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Classification of hypersurfaces in the four dimensional Thurston geometry $Nil^3 \times \mathbb{R}$

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Abstract. We investigate hypersurfaces in the four-dimensional Thurston geometry $Nil^3 \times \mathbb{R}$, by giving a complete classification of hypersurfaces whose second fundamental form is a Codazzi tensor, they are either parallel or totally geodesic. Furthermore, we prove that the totally umbilical hypersurfaces in $Nil^3 \times \mathbb{R}$ are totally geodesic.

1. Introduction

A Thurston geometry (M, G) in n dimensions pairs a complete, simply connected n-dimensional Riemannian manifold M with a Lie group G that acts isometrically, transitively, and effectively on M. The group G includes discrete subgroups Γ acting freely, producing finite-volume quotient spaces $\Gamma \setminus M$ (see [9, 11]). Thurston classified 3-dimensional geometries [12, 13]. Filipkiewicz classified four-dimensional geometries in [9], his work yields 19 geometries through stabilizer subgroup analysis (subgroups of $SO(4)$), with many model spaces M homeomorphic to $\mathbb{R}^4$. Six of these are solvable Lie groups: $\mathbb{E}^4$, Nil^4, $Nil^3 \times \mathbb{R}$, $Sol^4_{m,n}$, Sol^4_0, and Sol^4_1. Four of them admit $SO(2)$ as their stabilizer of the connected component of the identity of the group G that acts on them: $Nil^3 \times \mathbb{R}$, $Sl_2(\mathbb{R}) \times \mathbb{R}$, F^4 and Sol^4_0.

A hypersurface M with the second fundamental form B is Codazzi if its covariant derivative ∇B (via the Levi-Civita connection) is totally symmetric. This class includes parallel hypersurfaces ($\nabla B = 0$), such as totally geodesic ones ($B = 0$), and totally umbilical hypersurfaces, where $B = g \cdot H$ (with g the induced metric and H the mean curvature vector).

The study of geometric properties of hypersurfaces formed by a curve γ in $\mathbb{R}^2$ and the hyperbolic plane in F^4 has been done by Belkhelfa et al. [1].

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De Leo and Van der Veken [5] proved the non-existence of totally geodesic hypersurfaces in the model space F^4. Djellal et al. [6] classified Codazzi hypersurfaces in Nil^4 and characterized minimal hypersurfaces. In [7] D'haene et al. studied Codazzi, totally geodesic, parallel, and totally umbilical hypersurfaces in Sol_0^4. We proved the non-existence of Codazzi hypersurfaces nor umbilical hypersurfaces in Sol_1^4 [3]. Erjavec and Inoguchi confirmed these results in [8]. The non-existence with another metric in Sol_1^4 has been obtained by D'haene according to personal communications of the first author and J. Van der Veken during his visit to KULeuven (September 2024). In [2] and [4] we classify Codazzi and totally umbilical hypersurfaces in $Sol_{m,n}^4$ and $Sl_2(\mathbb{R}) \times \mathbb{R}$, respectively. See the table below for a comparative perspective with our previous works (E_i denote the vectors from the orthonormal basis according to the corresponding metric on each space).

In this paper, we focus on the Codazzi hypersurfaces in $Nil^3 \times \mathbb{R}$. In Section 2 we review the geometric properties of the Thurston geometry $Nil^3 \times \mathbb{R}$. In Section 3 we give classification of the Codazzi, totally geodesic, and parallel hypersurfaces. We show that all Codazzi hypersurfaces are parallel. In the last section, we prove that totally umbilical hypersurfaces are totally geodesic.

Table 1. Comparison of hypersurface classifications in 4D Thurston geometries.

Property	$\boldsymbol{Nil^3} \times \mathbb{R}$	$\boldsymbol{Sol^4_{m,n}}$	$\boldsymbol{Nil^4}$
Codazzi	$N = \cos\theta E_1 + \sin\theta E_3$ or $\pm E_4$	$N = E_1, E_2, E_3, E_4$	Only $N = E_4$
Totally geodesic	Only $N = \pm E_4$	$N = E_1, E_2, E_3$	None exist
Parallel	Codazzi ⇔ Parallel	Codazzi ⇔ Parallel	Codazzi ⇔ Parallel
Totally umbilical	Only totally geodesic	Only totally geodesic	Still not studied

2. Preliminaries

The metric of $Nil^3 \times \mathbb{R}$ can be expressed as [10]

$$\mathrm{d}s^2 = dx^2 + dy^2 - 2xdydz + (x^2+1)dz^2 + dw^2, \tag{1}$$

where (x, y, z, w) are the usual coordinates of $\mathbb{R}^4$. The non-zero Christoffel symbols are:

$$\Gamma^2_{12} = \Gamma^2_{21} = -\tfrac{1}{2}x; \quad \Gamma^3_{12} = \Gamma^3_{21} = -\tfrac{1}{2}; \quad \Gamma^2_{13} = \Gamma^2_{31} = \tfrac{1}{2}x^2 - \tfrac{1}{2};$$
$$\Gamma^3_{13} = \Gamma^3_{31} = \tfrac{1}{2}x; \quad \Gamma^1_{23} = \Gamma^1_{32} = \tfrac{1}{2}; \quad \Gamma^1_{33} = -x.$$

The left-invariant orthonormal frame $\{E_1, E_2, E_3, E_4\}$ in $Nil_3 \times R$ is given by

$$E_1 = \frac{\partial}{\partial x}\ ;\ E_2 = \frac{\partial}{\partial y}\ ;\ E_3 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\ ;\ E_4 = \frac{\partial}{\partial w}. \tag{2}$$

Consequently, the Levi-Civita connection $\widetilde{\nabla}$ of (1) with respect to $\{E_1, E_2, E_3, E_4\}$ is given by:

$$\begin{array}{llll}
\widetilde{\nabla}_{E_1}E_1 = 0\ ; & \widetilde{\nabla}_{E_1}E_2 = -\dfrac{1}{2}E_3\ ; & \widetilde{\nabla}_{E_1}E_3 = \dfrac{1}{2}E_2\ ; & \widetilde{\nabla}_{E_1}E_4 = 0\ ; \\
\widetilde{\nabla}_{E_2}E_1 = -\dfrac{1}{2}E_3\ ; & \widetilde{\nabla}_{E_2}E_2 = 0\ ; & \widetilde{\nabla}_{E_2}E_3 = \dfrac{1}{2}E_1\ ; & \widetilde{\nabla}_{E_2}E_4 = 0\ ; \\
\widetilde{\nabla}_{E_3}E_1 = -\dfrac{1}{2}E_2\ ; & \widetilde{\nabla}_{E_3}E_2 = \dfrac{1}{2}E_1\ ; & \widetilde{\nabla}_{E_3}E_3 = 0\ ; & \widetilde{\nabla}_{E_3}E_4 = 0\ ; \\
\widetilde{\nabla}_{E_4}E_1 = 0\ ; & \widetilde{\nabla}_{E_4}E_2 = 0\ ; & \widetilde{\nabla}_{E_4}E_3 = 0\ ; & \widetilde{\nabla}_{E_4}E_4 = 0.
\end{array}$$

The resulting non zero Lie brackets are

$$[E_1, E_3] = E_2. \tag{3}$$

By using the Levi-Civita connection, we can compute the Riemann curvature tensor $\widetilde{R}$ of $Nil^3 \times \mathbb{R}$. The non-zero components of $\widetilde{R}$ are:

$$\begin{array}{ccc}
\tilde{R}(E_1, E_2)E_1 = -\frac{1}{4}E_2\ ; & \tilde{R}(E_2, E_1)E_1 = \frac{1}{4}E_2\ ; & \tilde{R}(E_3, E_1)E_1 = -\frac{3}{4}E_3 \\
\tilde{R}(E_1, E_2)E_2 = \frac{1}{4}E_1\ ; & \tilde{R}(E_2, E_1)E_2 = -\frac{1}{4}E_1\ ; & \tilde{R}(E_3, E_1)E_3 = -\frac{3}{4}E_1 \\
\tilde{R}(E_1, E_3)E_1 = \frac{3}{4}E_3\ ; & \tilde{R}(E_2, E_3)E_2 = -\frac{1}{4}E_3\ ; & \tilde{R}(E_3, E_2)E_2 = \frac{1}{4}E_3 \\
\tilde{R}(E_1, E_3)E_3 = -\frac{3}{4}E_1\ ; & \tilde{R}(E_2, E_3)E_3 = \frac{1}{4}E_2\ ; & \tilde{R}(E_3, E_2)E_3 = -\frac{1}{4}E_2.
\end{array}$$

3. Codazzi, totally geodesic, and parallel hypersurfaces

In this section we are going to classify hypersurfaces in $Nil^3 \times \mathbb{R}$ which are Codazzi in the first place then we classify totally geodesic and parallel hypersurfaces. But first, let us begin by reviewing general concepts from Riemannian geometry and recall the necessary definitions for this work.

3.1. Properties of hypersurface in Riemannian manifold. Let $\widetilde{M}$ be an $(m+1)$-dimensional Riemannian manifold and M an m-dimensional hypersurface isometrically immersed in $\widetilde{M}$, with Levi-Civita connections $\widetilde{\nabla}$ and ∇, respectively. For vector fields X, Y, Z, W on M, the second fundamental form B, a symmetric tensor valued in the normal bundle, is defined as $B(X, Y) = h(X, Y)N$, where N is a unit normal vector field. The Gauss formula relates the connections

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)N, \tag{4}$$

where $h(X, Y) = \tilde{g}(\widetilde{\nabla}_X Y, N)$, with $\tilde{g}$ and g the metrics on $\widetilde{M}$ and M.

A normal vector field ξ induces the shape operator A_ξ, a self-adjoint endomorphism on M's tangent space, via the Weingarten formula

$$\widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{5}$$

where $\nabla^\perp$ is the normal connection and $\tilde{g}(B(X,Y),\xi) = g(A_\xi X, Y)$. For a unit normal N, this reduces to

$$\widetilde{\nabla}_X N = -A_N X. \tag{6}$$

The mean curvature vector is $H = \frac{1}{m}\mathrm{trace}_g(B)$, and with a local orthonormal frame $\{E_1, \ldots, E_m\}$ on M, the mean curvature function $\lambda = \tilde{g}(B, N)$ satisfies $\lambda = \frac{1}{m}\sum_{i=1}^m h(E_i, E_i)$. The covariant derivative of B is:

$$\nabla B(X,Y,Z) = \nabla^\perp_X B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \tag{7}$$

Hypersurfaces are classified as follows.

- *Codazzi*: ∇B is symmetric in all arguments.
- *Parallel*: $\nabla B = 0$.
- *Totally geodesic*: $B = 0$.
- *Totally umbilical*: $B(X,Y) = g(X,Y)H$.

The curvature tensors R and $\widetilde{R}$ on M and $\widetilde{M}$ are given by the following:

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X(\widetilde{\nabla}_Y Z) - \widetilde{\nabla}_Y(\widetilde{\nabla}_X Z) - \widetilde{\nabla}_{[X,Y]}Z. \tag{8}$$

These are related by the Gauss equation

$$\tilde{g}(\widetilde{R}(X,Y)Z,W) = g(R(X,Y)Z,W) - h(X,W)h(Y,Z) + h(X,Z)h(Y,W), \tag{9}$$

and the Codazzi equation

$$[\widetilde{R}(X,Y)Z]^\perp = (\nabla B)(X,Y,Z) - (\nabla B)(Y,X,Z). \tag{10}$$

See Willmore [16].

3.2. Classification of Codazzi hypersurfaces. Consider a hypersurface M of $Nil^3 \times \mathbb{R}$. If N is the unit normal on M, we can write N as $N = aE_1 + bE_2 + cE_3 + dE_4$, where a, b, c, and d are local functions on M. According to [5] we reduce our problem to a purely algebraic one, taking advantage of the Codazzi equation.

Consider the following tangent vectors:

$$\begin{aligned} T_1 &= bE_1 - aE_2 + dE_3 - cE_4, \\ T_2 &= cE_1 - dE_2 - aE_3 + bE_4, \\ T_3 &= dE_1 + cE_2 - bE_3 - aE_4. \end{aligned}$$

We can verify that T_1, T_2, T_3 is a local orthonormal frame of the tangent space of M.

Lemma 1. *The second fundamental form B is a Codazzi tensor if and only if $a = b = c = 0$, $a = c = d = 0$, or $b = d = 0$.*

Proof. Changing X, Y and Z in equation (10) by T_i, T_j and T_k for i, j, k in $\{1, 2, 3\}$, by the Codazzi condition, the right hand side being equal to zero, then the left hand side must be zero. Taking $X = T_1$, $Y = T_2$, and $Z = T_3$, we get the equation

$$\tilde{g}(\widetilde{R}(T_1, T_3)T_2, N) = -(a^2 + c^2)(b^2 + d^2) = 0,$$

which implies that $a = c = 0$ or $b = d = 0$.

For the case $a = c = 0$, we consider the equation (after replacing a and c by zero)

$$\tilde{g}(\widetilde{R}(T_1, T_2)T_1, N) = -\frac{1}{4}b^3 d = 0,$$

which implies that $a = b = c = 0$ or $a = c = d = 0$.

In the second case where $b = d = 0$, we obtain $\tilde{g}(\widetilde{R}(T_i, T_j)T_k, N) = 0$, for all i, j, k in $\{1, 2, 3\}$. □

Now we state the main theorem of this section.

Theorem 1. *Let M be a Codazzi hypersurface of $Nil^3 \times \mathbb{R}$. Then a local unit normal vector field N to M takes one of the following forms with respect to the frame* (2)*:*
(*a*) $N = \cos\theta E_1 + \sin\theta E_3$*, for some constant* θ*;*
(*b*) $N = \pm E_4$.
Conversely, there exist hypersurfaces of $Nil^3 \times \mathbb{R}$ admitting a local unit normal vector field of any of the forms above.

Proof. Let M be a connected Codazzi hypersurface of $Nil^3 \times \mathbb{R}$, then according to Lemma 1 there are three possible cases for N.

(a) $N = aE_1 + cE_3$. Knowing that $a^2 + c^2 = 1$, we can replace a by $\cos\theta$ and c by $\sin\theta$ for some function θ. In this case, we choose the following orthonormal frame of the tangent space :

$$\begin{aligned} T_1 &= \sin\theta E_1 - \cos\theta E_3, \\ T_2 &= E_2, \\ T_3 &= E_4, \end{aligned}$$

and $N = \cos\theta E_1 + \sin\theta E_3$. Checking the integrability of the vectors of the orthonormal frame, we obtain $\tilde{g}([T_2, T_1], N) = \partial_y\theta$, $\tilde{g}([T_3, T_1], N) = \partial_w\theta$, and $\tilde{g}([T_3, T_2], N) = 0$. We conclude that the function θ must be a function of x and z only.

(b) $N = \pm E_4$. In this case the orthonormal frame consists of E_1, E_2, and E_3, so according to the Lie brackets (3) one can verify the integrability of the distribution.

(c) $N = \pm E_2$. In this case the orthonormal frame consists of E_1, E_3, and E_4, but with respect to the Lie bracket $[E_1, E_3] = E_2$ we deduce that the distribution is not integrable, so we have to eliminate this case.

Now we prove the converse of the statement.

(a) We compute the second fundamental form according to the frame $\{T_1, T_2, T_3\}$ and taking into account the integrability condition, namely that θ is only a function of x and z, we obtain:

$$\begin{array}{lll} h(T_1,T_1) = (\sin\theta)\partial_x\theta - (\cos\theta)\partial_z\theta; & h(T_1,T_2) = -\frac{1}{2}; & h(T_1,T_3) = 0 \\ h(T_2,T_2) = 0; & h(T_2,T_3) = 0; & h(T_3,T_3) = 0. \end{array}$$

Recall that $h = \tilde{g}(B, N)$, where B is the second fundamental form. When checking ∇B to be symmetric, one obtains

$$0 = \nabla B(T_1, T_1, T_2) = \nabla B(T_1, T_2, T_1) = \frac{1}{2}(\sin\theta\partial_z\theta + \cos\theta\partial_x\theta).$$

We deduce that θ must be constant.

(b) Computing the second fundamental form according to the frame $\{E_1, E_2, E_3\}$, we notice that it vanishes. We conclude that the hypersurface is totally geodesic and therefore it is Codazzi. □

3.3. Classification of totally geodesic and parallel hypersurfaces. The totally geodesic and parallel hypersurfaces are a particular case of Codazzi hypersurfaces. So, in order to classify totally geodesic and parallel hypersurfaces we have to compute the second fundamental form, using the Gauss formula (4), which is already done in the proof of the previous theorem. Since there are only two cases for the Codazzi hypersurfaces we have to check each case in order to find the totally geodesic and the parallel classes.

The second case of Theorem 1 gives us a totally geodesic hypersurface although the first case cannot be totally geodesic.

Corollary 1. *Let M be a connected totally geodesic hypersurface of $Nil^3 \times \mathbb{R}$. Then M is an open subset of $\{M(x, y, z, w) \in Nil_3 \times \mathbb{R}\ ;\ w = \alpha\}$ for some $\alpha \in \mathbb{R}$.*

Remark 1. The totally geodesic hypersurface $w = c$ is isometric to Nil^3, with sectional curvatures $K(E_1, E_2) = \frac{1}{4}$, $K(E_1, E_3) = -\frac{3}{4}$, $K(E_2, E_3) = \frac{1}{4}$.

For the first case of Theorem 1 we have to compute the covariant derivative of B using (7) and the fact that θ is constant. We obtain for all i, j, k in $\{1, 2, 3\}$

$$\nabla(B)(T_i, T_j, T_k) = 0.$$

Thus, the hypersurface is parallel.

We now state the classification of parallel and non-totally geodesic hypersurfaces in $Nil^3 \times \mathbb{R}$.

Theorem 2. *Let M be a connected parallel and non-totally geodesic hypersurface of $Nil^3 \times \mathbb{R}$. Then M is an open subset of $\{M(x, y, z, w) \in Nil_3 \times \mathbb{R} \mid \alpha x + \beta z = \gamma\}$ (where α, β, and γ are constants).*

Conversely, The hypersurface above is a parallel and non-totally geodesic hypersurface in $Nil^3 \times \mathbb{R}$.

Remark 2 (*Parameterization of parallel hypersurfaces*). For a parallel, non-totally geodesic hypersurface in $Nil^3 \times \mathbb{R}$ defined by $ax + cz = e$, where $a = \cos\theta$, $c = \sin\theta$, θ is constant, and e is constant, the following parameterizations can be used.

- For $a \neq 0$, the hypersurface can be parameterized as
$$\phi(u, v, t) = \left(\frac{e - cu}{a}, v, u, t\right),$$
where $u, v, t \in \mathbb{R}$ are parameters corresponding to coordinates $x = \frac{e-cu}{a}$, $y = v$, $z = u$, $w = t$. This satisfies $ax + cz = e$, and the tangent vectors are orthogonal to the normal $N = aE_1 + cE_3$.
- For $a = 0$ (i.e., $c = \pm 1$), the hypersurface is $z = e$, parameterized as
$$\phi(u, v, t) = (u, v, e, t).$$
- For $c = 0$ (i.e., $a = \pm 1$), the hypersurface is $x = e$, parameterized as
$$\phi(v, s, t) = (e, v, s, t).$$

Remark 3. The parallel hypersurface $ax + cz = e$ has non-constant sectional curvature, resembling a tilted plane in the xz-plane of $Nil^3 \times \mathbb{R}$. These hypersurfaces are congruent under the isometries of $Nil^3 \times \mathbb{R}$, which include translations in y, w and rotations in the xz-plane.

Corollary 2. *Let M be a hypersurface in $Nil^3 \times \mathbb{R}$. Then M is Codazzi if and only if M is parallel.*

4. Classification of totally umbilical hypersurfaces

In this section, we give a classification for totally umbilical hypersurfaces in $Nil^3 \times \mathbb{R}$. From the Codazzi equation (10) and assuming the hypersurface is totally umbilical, we get

$$\tilde{g}(\widetilde{R}(X, Y)Z, N) = g(Y, Z)X(\lambda) - g(X, Z)Y(\lambda). \tag{11}$$

Lemma 2. *If $N = aE_1 + bE_2 + cE_3 + dE_4$ is a unit normal vector of a totally umbilical hypersurface M, then $a = c = 0$ or $b = d = 0$.*

Proof. Consider the following local orthonormal frame in the hypersurface:

$$\begin{aligned} T_1 &= bE_1 - aE_2 + dE_3 - cE_4, \\ T_2 &= cE_1 - dE_2 - aE_3 + bE_4, \\ T_3 &= dE_1 + cE_2 - bE_3 - aE_4, \end{aligned}$$

where a, b, c, d are smooth functions on the hypersurface. Substituting $X = T_i$, $Y = T_j$, and $Z = T_k$ in (11) for distinct i, j, k in $\{1, 2, 3\}$, and denoting $\tilde{g}(\widetilde{R}(T_i, T_j)T_k, N)$ by T_{ijk}, we obtain the following three equations:
$T_{213} = (ab + cd)^2 = 0$,
$T_{321} = (ad - bc)^2 = 0$,
$T_{312} = (a^2 + c^2)(b^2 + d^2) = 0$.
We conclude that either $a = c = 0$ or $b = d = 0$. □

We obtain the following theorem.

Theorem 3. *Let M be an hypersurface of $Nil^3 \times \mathbb{R}$. Then M is totally umbilical if and only it is totally geodesic.*

Proof. If M is a totally umbilical hypersurface of $Nil^3 \times \mathbb{R}$, then by the previous computations there are two possible cases for the normal unit N.

(a) For the case $N = aE_1 + cE_3$, as in the proof of Theorem 1 we replace a by $\cos\theta$ and c by $\sin\theta$ for some function θ. We choose the following orthonormal frame of the tangent space:

$$\begin{aligned} T_1 &= \sin\theta E_1 - \cos\theta E_3, \\ T_2 &= E_2, \\ T_3 &= E_4. \end{aligned}$$

By the integrability conditions we have $\partial_y\theta = \partial_w\theta = 0$. The previous calculations of the second fundamental form give:

$$\begin{array}{ccc} h(T_1, T_1) = (\sin\theta)\partial_x\theta - (\cos\theta)\partial_z\theta; & h(T_1, T_2) = -\frac{1}{2}; & h(T_1, T_3) = 0 \\ h(T_2, T_2) = 0; & h(T_2, T_3) = 0; & h(T_3, T_3) = 0. \end{array}$$

Thus the matrix of h admits three different eigenvalues. We conclude that there is no totally umbilical hypersurface in this case.

(b) For the case $N = bE_2 + dE_4$, with $b^2 + d^2 = 1$, as above, we replace b by $\cos\theta$ and d by $\sin\theta$. We consider the following orthonormal frame in the tangent space $T_1 = \sin\theta E_2 - \cos\theta E_4$, $T_2 = E_1$, and $T_3 = E_3$, so according to the Lie brackets (3) the integrability of the distribution we obtain the following conditions:

$$\left\{ \begin{aligned} \theta_x &= 0, \\ x\theta_y + \theta_z &= 0, \\ \cos\theta &= 0. \end{aligned} \right.$$

We conclude that $N = E_4$, by the second case of the Theorem 1, M is totally geodesic. □

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