes\mathcal{S}^\vee)\otimes\mathcal{S})[1]. \tag{21}$$

Consider the pullback diagram

$$\begin{array}{ccc} X\times Y & \xrightarrow{\pi_Y} & Y \\ \downarrow{\scriptstyle\pi_X} & & \downarrow{\scriptstyle u} \\ X & \xrightarrow{v} & Spec(K)\,. \end{array}$$

From flat base change in general $R\pi_{X*}R\pi_Y^*(-)\simeq R\Gamma(X,-)\otimes\mathcal{O}_X$, with derived global section functor $R\Gamma(X,-)\simeq\bigoplus_j \mathrm{H}^j(X,-)[-j]$ ([13], section 3.3), setting $X=Y$ we get from (21) the first three terms as:

$$\bigoplus_j \mathrm{H}^j(X,\mathcal{S}^\vee\times\mathcal{E})[-j]\otimes\mathcal{S}\longrightarrow\mathcal{E}\longrightarrow T_{\mathcal{S}}(\mathcal{E}). \tag{22}$$

From Serre's duality and Yoneda construction at the level of derived category $D^b(X)$ ([3] 1.2.2), we obtain the following from (22):

$$\bigoplus_j \mathrm{Ext}^{n-j}(\mathcal{S}^\vee\otimes\mathcal{E},\omega_X)[-j]\otimes\mathcal{S}\longrightarrow\mathcal{E}\longrightarrow T_{\mathcal{S}}(\mathcal{E}). \tag{23}$$

Doing the same again on (23), and using the fact that $\mathcal{S}\otimes\omega_X\simeq\mathcal{S}\otimes\omega_X^*\simeq\mathcal{S}$, we get the result,

$$\bigoplus_j \mathrm{Ext}^j(\mathcal{S},\mathcal{E})[-j]\otimes\mathcal{S}\longrightarrow\mathcal{E}\longrightarrow T_{\mathcal{S}}(\mathcal{E}). \tag{24}$$

□

One must remark that we cannot simply apply ch(-) on (24) after applying the K-theoretic descent $[-]$ from (9) for two reasons. First of all, (24) is not an object in $D^b(X)$. Thus, the composition $D^b(X)\overset{[-]}{\to}K(X)\overset{\mathrm{ch}}{\to}\mathrm{H}^{2\cdot *}(X,\mathbb{Q})$ does not make sense on (24). It is an exact triple that only exists as a diagram at the level of $D^b(X)$. On the other hand, it may be considered as an element $\gamma\in\mathrm{Ext}^1(T_{\mathcal{S}}(\mathcal{E}),\bigoplus_j\mathrm{Ext}^j(\mathcal{S},\mathcal{E})[-j]\otimes\mathcal{S})$. Second, since f in (1) is not a morphism of complexes (i.e. $f\notin Mor(\mathrm{Kom(Coh(X))})$), standard constructions of mapping cone and mapping cylinder, which help determine these objects from Kom(Coh(X)) up to homotopy, are not directly applicable (cf. [10] III.3 for the details of mapping cones and mapping cylinder which determine a distinguished triangle up to non-unique isomorphisms from the information given in Kom(Coh(X))).

Proof. **(Proposition 2)** Since $f \in \mathrm{Hom}_{D^b(X)}(\mathcal{S}^\vee \boxtimes \mathcal{S}, \mathcal{O}_\triangle)$, we may express f in $D^b(X)$ as a left S-roof

$$\begin{array}{ccc} & \mathcal{L}^\bullet & \\ {}^{s}\swarrow & & \searrow^{h} \\ \mathcal{S}^\vee \boxtimes \mathcal{S} & & \mathcal{O}_\triangle \end{array}$$

for some complex $\mathcal{L}^\bullet$ such that s is a quasi-isomorphism. Thus, $f = h/s$ as a left s-fraction with $h \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{L}^\bullet, \mathcal{O}_\triangle)$ determined up to homotopy and $\mathcal{A}$ =Kom(Coh(X)) ([10] III.2 and III.4). Then, identifying $h \equiv h/1$ as left s-fraction, we have

$$\mathcal{L}^\bullet \overset{h/1\equiv h}{\longrightarrow} \mathcal{O}_\triangle \longrightarrow \mathrm{Cone}(h) \longrightarrow \mathcal{L}^\bullet[1],$$

which gives us the following from (21):

$$\begin{array}{ccccccc} R\pi_{2*}(\pi_1^*(\mathcal{E}) \otimes \mathcal{L}^\bullet) & \longrightarrow & \mathcal{E} & \longrightarrow & \Phi_{\mathrm{Cone(h)}}(\mathcal{E}) & \longrightarrow & R\pi_{2*}(\pi_1^*(\mathcal{E}) \otimes \mathcal{L}^\bullet)[1] \\ \downarrow \simeq & & \Big| \mathrm{id} & & \downarrow h' & & \downarrow \simeq \\ \bigoplus_j \mathrm{Ext}^j(\mathcal{S}, \mathcal{E})[-j] \otimes \mathcal{S} & \longrightarrow & \mathcal{E} & \longrightarrow & T_{\mathcal{S}}(\mathcal{E}) & \longrightarrow & \bigoplus_j \mathrm{Ext}^j(\mathcal{S}, \mathcal{E})[-j] \otimes \mathcal{S}[1], \end{array} \tag{25}$$

where $\Phi_{\mathrm{Cone(h)}}$ is the Fourier–Mukai functor with kernel Cone(h). Diagram (25) forces h' to be an isomorphism as well with all squares commutative up to quasi-isomorphism. Then, up to isomorphism,

$$\begin{aligned} \Phi_{\mathrm{Cone(h)}}(\mathcal{E}) &\simeq R\pi_{2*}(\pi_1^*(\mathcal{E}) \otimes (\mathcal{L}^\bullet[1] \oplus \mathcal{O}_\triangle)) \\ &\simeq R\pi_{2*}(\pi_1^*(\mathcal{E}) \otimes ((\mathcal{S}^\vee \boxtimes \mathcal{S})[1] \oplus \mathcal{O}_\triangle)) \\ &\simeq R\pi_{2*}(\pi_1^*(\mathcal{E}) \otimes ((\pi_1^*(\mathcal{S}^\vee) \otimes \pi_2^*(\mathcal{S}))[1] \oplus i_*(\mathcal{O}_X))) \\ &\simeq (\bigoplus_j \mathrm{Ext}^j(\mathcal{S}, \mathcal{E})[-j] \otimes \mathcal{S})[1] \oplus \mathcal{E} \\ &\simeq T_{\mathcal{S}}(\mathcal{E}), \end{aligned} \tag{26}$$

which is obtained after repeated application of projection formula, assuming that derived pushforward commutes with direct sum and the flat base change with the pullback diagram which had given us (22).

We can now apply ch(-) on (26) (after applying [-]) and obtain the following, for $0 \le j \le \dim(X)$,

$$\begin{aligned} \mathrm{ch}(T_{\mathcal{S}}(\mathcal{E})) &= \mathrm{ch}(\mathcal{E}) + \Sigma_j \mathrm{ch}((\mathrm{Ext}^j(\mathcal{S}, \mathcal{E})[-j] \otimes \mathcal{S})[1]) \\ &= \mathrm{ch}(\mathcal{E}) + \Sigma_j(-1)(\mathrm{ch}(\mathrm{Ext}^j(\mathcal{S}, \mathcal{E})) \cdot (-1)^j \cdot \mathrm{ch}(\mathcal{S})) \\ &= \mathrm{ch}(\mathcal{E}) - \Sigma_j(-1)^j(\mathrm{ch}(\mathrm{Ext}^j(\mathcal{O}_X, \mathcal{S}^\vee \otimes \mathcal{E})) \cdot \mathrm{ch}(\mathcal{S})) \\ &= \mathrm{ch}(\mathcal{E}) - \Sigma_j(-1)^j(\mathrm{ch}(\mathrm{H}^j(X, \mathcal{S}^\vee \otimes \mathcal{E})) \cdot \mathrm{ch}(\mathcal{S})) \end{aligned}$$

$$= \mathrm{ch}(\mathcal{E}) - \Sigma_j(-1)^j(h^j(X, \mathcal{S}^\vee \otimes \mathcal{E})) \cdot \mathrm{ch}(\mathcal{S})).$$

□

We thus have the following consequence.

Lemma 2. *Let $\mathcal{S}$ be a spherical object in $D^b(X)$. Then $ch(T_{\mathcal{S}}(\mathcal{S})) = (-1)^{1-dim(X)} ch(\mathcal{S})$. If $\mathcal{E} \in < \mathcal{S} >^\perp$, then $ch(T_{\mathcal{S}}(\mathcal{E})) = ch(\mathcal{E})$.*

Before we prove Lemma 2, we note that from Lemma 1 we can describe the action of $T_{\mathcal{S}}$ in a very concrete way, though only up to a non-unique isomorphism in $D^b(X)$, on $\mathcal{S}$ and its orthogonal complement $< \mathcal{S} >^\perp$, where $< \mathcal{S} >$ is the triangulated subcategory of $D^b(X)$ generated by $\mathcal{S}$. Seidel and Thomas [22] used part of the statement of the following lemma to show that every spherical twist is an autoequivalene. We present a more elaborate statement and our corresponding proof here. We will prove the statement of Lemma 2 after applying the following.

Lemma 3. *If $\mathcal{S}$ is any spherical object in $D^b(X)$, then its twist is determined by the general formula*

$$(-)[1] \oplus (-)[1-n] \oplus (-) \simeq (-)[1-n], \tag{27}$$

where $n = dim(X)$. Moreover, the restriction of spherical twist $T_{\mathcal{S}}$ onto $< \mathcal{S} >^\perp$ is naturally isomorphic to the identity functor on the triangulated category $< \mathcal{S} >^\perp$.

Proof. Applying the second condition in the definition of spherical object (cf. Definition 2 (ii)) in Lemma 1, we obtain the exact triangle

$$\mathrm{H}^*(\mathbb{S}^n, \mathbb{C}) \otimes \mathcal{S} \xrightarrow{g} \mathcal{S} \longrightarrow T_{\mathcal{S}}(\mathcal{S}).$$

Since the definition of g involves the definition of f, we again do the same with g that we did with f and obtain the following in $D^b(X)$ from (25) and (26) (cf. the proof of 2 and [10] III.3.3):

$$\begin{aligned} T_{\mathcal{S}}(\mathcal{S}) &\simeq ((\mathbb{C} \oplus \mathbb{C}[-n])[1] \otimes \mathcal{S}) \oplus \mathcal{S} \\ &\simeq \mathcal{S}[1] \oplus \mathcal{S}[1-n] \oplus \mathcal{S}. \end{aligned} \tag{28}$$

Consider the following split exact sequence in Kom(Coh(X)):

$$0 \longrightarrow \mathcal{S}[1-n] \longrightarrow \mathcal{S} \oplus \mathcal{S}[1-n] \longrightarrow \mathcal{S} \longrightarrow 0.$$

Completing this to the distinguished triangle and then forming the double helix (cf. [10] IV.1.2), we obtain the following diagram in $D^b(X)$:

$$\cdots \to \mathcal{S}[-1] \to \mathcal{S}[1-n] \to \mathcal{S} \oplus \mathcal{S}[1-n] \to \mathcal{S} \xrightarrow{0} \mathcal{S}[1-n] \to \mathrm{Cone}(0) \to \cdots . \tag{29}$$

Every four terms of this diagram is a distinguished triangle. If Cyl(0) denotes the mapping cylinder of the trivial morphism 0, then we get (cf. [10], III.3.2)

$$\mathrm{Cyl}(0) \simeq \mathcal{S} \oplus \mathcal{S}[1] \oplus \mathcal{S}[1-n] \simeq \mathcal{S}[1-n]. \tag{30}$$

Note that (30) makes sense only in $D^b(X)$ and it is certainly false in the abelian category Kom(Coh(X)). From (28) and (30) we get the formula (27).

Now let $\mathcal{E} \in < \mathcal{S} >^{\perp}$, we have $\mathrm{Ext}^j(\mathcal{S}, \mathcal{E}) \simeq 0$ for every j, $0 \leq j \leq n$. From Lemma 1 we get

$$0 \longrightarrow \mathcal{E} \xrightarrow{\simeq} T_{\mathcal{S}}(\mathcal{E}) \longrightarrow 0, \tag{31}$$

giving us the twist as identity on $\mathrm{Ob}(D^b(X))$. Suppose $\mathcal{E}_1, \mathcal{E}_2 \in < \mathcal{S} >^{\perp}$ with $f \in \mathrm{Hom}_{<\mathcal{S}>^{\perp}}(\mathcal{E}_1, \mathcal{E}_2)$. Then the natural isomorphism would follow if we can show that the diagram

$$\begin{array}{ccc} T_{\mathcal{S}}(\mathcal{E}_1) & \xrightarrow{\mathrm{id}} & \mathcal{E}_1 \\ {\scriptstyle T_{\mathcal{S}}(f)}\downarrow & & \downarrow{\scriptstyle f} \\ T_{\mathcal{S}}(\mathcal{E}_1) & \xrightarrow{\mathrm{id}} & \mathcal{E}_2 \end{array}$$

commutes, i.e, we must show that $T_{\mathcal{S}}(f) = f$. This easily follows from (31) and the fact that spherical twists are exact functors and exact functors map distinguished triangles to distinguished triangles with all squares commutative ([10], Chapter IV). □

Proof. **(Lemma 2)** Proof of Lemma 2 is a consequence of Lemma 3 as follows. Consider any complex $\mathcal{E} \in D^b(X)$, then $[\mathcal{E}] \in K(X)$ such that $[\mathcal{E}[j]] = (-1)^j[\mathcal{E}]$ with $[\mathrm{H}(\mathcal{E})] = [\mathcal{E}]$ in $K(X)$ where H corresponds to the cohomology functor $D^b(X) \to \mathrm{Kom}(\mathrm{Coh}((X))$ ([13] 5.2, [10] III, [9] Ch. 1). Then the statement follows immediately. □

4. Conclusion

This paper presented an exposition on derived category and Fourier–Mukai technique in algebraic geometry with a computational emphasis involving characteristic classes from rational Chow ring $\mathrm{CH}^*(X) \otimes \mathbb{Q}$ and the ring of singular rational cohomology $\mathrm{H}^*(X, \mathbb{Q})$, where X is some smooth projective variety. Cohomological descent was used to view these classes as the image of objects in the bounded derived category of complexes of coherent sheaves on X, i.e. $D^b(X)$. Proposition 1 showed how this descent could be applied to the classical algebraic geometric problem of calculating characteristic classes of projective spaces, in particular the Chern characteristic of the diagonal. Limitations of the classical approach were presented in case one goes beyond projective spaces to include general smooth projective varieties. Proposition 2, which may be seen as the Fourier–Mukai version of the classical Grothendieck–Rieman–Roch theorem, presented a new approach to calculating characteristic classes, in particular Chern and its connection with Euler characteristic, based upon the application of cohomological descent on spherical twist. Besides describing the action of spherical twist in general, Lemma 1 helped in proving Proposition 2. Lemma 2 based upon Lemma 3, significantly simplified the calculation of Chern characteristic of the spherical twist over the orthogonal complement of the spherical object. More importantly, it showed how the structure of derived category $D^b(X)$ is reflected in the simplification of the calculation of characteristic classes in the rational cohomology ring $\mathrm{H}^*(X, \mathbb{Q})$.

References

[1] N. Addington, *New derived symmetries of some hyperkähler varieties*, Algebraic Geometry **3** (2016), 223–260. DOI

[2] R. Anno and T. Logvinenko, *Spherical dg-functors*, J. Eur. Math. Soc. (JEMS) **19** (2017), 2577–2656. DOI

[3] C. Bartocci, U. Bruzzo, and D. H. Ruiperez, *Fourier–Mukai and Nahm Transforms in Geometry and Mathematical Physics*, Birkhäuser, Dordrecht–Heidelberg, 2009.

[4] A. Bondal and D. Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, Composit. Math. **125** (2001), 327–344. DOI

[5] A. Canonaco and P. Stellari, *Fourier–Mukai functors: a survey*, in: *Derived Categories in Algebraic Geometry*, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2013, 27–60. DOI

[6] D. Eisenbud and J. Harris, *A Second Course in Algebraic Geometry*, Cambridge Univ. Press, Cambridge, 2016.

[7] R. Hartshorne, *Algebraic Geometry*, Springer, New York, 1977.

[8] W. Fulton, *Intersection Theory*, Springer, Berlin, 1998.

[9] W. Fulton and S. Lang, *Riemann–Roch Algebra*, Springer, New York, 1985.

[10] S. I. Gelfand and Y. I. Manin, *Methods of Homological Algebra*, Springer, Berlin, 1996.

[11] H. S. Husain and M. Sultana, *Principal parts of a vector bundle on projective line and the fractional derivative*, Turkish J. Math. **43** (2019), 2697–2706. DOI

[12] H. S. Husain and M. Sultana, *Grothendieck's Quot scheme and moduli of quotient sheaves*, Punjab Univ. J. Math. **52** (2020), 79–87.

[13] D. Huybrechts, *Fourier–Mukai Transforms in Algebraic Geometry*, Clarendon Press, Oxford, 2006.

[14] Y. Kawamata, *D-equivalence and K-equivalence*, J. Differential Geom. **61** (2002), 147–171. DOI

[15] M. Kontsevich, *Homological algebra of mirror symmetry*. In: Proceedings of the International Congress of Mathematicians, 120–139, Birkhäuser, Basel, 1995.

[16] S. Mukai, *Moduli of vector bundles on K3 surfaces, and symplectic manifolds*, Sugaku Expositions **1** (1988), 139–174.

[17] D. Orlov, *Equivalences of derived categories and K3 surfaces*, J. Math. Sci. (New York) **84** (1997), 1361–1381. DOI

[18] P. Pragacz, V. Srinivas, and V. Pati, *Diagonal subschemes and vector bundles*, Pure Appl. Math. Q. **4** (2008), 1233–1278. DOI

[19] A. Rudakov (ed.), *Helices and Vector Bundles*, London Math. Soc., Cambridge Univ. Press, Cambridge, 1990.

[20] J. Sawon, *Moduli spaces of sheaves on K3 surfaces*, J. Geom. Phys. **109** (2016), 68–82. DOI

[21] E. Segal, *All autoequivalences are spherical twists*, Int. Math. Res. Not. IMRN **2018** (2018), 3137–3154. DOI

[22] P. Seidel and R. P. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. **108** (2001), 37–108. DOI

Department of Mathematical Sciences, Federal Urdu University of Arts, Science & Technology, Karachi, Pakistan
Department of Mathematics, University of Karachi, Karachi, Pakistan

ORCID ID: https://orcid.org/0000-0002-4043-3658
E-mail address: hsyed.hussain@fuuast.edu.pk