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On a generalization of quasi-metric space

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Abstract. We introduce a distinctive metric structure for generalized topology by extending the quasi-metric. This extension (to be called g-quasi metric) naturally induces a generalized topology, yet it may diverge from forming a topology. We demonstrate that g-quasi metrizability remains an invariant property of generalized topological spaces. Expanding the concepts of metric product and uniform continuity within g-quasi metric spaces, we observe an instance where a g-quasi metric may not exhibit uniform continuity like standard metrics. Additionally, we study completeness, Lebesgue property, and weak G-completeness for g-quasi metric spaces.

1. Introduction

Császár [5] proposed a notion of generalized topology by taking into account the idea of monotone mappings. It accommodates various open-like sets that existed in literature [4, 12, 14, 15]. Given a nonempty set X, it is defined as a subcollection of $\mathcal{P}(X)$ which contains $\emptyset$ and is closed under arbitrary union. Considering the members of a generalized topology as open, it then became natural to study the usual topological notions for generalized topology. Accordingly, analogoues of closed set, closure, interior, product and subspaces, continuous functions, countability and separation axioms, compactness and connectedness have been studied for generalized topological spaces. Additionally, multiple weaker versions of the above notions have also been investigated in the light of weaker forms of open sets in the context of generalized topology. The interested readers may consult [18] and references therein.

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In view of the facts that the associated open balls in any metric space form a base for a natural topology and generalized topology is a generalization of usual topology, it is natural to search for a generalization of metric structure that under standard approach induces a generalized topology and fails to produce a topology, in general. This paper addresses this question.

Here, in Section 3, we obtain the related spaces (termed as g-quasi metric spaces) by generalizing Wilson's widely studied notion of the quasi-metric structure [17]. Subsequently, we discuss certain separation properties of the generalized topology induced by a g-quasi metric. We demonstrate that g-quasi metrizability is an invariant property of the generalized topology. Next, in Section 4, we propose the natural extensions of the notions of metric product and uniform continuity in the context of g-quasi metric spaces. It is noted that, unlike usual metric, a g-quasi metric may fail to be uniformly continuous in the extended sense while considered as a mapping from the product space to $\mathbb{R}$.

The study made in Section 3 and Section 4 pave the way for the related investigations on Cauchyness and completeness. Accordingly, in Section 5, we take up the task of extending them in g-quasi metric spaces. Apart from the usual completeness, we introduce two stronger forms of completenesses viz. Lebesgue property [2, 11, 16] and weak G-completeness [1, 9, 10] in g-quasi metric spaces. It is known that both Lebesgue property and weak G-completeness are intermediate metric properties between compactness and completeness that can be characterized in terms of pseudo-Cauchy [16] and G-Cauchy [9] sequences, respectively. In what follows, we explore the mutual dependence of those completenesses for g-quasi metrics and enquire their behavior in the product spaces through Cauchy, pseudo-Cauchy and G-Cauchy sequences.

2. Preliminaries

This section discusses the prerequisites that will be required subsequently.

Definition 1 ([5, 3, 13])**.** A *generalized topology* μ on a nonempty set X is a collection of its subsets such that $\emptyset \in \mu$ and μ is closed under arbitrary unions. The pair (X, μ) is called a *generalized topological space*. Moreover, if $X \in \mu$ then μ is called a *supratopology* or a *strong generalized topology*.

Given a generalized topological space (X, μ), the elements of μ are called *generalized open sets* (or *μ-open sets*) and their complements are called *generalized closed sets* (or *μ-closed sets*) in (X, μ).

Definition 2 ([6])**.** Given a nonempty set X and $\mathcal{B} \subset \mathcal{P}(X)$ with $\emptyset \in \mathcal{B}$, all possible unions of elements of $\mathcal{B}$ form a generalized topology $\mu(\mathcal{B})$ on X. Here $\mathcal{B}$ is called a *base* for $\mu(\mathcal{B})$. Equivalently, $\mu(\mathcal{B})$ is said to be *generated by the base $\mathcal{B}$*.

Definition 3 ([5, 7]). Let (X, μ) and (Y, μ') be two generalized topological spaces.

(a) A mapping $f : X \to Y$ is called *generalized continuous* or (μ, μ')-*continuous* if $f^{-1}(G) \in \mu$, for all $G \in \mu'$.

(b) A mapping $f : X \to Y$ is called a *generalized homeomorphism* or (μ, μ')-*homeomorphism* if f is bijective and f, f^{-1} are generalized continuous.

(c) A property of generalized topological spaces that is invariant under generalized homeomorphism is called a *g-topological invariant*.

Definition 4 ([19]). Let (X, μ) be a strong generalized topological space.

(a) (X, μ) is called μ-T_0 if for $x, y \in X$ with $x \neq y$ there exists $B \in \mu$ such that exactly one of x and y is in B.

(b) (X, μ) is called μ-T_1 if for $x, y \in X$ with $x \neq y$ there exist $B_1, B_2 \in \mu$ such that B_1 contains x but not y and B_2 contains y but not x.

Clearly every μ-T_1 strong generalized topological space is μ-T_0.

Theorem 1 ([19]). *Every singleton set in a μ-T_1 strong generalized topological space is μ-closed.*

Definition 5 ([8]). Let X be a nonempty set. A nonempty subset $\mathcal{U}$ of $\mathcal{P}(X \times X)$ is called a *generalized quasi-uniformity* (or *g-quasi uniformity*) if

(i) each member of $\mathcal{U}$ contains the diagonal of X,

(ii) $\mathcal{U}$ is closed under supersets,

(iii) for $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$.

In this case, $(X, \mathcal{U})$ is called a *generalized quasi-uniform space* (or *g-quasi uniform space*).

Theorem 2 ([8]). *Given a nonempty set X and $\mathcal{B}$ $(\neq \emptyset) \subset \mathcal{P}(X \times X)$, $\mathcal{B}$ forms a base for some g-quasi uniformity on X if and only if (i) $\Delta(X) \subset B$, for every $B \in \mathcal{B}$, and (ii) if $B \in \mathcal{B}$, then there exists $V \in \mathcal{B}$ such that $V \circ V \subset B$.*

Moreover, such $\mathcal{B}$ is a base for the g-quasi uniformity $\{V \subset X : B \subset V$ for some $B \in \mathcal{B}\}$ on X.

We finish this section by recalling certain preliminaries on Lebesgue property and weak G-completeness for metric spaces.

Definition 6 ([11]). A metric space on which every real-valued continuous function is uniformly continuous is said to be *Lebesgue* (or an *Atsuji space*).

Definition 7 ([11]). A sequence (x_n) in a metric space (X, d) is said to be *pseudo-Cauchy* if given $\epsilon > 0, k \in \mathbb{N}$ there exist distinct m, n $(> k) \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$.

Theorem 3 ([11, 16]). *A metric space is Lebesgue if and only if every pseudo-Cauchy sequence having distinct terms clusters in it.*

Definition 8 ([1, 9, 10])**.** A sequence (x_n) in a metric space (X, d) is called *G-Cauchy* if $\lim_{n\to\infty} d(x_{n+p}, x_n) = 0$, for every $p \in \mathbb{N}$ (or equivalently, $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$). A metric space in which every G-Cauchy sequence converges is said to be *weakly G-complete*.

Both Lebesgue property and weak G-completeness are strictly intermediate between compactness and completeness of metric spaces [9, 11].

3. g-Quasi metric spaces and the induced generalized topology

Definition 9 ([17])**.** Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}$ is called a *quasi-metric* on X, if

(a) for all $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;
(b) for all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

Here d is a quasi-metric on X. Moreover, the pair (X, d) is called a *quasi-metric space.*

Definition 10. Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}$ is called a *g-quasi metric* on X, if there exists $r \geq 0$ such that

(a) for all $x, y \in X$, $d(x, y) \geq r$ and $d(x, y) = r$ if and only if $x = y$;
(b) for all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

Here d is a g-quasi metric on X and r the index of d. Moreover, the pair (X, d) is called a *g-quasi metric space* (of *index r*).

Clearly a g-quasi metric of index 0 is a quasi-metric and vice versa.

The condition $d(x, x) > 0$ in a g-quasi metric space may arise naturally in practical situations. Suppose X is a set of states of a system, and assume that every operation on the system requires paying a fixed *initial activation cost* $c > 0$ (for example, the start-up energy of a device, the fixed overhead cost of running a program, or the minimum charge for using a service). Let $\rho : X \times X \to [0, \infty)$ be any quasi-metric that measures the additional cost of moving from one state to another. We define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} c, & \text{if } x = y, \\ c + \rho(x, y), & \text{if } x \neq y. \end{cases}$$

Then $d(x, y) \geq c > 0$ for all $x, y \in X$, and $d(x, y) = c$ occurs exactly when $x = y$. Thus d is a g-quasi metric on X with index $r = c$.

This example shows that $d(x, x) > 0$ is meaningful: even if one starts and ends at the same state, the system must still pay the fixed activation cost c. Hence a positive self-distance is a natural reflection of the underlying practical setup, rather than an artificial condition.

Remark 1. For every nonzero value of the index r, the g-quasi metric introduced in Definition 10 induces a generalized topology that is not a topology;

indeed, Example 2 provides such a construction for each $r > 0$. When $r = 0$, the notion reduces to the classical quasi-metric, and the induced generalized topology necessarily becomes a topology. In this sense, the definition can be regarded as minimal. We emphasize, however, that this minimality is not intended in any categorical or universal sense. The problem of identifying a universal or categorically minimal generalized metric that induces Császár-type generalized topologies remains open.

Definition 11. A g-quasi metric d on a nonempty set X (and hence the related g-quasi metric space (X, d)) is said to be *symmetric* if $d(x, y) = d(y, x)$, for all $x, y \in X$.

Clearly a symmetric g-quasi metric of index 0 is a metric and vice versa.

Theorem 4. *Let (X, d) be a quasi-metric space. Then, for $r \geq 0$, $d' = d + r$ forms a g-quasi metric on X of index r.*

Proof. (a) Clearly, for all $x, y \in X$, $d'(x, y) \geq r$ and $d'(x, y) = r$ if and only if $x = y$.

(b) Choose $x, y, z \in X$. Then $d'(x, y) = d(x, y) + r \leq d(x, z) + d(z, y) + r \leq \{d(x, z) + r\} + \{d(z, y) + r\} \leq d'(x, z) + d'(z, y)$.

Hence the result follows. □

However given a g-quasi metric d on a nonempty set X, $d' = d - r$ may not form a quasi-metric on it for all choices of $r \geq 0$.

Example 1. Let $X = [2, 4]$ and let $d : X \times X \to \mathbb{R}$ be given by $d(x, y) = (x - y)^2 + 100$, for all $x, y \in X$. Then, for no values of $r \geq 0$, $d' = d - r$ forms a quasi-metric on X. It follows by observing that for $r = 100$, $d'(2, 4) > d'(2, 3) + d'(3, 4)$, while for all other values of r, $d'(x, x) \neq 0$, if $x \in X$.

However d is a g-quasi metric on X of index 100:

(a) for all $x, y \in X$, $d(x, y) \geq 100$ and $d(x, y) = 100$ if and only if $x = y$,

(b) for all $x, y, z \in X$, $d(x, y) + d(y, z) \geq 100 + 100 \geq (x - z)^2 + 100 = d(x, z)$.

Definition 12. Let (X, d) be a g-quasi metric space of index $r \geq 0$. Given $x \in X$ and $p > 0$, we denote the set $\{y \in X : d(x, y) < p\}$ by $B_d(x, p)$ (or simply by $B(x, p)$). Clearly $\mathcal{B}(d) = \{B(x, p) : x \in X, p > 0\} \bigcup \{\emptyset\}$ forms a base for some strong generalized topology $\mu(d)$ on X. It is called the *generalized topology induced by d.*

Note 1. It should be noted, at this stage, that if (X, d) is a symmetric g-quasi metric space of index 0 (i.e., a metric space) then $\mathcal{B}(d)$, defined as before, forms a base for the topology induced by d.

In what follows, we show that, for all positive values of r, a symmetric g-quasi metric space (X, d) can be found so that (i) d is of index r, (ii) $\mu(d)$ does not form a topology on X. We consider the following example.

Example 2. Let $r > 0$ and $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$d(x,y) = \begin{cases} r, & \text{if } x = y, \\ 2r, & \text{if } 0 < |x-y| \leq r, \\ |x-y|, & \text{if } |x-y| > r. \end{cases}$$

We show that $(\mathbb{R}, d)$ is a g-quasi metric space of index r though $\mu(d)$ does not form a topology on $\mathbb{R}$.

(a) Clearly $d(x,y) \geq r$ and $d(x,y) = r$ if and only if $x = y$, for all $x, y \in \mathbb{R}$.

(b) Let $x, y, z \in \mathbb{R}$. We show that $d(x,y) \leq d(x,z) + d(z,y)$. Note that for $0 \leq |x-y| \leq r$ the above inequality follows from (a). So let us assume $|x-y| > r$.

If $|x-z| = 0$ or $|z-y| = 0$, then the inequality is immediate.

If $|x-z|, |z-y| > r$, then it follows from the order property of $\mathbb{R}$.

If $0 < |x-z|, |z-y| \leq r$, then $d(x,y) = |x-y| \leq |x-z| + |z-y| \leq 4r = d(x,z) + d(z,y)$.

If $|x-z| > r$ and $0 < |z-y| \leq r$, then $d(x,y) \leq d(x,z) + |z-y| \leq d(x,z) + r \leq d(x,z) + 2r = d(x,z) + d(z,y)$.

If $|z-y| > r$ and $0 < |x-z| \leq r$, then it follows similarly as before.
Thus d forms a g-quasi metric on $\mathbb{R}$ of index r.

We now show that $\mu(d)$ does not form a topology on $\mathbb{R}$. Since

$$\begin{aligned} &B\left(r, 2r + \tfrac{r}{10}\right) \bigcap B\left(\tfrac{21r}{5}, 2r + \tfrac{r}{10}\right) \\ &= \left(r - \tfrac{21r}{10}, r + \tfrac{21r}{10}\right) \bigcap \left(\tfrac{21r}{5} - \tfrac{21r}{10}, \tfrac{21r}{5} + \tfrac{21r}{10}\right) = \left(\tfrac{21r}{10}, r + \tfrac{21r}{10}\right), \end{aligned}$$

it suffices to show that $(\frac{21r}{10}, r + \frac{21r}{10})$ does not contain any nonempty set of the form $B(x,c)$ where $x \in \mathbb{R}, c > 0$.

Suppose otherwise. Then $B(x,c) \subset \left(\frac{21r}{10}, r + \frac{21r}{10}\right)$ for some $x \in \mathbb{R}$ and $c > r$.

Case I: $r < c \leq 2r$. Then, for chosen $y \in \mathbb{R}$ with $r < |x-y| < c$, $y \in B(x,c)$. Also $x \in B(x,c)$. Thus $x, y \in B(x,c) \subset \left(\frac{21r}{10}, r + \frac{21r}{10}\right)$, a contradiction to $|x-y| > r$.

Case II: $c > 2r$. Then, for each $y \in (x-r, x+r)$, $d(x,y) < c$. Consequently $(x-r, x+r) \subset B(x,c)$, hence $(x-r, x+r) \subset \left(\frac{21r}{10}, r + \frac{21r}{10}\right)$, a contradiction to $\left|\left(\frac{21r}{10}, r + \frac{21r}{10}\right)\right| = r$.

The contradictions arrived at both the cases prove our claim.

Remark 2. Let $(X, \overline{d})$ be a g-quasi metric space with index r. For $\epsilon > r$, set $V_\epsilon = \{(x,y) \in X \times X : \overline{d}(x,y) < \epsilon\}$.

If $r = 0$, then it is clear that $\mathcal{B}_\mathcal{U} = \{V_\epsilon : \epsilon > r\}$ forms a base for a g-quasi uniformity on X alike the classical case.

However for all other non-negative values of r, there is some g-quasi metric space with index r such that $\mathcal{B}_\mathcal{U}$ fails to form a base for some g-quasi uniformity on X as we will see now.

Consider the g-quasi metric space $(\mathbb{R}, d)$, as defined in Example 2, with index $r > 0$. If possible, let $\mathcal{B}_{\mathcal{U}} = \{V_\epsilon : \epsilon > r\}$ form a base for a g-quasi uniformity on $\mathbb{R}$ where $V_\epsilon = \{(x, y) \in X \times X : d(x, y) < \epsilon\}$, for every $\epsilon > r$. Then there is $\delta > r$ such that $V_\delta \circ V_\delta \subset V_{\frac{3r}{2}}$.

Set $x = 0, y = r + \frac{\delta - r}{2}, z = r + \delta$. Then $d(x, y) = d(y, z) = r + \frac{\delta - r}{2} < \delta$ and hence $(x, y), (y, z) \in V_\delta$ implies $(x, z) \in V_{\frac{3r}{2}}$, i.e., $d(x, z) < \frac{3r}{2}$, i.e., $r + \delta < \frac{3r}{2}$, i.e., $\delta < \frac{r}{2}$, a contradiction. Hence $\mathcal{B}_{\mathcal{U}}$ is not a base for a g-quasi uniformity on $\mathbb{R}$.

Remark 3. In the classical setting, every metric induces a uniformity, and consequently a proximity structure, through the standard entourage system. For g-quasi metrics of index $r > 0$, however, Remark 2 shows that the family $\{V_\varepsilon : \varepsilon > r\}$ may fail to generate even a generalized quasi-uniformity. Hence a proximity structure in the classical sense need not arise from a g-quasi metric. Developing a suitable analogue of proximity, compatible with the generalized topologies induced by g-quasi metrics, is therefore nontrivial and may be pursued in future work.

Theorem 5. *Let d be a g-quasi metric on X. Then $(X, \mu(d))$ is μ-T_1.*

Proof. Let d be of index r. Choose $x, y \in X$ such that $x \neq y$. Clearly $d(x, y),\ d(y, x) > r$. Choose $p \in \mathbb{R}$ such that $r < p < \min\{d(x, y), d(y, x)\}$. Then $B(x, p),\ B(y, p)$ are generalized open sets in $(X, \mu(d))$ containing $x,\ y$ respectively such that $x \notin B(y, p)$ and $y \notin B(x, p)$. □

Let d be a g-quasi metric on X. Then we may conclude the following, stated as corollaries.

Corollary 1. *Each singleton set in $(X, \mu(d))$ is μ-closed.*

Corollary 2. *$(X, \mu(d))$ is μ-T_0.*

Remark 4. g-Quasi metrices of different indices may induce the same generalized topology. For example, choosing $X = \{x, y\}$ and $d_1, d_2 : X \times X \to \mathbb{R}$ as given by $d_1(x, x) = d_1(y, y) = 3, d_1(x, y) = d_1(y, x) = 4$ and $d_2(x, x) = d_2(y, y) = 5, d_2(x, y) = d_2(y, x) = 6$, we observe that both d_1, d_2 induce discrete topology on X though they have distinct indices.

Definition 13. A generalized topological space (X, μ) is called *g-quasi metrizable* if for some g-quasi metric d on X, $\mu(d) = \mu$.

Theorem 6. *Let $(X, \mu), (Y, \mu')$ be two generalized topological spaces and $f : X \to Y$ be a generalized homeomorphism. If (X, μ) is g-quasi metrizable, then so is (Y, μ').*

Proof. Let (X, μ) be g-quasi metrizable and $d : X \times X \to \mathbb{R}$ a g-quasi metric on X of index $r \geq 0$ such that $\mu(d) = \mu$.

Define $d' : Y \times Y \to \mathbb{R}$ by $d'(y_1, y_2) = d(f^{-1}(y_1), f^{-1}(y_2))$ for all $y_1, y_2 \in Y$. Then

(a) for all $y_1, y_2 \in Y$, $d'(y_1, y_2) \geq r$ and $d'(y_1, y_2) = r$ iff $f^{-1}(y_1) = f^{-1}(y_2) \iff y_1 = y_2$;

(b) for all $y_1, y_2, y_3 \in Y$, $d'(y_1, y_2) + d'(y_2, y_3) = d(f^{-1}(y_1), f^{-1}(y_2)) + d(f^{-1}(y_2),$
$f^{-1}(y_3)) \geq d(f^{-1}(y_1), f^{-1}(y_3)) = d'(y_1, y_3)$. Thus d' forms a g-quasi metric on Y of index r.

Choose $V \in \mu'$ and $y \in V$. Then, for some $x \in X$, $p > r$, we have $f^{-1}(y) \in B_d(x, p) \subset f^{-1}(V)$, hence $y \in f(B_d(x, p)) \subset V$ and therefore $y \in B_{d'}(f(x), p) \subset V$.

Thus $\mathcal{B}(d')$ forms a base for μ'. Hence the result follows. □

4. Product of g-quasi metrics

Theorem 7. *Let (X, d_X) and (Y, d_Y) be g-quasi metric spaces of the same index r. Define $d_{XY} : (X \times Y) \times (X \times Y) \to \mathbb{R}$ by*

$$d_{XY}((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\},$$

$x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then d_{XY} defines a g-quasi metric on $X \times Y$ of index r.

Proof. Straightforward. □

Definition 14. Given two g-quasi metric spaces (X, d_X) and (Y, d_Y) of the same index r, d_{XY} is called the *g-quasi metric product* of d_X with d_Y or simply *product g-quasi metric* on $X \times Y$.

Clearly if (X, d_X) and (Y, d_Y) are metric spaces, then d_{XY} defines the product metric on $X \times Y$.

Definition 15. Let (X, d_X) and (Y, d_Y) be g-quasi metric spaces of indices r_1 and r_2 respectively. A mapping $f : X \to Y$ is said to be *g-uniformly continuous* if for $\epsilon > r_2$ there exists $\delta > r_1$ such that $d_X(x_1, x_2) < \delta$ implies $d_Y(f(x_1), f(x_2)) < \epsilon$ for all $x_1, x_2 \in X$.

Clearly if (X, d_X) and (Y, d_Y) are metric spaces, then every g-uniformly continuous mapping from X to Y is uniformly continuous as a mapping between metric spaces.

It is known that if (X, d) is a metric space, then the distance function $d : X \times X \to \mathbb{R}$ is uniformly continuous where $X \times X$ is equipped with the product metric and $\mathbb{R}$ with the usual metric. However, given a g-quasi metric space (X, d), the mapping $d : X \times X \to \mathbb{R}$ may not be g-uniformly continuous where $X \times X$ is equipped with the product metric and $\mathbb{R}$ with the usual metric (recall that, it is a g-quasi metric of index 0). We consider the following example.

Example 3. Consider the g-quasi metric space $(\mathbb{R}, d)$, defined in Example 2, of index $r > 0$. We show that $d : (\mathbb{R} \times \mathbb{R}, d') \to (\mathbb{R}, d_u)$ is not g-uniformly continuous, where d' is the g-quasi metric product of d with itself on $\mathbb{R} \times \mathbb{R}$ and d_u is the usual metric on $\mathbb{R}$.

Suppose otherwise. Then, for $\epsilon = \frac{r}{2}$, there exists $\delta > r$ such that $d'((x, y), (x', y')) < \delta$ and hence $|d(x, y) - d(x', y')| < \epsilon$ for all $(x, y), (x', y') \in \mathbb{R} \times \mathbb{R}$. In particular, for $(x', y') = (0, 0)$, $d'((x, y), (0, 0)) < \delta$ and so $|d(x, y) - d(0, 0)| < \frac{r}{2}$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}$. That is,

$$\max\{d(x, 0), d(y, 0)\} < \delta \implies |d(x, y) - d(0, 0)| < \frac{r}{2}$$

for all $(x, y) \in \mathbb{R} \times \mathbb{R}$.

Choose $n \in \mathbb{N}\backslash\{1\}$ such that $\frac{\delta - r}{(n-1)^2} < \frac{r}{2}$. Then $\frac{\delta - r}{n(n-1)} < \frac{r}{2}$. Set $x = r + \frac{\delta - r}{n}$ and $y = r + \frac{\delta - r}{n-1}$. Then $|x - 0| = r + \frac{\delta - r}{n} > r$ and $|y - 0| = r + \frac{\delta - r}{n-1} > r$. Consequently, $d(x, 0) = r + \frac{\delta - r}{n} < \delta$ and $d(y, 0) = r + \frac{\delta - r}{n-1} < \delta$, whence $\max\{d(x, 0),\ d(y, 0)\} < \delta$. However, $|x - y| = \frac{\delta - r}{n(n-1)} < \frac{r}{2} \leq r$ and therefore $d(x, y) = 2r$, a contradiction since $|d(x, y) - d(0, 0)| < \frac{r}{2}$.

Hence $d : (\mathbb{R} \times \mathbb{R}, d') \to (\mathbb{R}, d_u)$ is not g-uniformly continuous.

5. Completeness, Lebesgue property and (weak) G-completeness in g-quasi metric spaces

In this section, we extend the study of completeness, Lebesgue property and weak G-completeness for g-quasi metric spaces using the extended notion of Cauchy, G-Cauchy and pseudo-Cauchy sequences.

Definition 16. Let (x_n) be a sequence in a g-quasi metric space (X, d) of index r and $c \in X$. Then

(i) (x_n) is said to be *convergent* to c in (X, d) if it is so in $(X, \mu(d))$;

(ii) c is called a *cluster point* of (x_n) in (X, d) if it is so in $(X, \mu(d))$.

Clearly if (x_n) is convergent to c, then c is a cluster point of (x_n) (in (X, d)).

Definition 17. Let (X, d) be a g-quasi metric space of index r and (x_n) be a sequence in X. Then

(i) (x_n) is called *Cauchy* if given $\epsilon > r$ there exists $k \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq k$;

(ii) (x_n) is called *G-Cauchy* if given $\epsilon > r$ there exists $k \in \mathbb{N}$ such that $d(x_n, x_{n+1}) < \epsilon$ for all $n \geq k$;

(iii) (x_n) is called *pseudo-Cauchy* if given $\epsilon > r$ and $k \in \mathbb{N}$ there exist p, q $(p \neq q) \in \mathbb{N}$ with $p, q \geq k$ such that $d(x_p, x_q) < \epsilon$.

Definition 18. A g-quasi metric space (X, d) is said to be

(i) *complete* if every Cauchy sequence converges to some point in it;

(ii) *G-complete* if every G-Cauchy sequence converges to some point in it;

(iii) *weakly G-complete* if every G-Cauchy sequence has a cluster point in it;

(iv) *Lebesgue* if every pseudo-Cauchy sequence having distinct terms has a cluster point in it;

(v) *strongly Lebesgue* if every pseudo-Cauchy sequence has a cluster point in it.

Clearly, for g-quasi metric spaces we have the following chain of implications:

$$\begin{array}{ccccc} \text{Strongly Lebesgue} & \longrightarrow & \text{Lebesgue} & \longrightarrow & \text{Weak } G\text{-completeness} \\ & & & & \uparrow \\ & & \text{Completeness} & \longleftarrow & G\text{-completeness} \end{array}$$

In what follows, we show that for each $r > 0$, $(\mathbb{R}, d)$ of index r, as defined in Example 2, is not weakly G-complete.

Example 4. Consider the sequence (x_n) in $(\mathbb{R}, d)$, where $x_n = rn - \frac{r}{n}$ for all $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$, $|x_{n+1} - x_n| = r + r\left(\frac{1}{n} - \frac{1}{n+1}\right) > r$, hence $d(x_{n+1}, x_n) = r + r\left(\frac{1}{n} - \frac{1}{n+1}\right)$ and so $d(x_n, x_{n+1}) = r + r\left(\frac{1}{n} - \frac{1}{n+1}\right)$.

Choose $\epsilon > r$. Then there is $k \in \mathbb{N}$ such that $\frac{r}{n(n+1)} < \epsilon - r$ for all $n \geq k$ and hence $d(x_n, x_{n+1}) < \epsilon$ for all $n \geq k$. Thus (x_n) is G-Cauchy in $(\mathbb{R}, d)$.

If possible, let there be a cluster point c of (x_n) in $(\mathbb{R}, d)$. Since (x_n) is a sequence of distinct terms, $B(c, \frac{3r}{2})$ contains infinitely many elements of (x_n). However
$B(c, \frac{3r}{2}) = \{y \in \mathbb{R} : d(c, y) < \frac{3r}{2}\} = \left(c - \frac{3r}{2}, c - r\right) \bigcup \left(c + r, c + \frac{3r}{2}\right) \bigcup \{c\}$
which clearly contains finitely many elements of (x_n), a contradiction.

Hence $(\mathbb{R}, d)$ is not weakly G-complete.

Lemma 1. *Let (X, d_X) and (Y, d_Y) be g-quasi metric spaces of the same index r. A sequence $((x_n, y_n))$ is Cauchy in $(X \times Y, d_{XY})$ if and only if (x_n) and (y_n) are Cauchy in (X, d_X) and (Y, d_Y), respectively.*

Proof. Let $((x_n, y_n))$ be Cauchy in $(X \times Y, d_{XY})$. Choose $\epsilon > r$. Then there exists $k \in \mathbb{N}$ such that $d_{XY}((x_m, y_m), (x_n, y_n)) < \epsilon$, for all $m, n \geq k$. That is, $d_X(x_m, x_n), d_Y(y_m, y_n) < \epsilon$ for all $m, n \geq k$. Then (x_n) and (y_n) are Cauchy in (X, d_X) and (Y, d_Y), respectively.

Conversely, let (x_n) and (y_n) be Cauchy in (X, d_X) and (Y, d_Y), respectively. Choose $\epsilon > r$. Then there exist $p, q \in \mathbb{N}$ such that $d_X(x_m, x_n) < \epsilon$ for all $m, n \geq p$ and $d_Y(y_m, y_n) < \epsilon$ for all $m, n \geq q$.

Set $r = \max\{p, q\}$. Then $d_{XY}((x_m, y_m), (x_n, y_n)) < \epsilon$ for all $m, n \geq r$. Hence $((x_n, y_n))$ is Cauchy in $(X \times Y, d_{XY})$. $\square$

Similar chains of arguments yield the following results that we state without proof.

Lemma 2. *Let (X, d_X) and (Y, d_Y) be g-quasi metric spaces of the same index r. A sequence $((x_n, y_n))$ is G-Cauchy in $(X \times Y, d_{XY})$ if and only if (x_n) and (y_n) are G-Cauchy in (X, d_X) and (Y, d_Y), respectively.*

Lemma 3. *Let (X, d_X) and (Y, d_Y) be g-quasi metric spaces of the same index r. If $((x_n, y_n))$ is a pseudo-Cauchy sequence in $(X \times Y, d_{XY})$, then (x_n) and (y_n) are pseudo-Cauchy in (X, d_X) and (Y, d_Y), respectively.*

The converse of Lemma 3 is not true. In support, we produce the following example.

Example 5. Consider the g-quasi metric space $(\mathbb{R}, d)$, as defined in Example 2, of index 1. Then the sequences (x_n) and (y_n) in $\mathbb{R}$, defined by

$$x_n = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 10^n, & \text{if } n \text{ is even,} \end{cases}$$

and

$$y_n = \begin{cases} 10^n, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even,} \end{cases}$$

for all $n \in \mathbb{N}$, are pseudo-Cauchy in $(\mathbb{R}, d)$.

However $((x_n, y_n))$ is not pseudo-Cauchy in $\mathbb{R} \times \mathbb{R}$ (where $\mathbb{R} \times \mathbb{R}$ is equipped with d', the g-quasi metric product of d with itself). In fact, for any pair of positive integers m, q $(m \neq q)$ with $m, q \geq 1$, we get $d'((x_m, y_m), (x_q, y_q)) > 2$, by considering even and odd cases separately for m and q. Thus $((x_n, y_n))$ is not pseudo-Cauchy in $(\mathbb{R} \times \mathbb{R}, d')$.

Theorem 8. *Let (X, d_X) and (Y, d_Y) be g-quasi metric spaces of the same index r. Then $(X \times Y, d_{XY})$ is complete if and only if (X, d_X) and (Y, d_Y) are complete.*

Proof. Let $(X \times Y, d_{XY})$ be complete.

We first show that (X, d_X) is complete. Choose a Cauchy sequence (x_n) in (X, d_X) and fix $y \in Y$. Then, according to Lemma 1, $((x_n, y))$ is Cauchy in $(X \times Y, d_{XY})$. Since $(X \times Y, d_{XY})$ is complete, $((x_n, y))$ converges to a point (a, b) in $(X \times Y, d_{XY})$.

We claim that (x_n) is convergent to a in (X, d_X). Let V be a generalized open set in $(X, \mu(d_X))$ containing a. Then there exist $p \in X, \delta > r$ such that $a \in B_{d_X}(p, \delta) \subset V$. Since $B_{d_{XY}}((p, b), \delta)$ is an open set in $(X \times Y, \mu(d_{XY}))$ containing (a, b), there exists $k \in \mathbb{N}$ such that $(x_n, y) \in B_{d_{XY}}((p, b), \delta)$ for all $n \geq k$. Thus $\max\{d_X(p, x_n), d_Y(b, y)\} < \delta$ for all $n \geq k$, therefore $d_X(p, x_n) < \delta$ for all $n \geq k$, that is, $x_n \in B_{d_X}(p, \delta) \subset V$ for all $n \geq k$. Hence (x_n) converges to a in (X, d_X) and so (X, d_X) is complete. Similarly (Y, d_Y) is complete.

Conversely, let (X, d_X) and (Y, d_Y) be complete.

Choose a Cauchy sequence $((x_n, y_n))$ in $(X \times Y, d_{XY})$. By Lemma 1, (x_n) and (y_n) are Cauchy in (X, d_X) and (Y, d_Y), respectively. So, by hypothesis, there exist $a \in X, b \in Y$ such that (x_n) converges to a in (X, d_X) and (y_n) to b in (Y, d_Y).

We show that $((x_n, y_n))$ converges to (a, b) in $(X \times Y, d_{XY})$. Let W be a generalized open set in $(X \times Y, \mu(d_{XY}))$ containing (a, b). Then there exist $(p, q) \in X \times Y, \delta > r$ such that $(a, b) \in B_{d_{XY}}((p, q), \delta) \subset W$. Consequently $a \in B_{d_X}(p, \delta)$ and $b \in B_{d_Y}(q, \delta)$. Since (x_n) converges to a and (y_n) converges to b, there exist $k_1, k_2 \in \mathbb{N}$ such that $x_n \in B_{d_X}(p, \delta)$ for every $n \geq k_1$ and $y_n \in B_{d_Y}(q, \delta)$ for every $n \geq k_2$. Set $k = \max\{k_1, k_2\}$. Then $(x_n, y_n) \in B_{d_{XY}}((p, q), \delta) \subset W$ for every $n \geq k$. Therefore $((x_n, y_n))$ converges to (a, b) in $(X \times Y, d_{XY})$. Thus $(X \times Y, d_{XY})$ is complete. □

Similar chains of arguments yield the following results that we state without proof.

Theorem 9. *Let (X, d_X) and (Y, d_Y) be g-quasi metric spaces of the same index r. Then $(X \times Y, d_{XY})$ is G-complete if and only if (X, d_X) and (Y, d_Y) are G-complete.*

Theorem 10. *Let (X, d_X) and (Y, d_Y) be g-quasi metric spaces of the same index r. If $(X \times Y, d_{XY})$ is weakly G-complete then (X, d_X) and (Y, d_Y) are weakly G-complete.*

Theorem 11. *Let (X, d_X) and (Y, d_Y) be g-quasi metric spaces of the same index r. If $(X \times Y, d_{XY})$ is (strongly) Lebesgue then (X, d_X) and (Y, d_Y) are (strongly) Lebesgue.*

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