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# Convergence analysis of an inertial method for a system of general quasi-variational inequalities under mild conditions

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Abstract. In this paper, we propose an efficient inertial iterative algorithm for solving a system of generalized quasi-variational inequalities (SGQVI) in Hilbert spaces. Using the projection operator technique, we establish an equivalence between SGQVI and fixed-point problems, thus developing a novel inertial method. The algorithm introduces an inertial term to accelerate convergence, and its performance is rigorously analyzed under some mild conditions, including relaxed co-coercivity and Lipschitz continuity of the involved mappings. Our framework unifies and extends several existing models, such as classical variational inequalities, quasi-variational inequalities, and related optimization problems. Some experiments demonstrate the effectiveness of the inertial method, which shows an improvement in convergence speed compared to non-inertial methods. Our results generalize and enhance previous research results in the literature, making it more widely applicable in computational mathematics, engineering, and economics.

## 1. Introduction

The theory of variational inequalities has become a significant field with wide applications in disciplines such as computational science, engineering, and economics [1, 30, 2]. The study of variational inequalities has undergone multiple developments, with novel approaches continually being adopted to expand their scope of application. Among these developments, quasi-variational inequalities stand out as a key extension, characterized by their

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reliance on solutions over implicitly or explicitly defined convex sets. These inequalities provide a powerful framework for solving a wide range of problems in both theoretical and applied disciplines. However, quasi-variational inequalities present a higher complexity compared to classical variational inequalities, which poses a significant challenge in designing efficient solution strategies. A widely adopted approach is to establish their equivalence with fixed-point problems, which facilitates the application of projection-based techniques. For further insights, see [20] and related references, as well as [22, 23, 25, 29] and other sources cited therein.

Previous research in this area has predominantly relied on the assumption of convexity of the underlying set. However, practical applications often involve non-convex sets, necessitating alternative approaches. To address this limitation, Noor [25] pioneered the theory of generalized convex sets and functions defined through arbitrary mappings $\eth$. Building on this framework, Cristescu et al. [11], together with further contributions by Noor [25], established the fundamental properties of $\eth$-convex sets. Advancing this line of research, Noor [26] proved that the extreme values of differentiable generalized convex functions on such sets can be characterized by a special class of variational inequalities, known as generalized variational inequalities. Currently, gradient projection and extra gradient methods for solving quasi-variational inequalities with the assumptions of strong monotonicity and Lipschitz continuity of the related mappings are studied by Antipin et al. [6]. Mijajlovic et al. [19] introduced a more general gradient projection method with strong convergence for solving quasi-variational inequalities in a real Hilbert space. This method works well in various practical applications, so it has great potential.

It is very important to establish an iterative scheme with a more accelerated rate of convergence. The inertial method is obtained from the oscillator equation with damping and conservative restoring force. It has become a vital source for refining the performance of the method and has nice convergence characteristics. The general foremost features of inertial-type alternatives are that we use previous iterations to construct the next. For constructing inertial methods, many authors have combined the inertial term $\{\Theta_n(\mu_n - \mu_{n-1})\}$ into various kinds of algorithms, such as Halpern, Kranoselski, Mann, Viscosity, etc. for finding solutions of optimization problems and fixed point problems. Here $\Theta_n$ is an extrapolating factor that stimulates the convergence rate of the method. Polyak [31] was the first author to suggest the heavy ball method. Alvarez et al. [4] used it to set up a proximal point algorithm. Also see [3, 7, 8, 9, 12, 18] and the references therein. In particular, Jabeen and coauthors investigated some inertial projection methods for some classes of general quasi-variational inequalities (general QVIs) [14, 15, 16]. In those works, the convergence of those techniques was rigorously investigated.

Motivated by these facts, we study a class of generalized quasi-variational inequalities system (SGQVI) and propose a new inertial iterative method to solve it. We establish convergence criteria under the relaxed co-coercivity and Lipschitz continuity conditions to ensure wider applicability. Since the SGQVI framework covers quasi-variational inequalities, classical variational inequalities, and related optimization problems as special cases, our results generalize and improve existing research results. The proposed method not only improves the computational efficiency through inertial acceleration, but also provides a unified approach for solving complex variational systems. Numerical experiments verify its effectiveness and show that its convergence speed is improved compared to the non-inertial scheme. This work advances the theoretical and practical understanding of variational inequalities and provides new insights for future research in optimization.

## 2. Preliminaries

Let $\mathcal{C}$ be a nonempty, closed and convex set in a real Hilbert space $\mathtt{H}$, with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $\mathtt{T}_1$, $\mathtt{T}_2 : \mathtt{H}\times\mathtt{H} \longrightarrow \mathtt{H}$ be multivalued operators and $\eth : \mathtt{H} \longrightarrow \mathtt{H}$ be a single-valued operator. Let $\mathcal{C} : \mathtt{H}\times\mathtt{H} \longrightarrow \mathtt{H}$ be a set-valued mapping, which for any element $(\mathsf{x},\mathsf{y}) \in \mathtt{H}\times\mathtt{H}$, associates a convex and closed set $\mathcal{C}(\mathsf{x},\mathsf{y}) \subset \mathtt{H}$. Under these circumstances, we consider the *system of general quasi-variational inequality* (SGQVI), which consists of finding $\mathsf{x}^*,\mathsf{y}^* \in \mathcal{C}(\mathsf{x}^*,\mathsf{y}^*)$ such that

$$\begin{cases}\left\langle \rho_1\,\mathtt{T}_1\,(\mathsf{y}^*,\mathsf{x}^*) + \mathsf{x}^* - \eth(\mathsf{y}^*)\,,\, \eth(\mathsf{x}) - \mathsf{x}^* \right\rangle \geq 0,\;\; \mathsf{x}\in\mathtt{H} : \eth(\mathsf{x}) \in \mathcal{C}(\mathsf{y}^*,\mathsf{x}^*),\\ \left\langle \rho_2\,\mathtt{T}_2\,(\mathsf{x}^*,\mathsf{y}^*) + \mathsf{y}^* - \eth(\mathsf{x}^*)\,,\, \eth(\mathsf{x}) - \mathsf{y}^* \right\rangle \geq 0,\;\; \mathsf{x}\in\mathtt{H} : \eth(\mathsf{x}) \in \mathcal{C}(\mathsf{x}^*,\mathsf{y}^*).\end{cases} \tag{1}$$

Here, $\rho_1 > 0$, $\rho_2 > 0$ are constants.

We now discuss some special cases of SGQVI (1).

**I.** If $\eth = I$ (the identity operator), then SGQVI (1) reduces to the *system of quasi-variational inequalities* (SQVI), which is equivalent to finding $\mathsf{x}^*,\mathsf{y}^* \in \mathcal{C}(\mathsf{x}^*,\mathsf{y}^*)$ such that

$$\begin{cases}\left\langle \rho_1\,\mathtt{T}_1\,(\mathsf{y}^*,\mathsf{x}^*) + \mathsf{x}^* - \mathsf{y}^*\,,\, \mathsf{x} - \mathsf{x}^* \right\rangle \geq 0,\;\; \mathsf{x}\in\mathtt{H} : \mathsf{x} \in \mathcal{C}(\mathsf{y}^*,\mathsf{x}^*),\\ \left\langle \rho_2\,\mathtt{T}_2\,(\mathsf{x}^*,\mathsf{y}^*) + \mathsf{y}^* - \mathsf{x}^*\,,\, \mathsf{x} - \mathsf{y}^* \right\rangle \geq 0,\;\; \mathsf{x}\in\mathtt{H} : \mathsf{x} \in \mathcal{C}(\mathsf{x}^*,\mathsf{y}^*).\end{cases} \tag{2}$$

This problem was studied by Noor et al. [27].

**II.** If $\mathtt{T}_1 = \mathtt{T}_2 = \mathtt{T}$, then SGQVI (1) reduces to the following *system of general quasi-variational inequalities* (SGQVI): find $\mathsf{x}^*,\mathsf{y}^* \in \mathcal{C}(\mathsf{x}^*,\mathsf{y}^*)$ with the property that

$$\begin{cases}\left\langle \rho_1\,\mathtt{T}\,(\mathsf{y}^*,\mathsf{x}^*) + \mathsf{x}^* - \mathsf{y}^*\,,\, \mathsf{x} - \mathsf{x}^* \right\rangle \geq 0, \quad \mathsf{x}\in\mathtt{H} : \mathsf{x} \in \mathcal{C}(\mathsf{y}^*,\mathsf{x}^*),\\ \left\langle \rho_2\,\mathtt{T}\,(\mathsf{x}^*,\mathsf{y}^*) + \mathsf{y}^* - \mathsf{x}^*\,,\, \mathsf{x} - \mathsf{y}^* \right\rangle \geq 0, \quad \mathsf{x}\in\mathtt{H} : \mathsf{x} \in \mathcal{C}(\mathsf{x}^*,\mathsf{y}^*).\end{cases} \tag{3}$$

**III.** If $\mathcal{C}(\mathsf{x}^*, \mathsf{y}^*) = \mathcal{C}(\mathsf{y}^*, \mathsf{x}^*) = \mathcal{C}$, then SGQVI (1) reduces to the following *system of general variational inequalities* (SGVI): find $\mathsf{x}^*, \mathsf{y}^* \in \mathcal{C}$ such that

$$\begin{cases} \left\langle \rho_1 \mathtt{T}_1 (\mathsf{y}^*, \mathsf{x}^*) + \mathsf{x}^* - \eth(\mathsf{y}^*) \, , \, \eth(\mathsf{x}) - \mathsf{y}^* \right\rangle \geq 0, & \mathsf{x} \in \mathtt{H} : \eth(\mathsf{x}) \in \mathcal{C}, \\ \left\langle \rho_2 \mathtt{T}_2 (\mathsf{x}^*, \mathsf{y}^*) + \mathsf{y}^* - \eth(\mathsf{x}^*) \, , \, \eth(\mathsf{x}) - \mathsf{x}^* \right\rangle \geq 0, & \mathsf{x} \in \mathtt{H} : \eth(\mathsf{x}) \in \mathcal{C}. \end{cases} \tag{4}$$

This problem has been considered and studied in [24].

**IV.** If $\eth = I$, then SGVI (4) reduces to the problem of finding $\mathsf{x}^*, \mathsf{y}^* \in \mathcal{C}$ with the property that

$$\begin{cases} \left\langle \rho_1 \mathtt{T}_1 (\mathsf{y}^*, \mathsf{x}^*) + \mathsf{x}^* - \mathsf{y}^* \, , \, \mathsf{x} - \mathsf{y}^* \right\rangle \geq 0, & \mathsf{x} \in \mathcal{C}, \\ \left\langle \rho_2 \mathtt{T}_2 (\mathsf{x}^*, \mathsf{y}^*) + \mathsf{y}^* - \mathsf{x}^* \, , \, \mathsf{x} - \mathsf{x}^* \right\rangle \geq 0, & \mathsf{x} \in \mathcal{C}. \end{cases} \tag{5}$$

This problem is called the *system of variational inequalities* (SVI), and it was considered and studied by Huang et al. [13].

To prove our result, the following definitions will be required.

**Definition 1.** Let $\mathtt{T} : \mathtt{H} \longrightarrow \mathtt{H}$ be a given mapping.

**i**. $\mathtt{T}$ is called *$r$-strongly monotone* ($r \geq 0$) if

$$\left\langle \mathtt{T}\delta - \mathtt{T}\gamma \, , \, \delta - \gamma \right\rangle \geq r \, \|\delta - \gamma\|^2, \quad \forall \, \delta, \, \gamma \, \in \, \mathtt{H}.$$

**ii**. $\mathtt{T}$ is called *$\xi$-cocoercive* ($\xi > 0$) if

$$\left\langle \mathtt{T}\delta - \mathtt{T}\gamma \, , \, \delta - \gamma \right\rangle \geq \xi \, \| \, \mathtt{T}\delta - \mathtt{T}\gamma \, \|^2, \quad \forall \, \delta, \, \gamma \, \in \, \mathtt{H}.$$

**iii**. $\mathtt{T}$ is called *relaxed* $(\xi, r)$*-cocoercive* ($r > 0$, $\xi > 0$) if

$$\left\langle \mathtt{T}\delta - \mathtt{T}\gamma \, , \, \delta - \gamma \right\rangle \geq \, -\xi \, \| \, \mathtt{T}\delta - \mathtt{T}\gamma \, \|^2 + r \, \|\delta - \gamma\|^2, \quad \forall \, \delta, \, \gamma \, \in \, \mathtt{H}.$$

For $\xi = 0$, the mapping $\mathtt{T}$ is *$r$-strongly monotone*. The class of relaxed $(\xi, r)$-cocoercive mappings is more general than the class of $r$-strongly monotone and $\xi$-cocoercive mappings.

**iv**. The mapping $\mathtt{T}$ is called *$\eta$-Lipschitz continuous* ($\eta > 0$) if

$$\| \, \mathtt{T}\delta - \mathtt{T}\gamma \, \| \leq \, \eta \, \|\delta - \gamma\|, \quad \forall \, \delta, \, \gamma \, \in \, \mathtt{H}.$$

The following projection lemma is the key to obtaining our results.

**Lemma 1** (Bensoussan and Lions [10])**.** *For a given $\omega \in \mathtt{H}$, there exists $\delta \in \mathcal{C}(\delta)$ such that*

$$\left\langle \delta - \omega, \gamma - \delta \right\rangle \geq 0, \ \ \forall \, \gamma \in \mathcal{C}(\delta),$$

*if and only if $\delta = \Pi_{\mathcal{C}(\delta)} [\omega]$, where $\Pi_{\mathcal{C}(\delta)}$ is the implicit projection of $\mathtt{H}$ onto the closed and convex-valued set $\mathcal{C}(\delta)$ in $\mathtt{H}$.*

To derive the convergence analysis of the iterative methods, we need the following assumption (see [22]).

**Assumption 1.** The projection operator $\Pi_{\mathcal{C}(\mathsf{x},\mathsf{y})}$ is not non-expansive but satisfies the following condition

$$\| \Pi_{\mathcal{C}(\mathsf{x},\mathsf{y})}[\omega] - \Pi_{\mathcal{C}(\mu,\nu)}[\omega] \| \leq \upsilon \ \| \mathsf{x} - \mu \| \quad \forall\, \mathsf{x},\ \mathsf{y},\ \omega,\ \mu,\ \nu \in \mathtt{H}, \tag{6}$$

where $\upsilon > 0$ is a constant.

The next result is very useful for our analysis.

**Lemma 2** (Xu [34])**.** *Consider a sequence of nonnegative real numbers* $\{\varrho_n\}$ *satisfying the inequality*

$$\varrho_{n+1} \leq (1 - \Upsilon_n)\varrho_n + \Upsilon_n\, \sigma_n + \varsigma_n, \quad \forall\, n \geq 1,$$

*where*

**i.** $\{\Upsilon_n\} \subset [0, 1]$, $\quad \sum\limits_{n=1}^{\infty} \Upsilon_n = \infty$;

**ii.** $\limsup \sigma_n \leq 0$;

**iii.** $\varsigma_n \geq 0$ $(n \geq 1)$, $\quad \sum\limits_{n=1}^{\infty} \varsigma_n < \infty$.

*Then* $\varrho_n \longrightarrow 0$ *as* $n \longrightarrow \infty$.

## 3. Iterative methods

In this section, we propose inertial iterative methods for solving the system of SGQVI (1). For this purpose, we establish the equivalence between the SGQVI (1) and fixed-point problems.

Using Lemma 1, we can easily show that the problem of finding a solution $\mathsf{x}^*, \mathsf{y}^* \in \mathcal{C}(\mathsf{x}^*, \mathsf{y}^*)$ of SGQVI (1) is equivalent to finding $\mathsf{x}^*, \mathsf{y}^* \in \mathcal{C}(\mathsf{x}^*, \mathsf{y}^*)$ with the property that

$$\begin{aligned} \mathsf{x}^* &= \Pi_{\mathcal{C}(\mathsf{y}^*,\mathsf{x}^*)}\left[\eth(\mathsf{y}^*) - \rho_1 \mathtt{T}_1(\mathsf{y}^*, \mathsf{x}^*)\right], \\ \mathsf{y}^* &= \Pi_{\mathcal{C}(\mathsf{x}^*,\mathsf{y}^*)}\left[\eth(\mathsf{x}^*) - \rho_2 \mathtt{T}_2(\mathsf{x}^*, \mathsf{y}^*)\right]. \end{aligned}$$

Notice that this problem can be equivalently written as

$$\mathsf{x}^* = (1 - \alpha_n)\mathsf{x}^* + \alpha_n \Pi_{\mathcal{C}(\mathsf{y}^*,\mathsf{x}^*)}\left[\eth(\mathsf{y}^*) - \rho_1 \mathtt{T}_1(\mathsf{y}^*, \mathsf{x}^*)\right], \tag{7}$$

$$\mathsf{y}^* = \Pi_{\mathcal{C}(\mathsf{x}^*,\mathsf{y}^*)}\left[\eth(\mathsf{x}^*) - \rho_2 \mathtt{T}_2(\mathsf{x}^*, \mathsf{y}^*)\right], \tag{8}$$

where $\alpha_n \in [0, 1]$ for all $n \geq 0$. This alternative form enables us to propose the following iterative methods for solving the SGQVI (1).

**Algorithm 1.** For arbitrarily chosen initial values $\mathsf{x}_1$, $\mathsf{y}_0$, $\mathsf{y}_1$, compute $\mathsf{x}_{n+1}$, $\mathsf{y}_{n+1}$ by the recurrence relation

$$\omega_n = (1 - \Theta_n)\mathsf{y}_n + \Theta_n \mathsf{y}_{n-1}, \tag{9}$$

$$\mathsf{x}_{n+1} = (1 - \alpha_n)\,\mathsf{x}_n + \alpha_n \Pi_{\mathcal{C}(\omega_n,\mathsf{x}_n)}\left[\eth(\omega_n) - \rho_1\, \mathtt{T}_1(\omega_n, \mathsf{x}_n)\right], \tag{10}$$

$$\mathsf{y}_{n+1} = \Pi_{\mathcal{C}(\mathsf{x}_{n+1},\omega_n)}\left[\eth(\mathsf{x}_{n+1}) - \rho_2\, \mathtt{T}_2(\mathsf{x}_{n+1}, \omega_n)\right], \quad n = 1, 2, \ldots, \tag{11}$$

where $\Theta_n, \alpha_n \in [0,1]$. This method is called the *forward-backward inertial method* for solving SGQVI (1).

For $\alpha_n = 1$, Algorithm 1 reduces to the following.

**Algorithm 2.** For arbitrary initial values $x_1$, $y_0$, $y_1$, compute $x_{n+1}$, $y_{n+1}$ through the recurrence relation

$$\begin{aligned}
\omega_n &= (1-\Theta_n)y_n + \Theta_n y_{n-1}, \\
x_{n+1} &= \Pi_{\mathcal{C}(\omega_n, x_n)}\left[\eth(\omega_n) - \rho_1 T_1(\omega_n, x_n)\right], \\
y_{n+1} &= \Pi_{\mathcal{C}(x_{n+1}, \omega_n)}\left[\eth(x_{n+1}) - \rho_2 T_2(x_{n+1}, \omega_n)\right], \quad n = 1, 2, \ldots,
\end{aligned}$$

where $\Theta_n \in [0,1]$.

For $T_1 = T_2 = T$, Algorithm 1 reduces to the following.

**Algorithm 3.** For arbitrarily chosen initial values $x_1$, $y_0$, $y_1$, compute $x_{n+1}$, $y_{n+1}$ by the recurrence relation

$$\begin{aligned}
\omega_n &= (1-\Theta_n)y_n + \Theta_n y_{n-1}, \\
x_{n+1} &= (1-\alpha_n)x_n + \alpha_n \Pi_{\mathcal{C}(\omega_n, x_n)}\left[\eth(\omega_n) - \rho_1 T(\omega_n, x_n)\right], \\
y_{n+1} &= \Pi_{\mathcal{C}(x_{n+1}, \omega_n)}\left[\eth(x_{n+1}) - \rho_2 T(x_{n+1}, \omega_n)\right], \quad n = 1, 2, \ldots,
\end{aligned}$$

where $\Theta_n, \alpha_n \in [0,1]$.

If $\eth = I$, then Algorithm 1 reduces to the following algorithm.

**Algorithm 4.** For arbitrary initial values $x_1$, $y_0$, $y_1$, compute $x_{n+1}$, $y_{n+1}$ by the recurrence relation

$$\begin{aligned}
\omega_n &= (1-\Theta_n)y_n + \Theta_n y_{n-1}, \\
x_{n+1} &= (1-\alpha_n)x_n + \alpha_n \Pi_{\mathcal{C}(\omega_n, x_n)}\left[\omega_n - \rho_1 T_1(\omega_n, x_n)\right], \\
y_{n+1} &= \Pi_{\mathcal{C}(x_{n+1}, \omega_n)}\left[x_{n+1} - \rho_2 T_2(x_{n+1}, \omega_n)\right], \quad n = 1, 2, \ldots,
\end{aligned}$$

where $\Theta_n, \alpha_n \in [0,1]$.

For $\mathcal{C}(x,y) = \mathcal{C}(y,x) = \mathcal{C}$, Algorithm 1 reduces to the following iterative method for solving a system of general variational inequalities (SGVI).

**Algorithm 5.** For arbitrary initial values $x_1$, $y_0$, $y_1$, compute $x_{n+1}$, $y_{n+1}$ by the recurrence relation

$$\begin{aligned}
\omega_n &= (1-\Theta_n)y_n + \Theta_n y_{n-1}, \\
x_{n+1} &= (1-\alpha_n)x_n + \alpha_n \Pi_{\mathcal{C}}\left[\eth(\omega_n) - \rho_1 T_1(\omega_n, x_n)\right], \\
y_{n+1} &= \Pi_{\mathcal{C}}\left[\eth(x_{n+1}) - \rho_2 T_2(x_{n+1}, \omega_n)\right], \quad n = 1, 2, \ldots,
\end{aligned}$$

where $\Theta_n, \alpha_n \in [0,1]$.

For $\mathcal{C}(x,y) = \mathcal{C}(x,y) = \mathcal{C}$, $\eth = I$, Algorithm 1 reduces to the following iterative method for solving a system of variational inequalities.

**Algorithm 6.** For arbitrary chosen initial values $x_1$, $y_0$, $y_1$, compute $x_{n+1}$, $y_{n+1}$ by the recurrence relation

$$\begin{aligned} \omega_n &= (1-\Theta_n) y_n + \Theta_n y_{n-1}, \\ x_{n+1} &= (1-\alpha_n) x_n + \alpha_n \Pi_{\mathcal{C}} \left[ \omega_n - \rho_1 \mathtt{T}_1(\omega_n, x_{n)} \right], \\ y_{n+1} &= \Pi_{\mathcal{C}} \left[ x_{n+1} - \rho_2 \mathtt{T}_2(x_{n+1}, \omega_n) \right], \quad n = 1, 2, \ldots, \end{aligned}$$

where $\Theta_n, \alpha_n \in [0,1]$.

Similarly, with an adequate and appropriate choice of spaces and operators, we can obtain numerous iterative methods to solve problem (1).

## 4. Convergence analysis

In this section, we derive convergence criteria for Algorithm 1 under some mild and appropriate conditions.

**Theorem 1.** *Let* $x^*, y^*$ *be the solution of SGQVI* (1)*. Let the mappings* $\mathtt{T}_1, \mathtt{T}_2 : \mathtt{H} \times \mathtt{H} \longrightarrow \mathtt{H}$ *be relaxed* $(\xi_1, r_1)$, $(\xi_2, r_2)$*-cocoercive and* $\eta_1$, $\eta_2$*-Lipschitz continuous in the first variable, respectively. The mapping* $\eth : \mathtt{H} \longrightarrow \mathtt{H}$ *is relaxed* $(\xi_3, r_3)$*-cocoercive and* $\eta_3$*-Lipschitz continuous in the first variable. Further, we suppose that*

***i.*** *Assumption 1 holds;*

***ii.*** *the constants* $\rho_1$, $\rho_2$ *satisfy the following conditions:* $\kappa < 1$ *and*

$$\left| \rho_1 - \frac{(r_1 - \xi_1 \eta_1^2)}{\eta_1^2} \right| < \frac{\sqrt{(r_1 - \xi_1 \eta_1^2)^2 - \eta_1^2 \kappa(2-\kappa)}}{\eta_1^2},$$

$$r_1 > \xi_1 \eta_1^2 + \eta_1 \sqrt{\kappa(2-\kappa)}, \tag{12}$$

$$\left| \rho_2 - \frac{(r_2 - \xi_2 \eta_2^2)}{\eta_2^2} \right| < \frac{\sqrt{(r_2 - \xi_2 \eta_2^2)^2 - \eta_2^2 \kappa(2-\kappa)}}{\eta_2^2},$$

$$r_2 > \xi_2 \eta_2^2 + \eta_2 \sqrt{\kappa(2-\kappa)}, \tag{13}$$

*where*

$$\kappa = \sqrt{1 + 2\xi_3\eta_3^2 - 2r_3 + \eta_3^2} + \upsilon; \tag{14}$$

***iii.*** $\Theta_n, \alpha_n \in [0,1]$ *for all* $n \geq 1$*, such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and*

$$\sum_{n=1}^{\infty} \Theta_n \|y_n - y_{n-1}\| < \infty.$$

*Then the sequences* $x_n$ *and* $y_n$*, obtained from the iterative scheme defined in Algorithm 1, converge strongly to the unique solutions* $x^*$ *and* $y^*$ *of the SGQVI* (1)*, respectively.*

*Proof.* Since $\mathsf{x}^*, \mathsf{y}^*$ is a solution of the SGQVI (1), equation (7) and equation (10) imply that

$$\begin{aligned}
\|\mathsf{x}_{n+1}-\mathsf{x}^*\| = \|&(1-\alpha_n)\,\mathsf{x}_n + \alpha_n \Pi_{\mathcal{C}(\omega_n,\mathsf{x}_n)}\,[\eth(\omega_n) - \rho_1\,\mathtt{T}_1(\omega_n,\mathsf{x}_n)] \\
&- (1-\alpha_n)\mathsf{x}^* - \alpha_n \Pi_{\mathcal{C}(\mathsf{y}^*,\mathsf{x}^*)}\,[\eth(\mathsf{y}^*) - \rho_1 \mathtt{T}_1(\mathsf{y}^*,\mathsf{x}^*)]\,\| \\
\le (1-\alpha_n)&\|\mathsf{x}_n - \mathsf{x}^*\| + \alpha_n \|\Pi_{\mathcal{C}(\omega_n,\mathsf{x}_n)}\,[\eth(\omega_n) - \rho_1\,\mathtt{T}_1(\omega_n,\mathsf{x}_n)\,] \\
&- \Pi_{\mathcal{C}(\mathsf{y}^*,\mathsf{x}^*)}\,[\eth(\mathsf{y}^*) - \rho_1 \mathtt{T}_1(\mathsf{y}^*,\mathsf{x}^*)]\,\| \\
\le (1-\alpha_n)&\|\mathsf{x}_n - \mathsf{x}^*\| + \alpha_n \|\Pi_{\mathcal{C}(\omega_n,\mathsf{x}_n)}\,[\eth(\omega_n) - \rho_1\,\mathtt{T}_1(\omega_n,\mathsf{x}_n)\,] \\
&- \Pi_{\mathcal{C}(\omega_n,\mathsf{x}_n)}\,[\eth(\mathsf{y}^*) - \rho_1 \mathtt{T}_1(\mathsf{y}^*,\mathsf{x}^*)]\,\| \\
&+ \alpha_n \|\Pi_{\mathcal{C}(\omega_n,\mathsf{x}_n)}\,[\eth(\mathsf{y}^*) - \rho_1 \mathtt{T}_1(\mathsf{y}^*,\mathsf{x}^*)] \\
&- \Pi_{\mathcal{C}(\mathsf{y}^*,\mathsf{x}^*)}\,[\eth(\mathsf{y}^*) - \rho_1 \mathtt{T}_1(\mathsf{y}^*,\mathsf{x}^*)]\,\| \\
\le (1-\alpha_n)&\|\mathsf{x}_n - \mathsf{x}^*\| + \alpha_n \upsilon \|\omega_n - \mathsf{y}^*\| \\
&+ \alpha_n \|\eth(\omega_n) - \eth(\mathsf{y}^*) - \rho_1\,[\,\mathtt{T}_1(\omega_n,\mathsf{x}_n) - \mathtt{T}_1(\mathsf{y}^*,\mathsf{x}^*)\,]\,\| \\
= (1-\alpha_n)&\|\mathsf{x}_n - \mathsf{x}^*\| + \alpha_n \| - (\omega_n - \mathsf{y}^*) + \eth(\omega_n) - \eth(\mathsf{y}^*) \\
&+ (\omega_n - \mathsf{y}^*) - \rho_1\,[\,\mathtt{T}_1(\omega_n,\mathsf{x}_n) - \mathtt{T}_1(\mathsf{y}^*,\mathsf{x}^*)\,]\,\| \\
&+ \alpha_n \upsilon \|\omega_n - \mathsf{y}^*\| \\
\le (1-\alpha_n)&\|\mathsf{x}_n - \mathsf{x}^*\| + \alpha_n \|\omega_n - \mathsf{y}^* - [\,\eth(\omega_n) - \eth(\mathsf{y}^*)\,]\,\| \\
&+ \alpha_n \|\omega_n - \mathsf{y}^* - \rho_1\,[\,\mathtt{T}_1(\omega_n,\mathsf{x}_n) - \mathtt{T}_1(\mathsf{y}^*,\mathsf{x}^*)\,]\,\| \\
&+ \alpha_n \upsilon \|\omega_n - \mathsf{y}^*\|,
\end{aligned} \tag{15}$$

where we have used Assumption 1. Since the operator $\mathtt{T}_1$ is the relaxed $(\xi_1\,, r_1)$-cocoercive and $\eta_1$-Lipschitzian, we obtain

$$\begin{aligned}
\|\omega_n - \mathsf{y}^* - &\rho_1\,[\,\mathtt{T}_1(\omega_n,\mathsf{x}_n) - \mathtt{T}_1(\mathsf{y}^*,\mathsf{x}^*)\,]\,\|^2 \\
&= \|\omega_n - \mathsf{y}^*\|^2 - 2\rho_1 \langle \mathtt{T}_1(\omega_n,\mathsf{x}_n) - \mathtt{T}_1(\mathsf{y}^*,\mathsf{x}^*)\,,\ \omega_n - \mathsf{y}^* \rangle \\
&\qquad + \rho_1^2 \|\mathtt{T}_1(\omega_n,\mathsf{x}_n) - \mathtt{T}_1(\mathsf{y}^*,\mathsf{x}^*)\|^2 \\
&\le \|\omega_n - \mathsf{y}^*\|^2 + 2\rho_1 \xi_1 \|\mathtt{T}_1(\omega_n,\mathsf{x}_n) - \mathtt{T}_1(\mathsf{y}^*,\mathsf{x}^*)\|^2 \\
&\qquad - 2\rho_1 r_1 \|\omega_n - \mathsf{y}^*\|^2 + \rho_1^2 \|\mathtt{T}_1(\omega_n,\mathsf{x}_n) - \mathtt{T}_1(\mathsf{y}^*,\mathsf{x}^*)\|^2 \\
&\le \|\omega_n - \mathsf{y}^*\|^2 + 2\rho_1 \xi_1 \eta_1^2 \|\omega_n - \mathsf{y}^*\|^2 - 2\rho_1 r_1 \|\omega_n - \mathsf{y}^*\|^2 \\
&\qquad + \rho_1^2 \eta_1^2 \|\omega_n - \mathsf{y}^*\|^2 \\
&= \left\{\,1 + 2\rho_1 \xi_1 \eta_1^2 - 2\rho_1 r_1 + \rho_1^2 \eta_1^2\,\right\} \|\omega_n - \mathsf{y}^*\|^2.
\end{aligned} \tag{16}$$

In a similar way, from the relaxed $(\xi_3\,, r_3)$-cocoercivity and $\eta_3$-Lipschitz continuity of the operator $\eth$ we have

$$\|\omega_n - \mathsf{y}^* - [\,\eth(\omega_n) - \eth(\mathsf{y}^*)\,]\,\|^2 \le \left\{\,1 + 2\xi_3 \eta_3^2 - 2r_3 + \eta_3^2\,\right\} \|\omega_n - \mathsf{y}^*\|^2. \tag{17}$$

From (15), (16), and (17), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\Big\{ \sqrt{1 + 2\xi_3\eta_3^2 - 2r_3 + \eta_3^2} \\ &\qquad + \sqrt{1 + 2\rho_1\xi_1\eta_1^2 - 2\rho_1 r_1 + \rho_1^2\eta_1^2} + \upsilon \Big\}\|\omega_n - y^*\| \\ &= (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\|\omega_n - y^*\|, \end{aligned} \tag{18}$$

with $\theta_1 = \kappa + \sqrt{1 + 2\rho_1\xi_1\eta_1^2 - 2\rho_1 r_1 + \rho_1^2\eta_1^2}$ and $\kappa$ given by (14). Since $T_2$ is relaxed $(\xi_2\,, r_2)$-cocoercive and $\eta_2$-Lipschitzian, we obtain

$$\begin{aligned} \|x_{n+1} - x^* - \rho_2\ [T_2(x_{n+1}, \omega_n) - T_2(x^*, y^*)]\,\|^2 \\ &= \|x_{n+1} - x^*\|^2 - 2\rho_2\langle T_2(x_{n+1}, \omega_n) - T_2(x^*, y^*)\,,\, x_{n+1} - x^*\rangle \\ &\quad + \rho_2^2\|T_2(x_{n+1}, \omega_n) - T_2(x^*, y^*)\|^2 \\ &\le \|x_{n+1} - x^*\|^2 + 2\rho_2\xi_2\|T_2(x_{n+1}, \omega_n) - T_2(x^*, y^*)\|^2 \\ &\quad - 2\rho_2 r_2\|x_{n+1} - x^*\|^2 + \rho_2^2\|T_2(x_{n+1}, \omega_n) - T_2(x^*, y^*)\|^2 \\ &\le \|x_{n+1} - x^*\|^2 + 2\rho_2\xi_2\eta_2^2\|x_{n+1} - x^*\|^2 \\ &\quad - 2\rho_2 r_2\|x_{n+1} - x^*\|^2 + \rho_2^2\eta_2^2\|x_{n+1} - x^*\|^2 \\ &= \Big\{\, 1 + 2\rho_2\xi_2\eta_2^2 - 2\rho_2 r_2 + \rho_2^2\eta_2^2 \,\Big\}\|x_{n+1} - x^*\|^2. \end{aligned} \tag{19}$$

In a similar fashion, we reach that

$$\|x_{n+1} - x^* - [\,\eth(x_{n+1}) - \eth(x^*)\,]\,\|^2 \le \Big\{\, 1 + 2\xi_3\eta_3^2 - 2r_3 + \eta_3^2 \,\Big\}\|x_{n+1} - x^*\|^2. \tag{20}$$

Now, from (8), (11), (19), (20), and using Assumption 1, we obtain

$$\begin{aligned} \|y_{n+1} - y^*\| &= \|\Pi_{\mathcal{C}(x_{n+1},\omega_n)}\,[\,\eth(x_{n+1}) - \rho_2\, T_2(x_{n+1}, \omega_n)\,] \\ &\qquad - \Pi_{\mathcal{C}(x^*,y^*)}\,[\eth(x^*) - \rho_2 T_2(x^*, y^*)]\,\| \\ &\le \|\Pi_{\mathcal{C}(x_{n+1},\omega_n)}\,[\,\eth(x_{n+1}) - \rho_2\, T_2(x_{n+1}, \omega_n)\,] \\ &\qquad - \Pi_{\mathcal{C}(x_{n+1},\omega_n)}\,[\eth(x^*) - \rho_2 T_2(x^*, y^*)]\,\| \\ &\qquad + \|\Pi_{\mathcal{C}(x_{n+1},\omega_n)}\,[\eth(x^*) - \rho_2 T_2(x^*, y^*)] \\ &\qquad - \Pi_{\mathcal{C}(x^*,y^*)}\,[\eth(x^*) - \rho_2 T_2(x^*, y^*)]\,\| \\ &\le \|\,[\eth(x_{n+1}) - \rho_2\, T_2(x_{n+1}, \omega_n)] - [\eth(x^*) - \rho_2 T_2(x^*, y^*)]\,\| + \upsilon\|x_{n+1} - x^*\| \\ &= \|-(x_{n+1} - x^*) + \eth(x_{n+1}) - \eth(x^*) + (x_{n+1} - x^*) \\ &\qquad - \rho_2\ [T_2(x_{n+1}, \omega_n) - T_2(x^*, y^*)]\,\| + \upsilon\|x_{n+1} - x^*\| \\ &\le \|x_{n+1} - x^* - [\eth(x_{n+1}) - \eth(x^*)]\,\| \\ &\qquad + \|x_{n+1} - x^* - \rho_2\ [T_2(x_{n+1}, \omega_n) - T_2(x^*, y^*)]\,\| + \upsilon\|x_{n+1} - x^*\| \\ &\le \theta_2\|x_{n+1} - x^*\| \le \|x_{n+1} - x^*\|, \end{aligned} \tag{21}$$

where $\theta_2 = \kappa + \sqrt{1 + 2\rho_2\xi_2\eta_2^2 - 2\rho_2 r_2 + \rho_2^2\eta_2^2} < 1$, from condition (13). From (9), we derive

$$\|\omega_n - y^*\| \leq \|y_n - y^*\| + \Theta_n \|y_n - y_{n-1}\|. \tag{22}$$

Using (18), (21), and (22), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\big[\|y_n - y^*\| + \Theta_n \|y_n - y_{n-1}\|\big] \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\big[\|x_n - x^*\| + \Theta_n \|y_n - y_{n-1}\|\big] \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\|x_n - x^*\| + \Theta_n \|y_n - y_{n-1}\| \\
&= \big[1 - \alpha_n(1-\theta_1)\big]\|x_n - x^*\| + \Theta_n \|y_n - y_{n-1}\|.
\end{aligned} \tag{23}$$

Using now condition (12), we obtain that $\theta_1 < 1$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let us set $\sigma_n = 0$ and $\varsigma_n = \sum_{n=1}^{\infty} \Theta_n \| y_n - y_{n-1} \| < \infty$. Using now Lemma 2, we obtain $\lim_{n\to\infty} \|x_n - x^*\| = 0$. The fact that $\lim_{n\to\infty} \|y_n - y^*\| = 0$ is derived from (21) in a similar fashion. Thus, the proof is complete. □

**Theorem 2.** *Let mappings* $\mathtt{T}_1, \mathtt{T}_2 : \mathtt{H} \times \mathtt{H} \longrightarrow \mathtt{H}$ *be* $r_1, r_2$*-strongly monotone and* $\eta_1, \eta_2$*-Lipschitz continuous in the first variable, respectively. The mapping* $\eth : \mathtt{H} \longrightarrow \mathtt{H}$ *is* $r_3$*-strongly monotonic and* $\eta_3$*-Lipschitz continuous in the first variable. Further, we suppose that*

**i**. *Assumption 1 holds;*

**ii**. *the constants* $\rho_1$, $\rho_2$ *satisfy the following conditions:* $\kappa < 1$ *and*

$$\left|\rho_1 - \frac{r_1}{\eta_1^2}\right| < \frac{\sqrt{r_1^2 - \eta_1^2\,\kappa(2-\kappa)}}{\eta_1^2},$$
$$r_1 > \eta_1\sqrt{\kappa(2-\kappa)},$$
$$\left|\rho_2 - \frac{r_2}{\eta_2^2}\right| < \frac{\sqrt{r_2^2 - \eta_2^2\,\kappa(2-\kappa)}}{\eta_2^2},$$
$$r_2 > \eta_2\sqrt{\kappa(2-\kappa)},$$

*where* $\kappa = \sqrt{1 - 2r_3 + \eta_3^2} + \upsilon$;

**iii**. $\Theta_n, \alpha_n \in [0,1]$ *for all* $n \geq 1$, *such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and*

$$\sum_{n=1}^{\infty} \Theta_n\|y_n - y_{n-1}\| < \infty.$$

*Then the approximate solution* $x_n$ *and* $y_n$, *obtained from the iterative scheme defined in Algorithm 1 converge strongly to the unique solution* $x^*$ *and* $y^*$ *of the SGQVI* (1), *respectively.*

*Proof.* The conclusion is reached from Theorem 1 by using Definition 1 with $\xi_1 = \xi_2 = 0$. $\square$

We discuss now some of the special cases of the SGQVI considered in the preliminaries.

**I.** If $\eth = I$, then the following result can be obtained from Theorem 1.

**Theorem 3.** *Let mappings* $\mathtt{T}_1$, $\mathtt{T}_2 : \mathtt{H}\times\mathtt{H} \longrightarrow \mathtt{H}$ *be relaxed* $(\xi_1, r_1)$-, $(\xi_2, r_2)$-*cocoercive and* $\eta_1$, $\eta_2$-*Lipschitz continuous in the first variable, respectively. Further, we suppose that*
(*a*) *Assumption 1 holds;*
(*b*) *the constants* $\rho_1$, $\rho_2$ *satisfy:* $\upsilon < 1$ *and*

$$\left|\rho_1 - \frac{(r_1 - \xi_1 \eta_1^2)}{\eta_1^2}\right| < \frac{\sqrt{(r_1 - \xi_1 \eta_1^2)^2 - \eta_1^2 \upsilon(2-\upsilon)}}{\eta_1^2},$$
$$r_1 > \xi_1 \eta_1^2 + \eta_1 \sqrt{\upsilon(2-\upsilon)},$$
$$\left|\rho_2 - \frac{(r_2 - \xi_2 \eta_2^2)}{\eta_2^2}\right| < \frac{\sqrt{(r_2 - \xi_2 \eta_2^2)^2 - \eta_2^2 \upsilon(2-\upsilon)}}{\eta_2^2},$$
$$r_2 > \xi_2 \eta_2^2 + \eta_2 \sqrt{\upsilon(2-\upsilon)};$$

(*c*) $\Theta_n, \alpha_n \in [0,1]$ *for all* $n \geq 1$, *such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and*

$$\sum_{n=1}^{\infty} \Theta_n \|\mathsf{y}_n - \mathsf{y}_{n-1}\| < \infty.$$

*Then the approximate solutions* $\mathsf{x}_n$ *and* $\mathsf{y}_n$, *obtained from the iterative scheme defined in Algorithm 4 converge strongly to the unique solution* $\mathsf{x}^*$ *and* $\mathsf{y}^*$ *of the SQVI* (2), *respectively.*

**II.** If $\mathcal{C}(\mathsf{x}) = \mathcal{C}(\mathsf{y}) = \mathcal{C}$, then we obtain the following from Theorem 1.

**Theorem 4.** *Let mappings* $\mathtt{T}_1$, $\mathtt{T}_2 : \mathtt{H}\times\mathtt{H} \longrightarrow \mathtt{H}$ *be relaxed* $(\xi_1, r_1)$-, $(\xi_2, r_2)$-*cocoercive and* $\eta_1$, $\eta_2$-*Lipschitz continuous in the first variable, respectively. The mapping* $\eth : \mathtt{H} \longrightarrow \mathtt{H}$ *is relaxed* $(\xi_3, r_3)$-*cocoercive and* $\eta_3$-*Lipschitz continuous in the first variable. Further, suppose that*
(*a*) *the constants* $\rho_1$, $\rho_2$ *satisfy*

$$\left|\rho_1 - \frac{(r_1 - \xi_1 \eta_1^2)}{\eta_1^2}\right| < \frac{\sqrt{(r_1 - \xi_1 \eta_1^2)^2 - \eta_1^2 \kappa(2-\kappa)}}{\eta_1^2},$$
$$r_1 > \xi_1 \eta_1^2 + \eta_1 \sqrt{\kappa(2-\kappa)},$$
$$\left|\rho_2 - \frac{(r_2 - \xi_2 \eta_2^2)}{\eta_2^2}\right| < \frac{\sqrt{(r_2 - \xi_2 \eta_2^2)^2 - \eta_2^2 \kappa(2-\kappa)}}{\eta_2^2},$$

$$r_2 > \xi_2 \eta_2^2 + \eta_2 \sqrt{\kappa(2-\kappa)},$$

*where* $\kappa = \sqrt{1 + 2\xi_3\eta_3^2 - 2r_3 + \eta_3^2}$, *and* $\kappa < 1$.

(*b*) $\Theta_n, \alpha_n \in [0,1]$ *for all* $n \geq 1$, *such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and*

$$\sum_{n=1}^{\infty} \Theta_n \|\mathsf{y}_n - \mathsf{y}_{n-1}\| < \infty.$$

*Then the approximate solutions* $\mathsf{x}_n$ *and* $\mathsf{y}_n$, *obtained from the iterative scheme defined in Algorithm 5 converge strongly to the unique solution* $\mathsf{x}^*$ *and* $\mathsf{y}^*$ *of the SGVI* (4), *respectively.*

**III.** For $\mathcal{C}(\mathsf{x}) = \mathcal{C}(\mathsf{y}) = \mathcal{C}$, $\eth = I$, we derive the following convergence criteria for systems of variational inequalities.

**Theorem 5.** *Let mappings* $\mathtt{T}_1$, $\mathtt{T}_2 : \mathtt{H}\times\mathtt{H} \longrightarrow \mathtt{H}$ *be relaxed* $(\xi_1, r_1)$-, $(\xi_2, r_2)$-*cocoercive and* $\eta_1$, $\eta_2$-*Lipschitz continuous in the first variable, respectively. Suppose that*

(*a*) *the constants* $\rho_1$, $\rho_2$ *satisfy the following conditions:*

$$0 < \rho_1 < \frac{2(r_1 - \xi_1 \eta_1^2)}{\eta_1^2},$$
$$0 < \rho_2 < \frac{2(r_2 - \xi_2 \eta_2^2)}{\eta_2^2};$$

(*b*) $\Theta_n, \alpha_n \in [0,1]$ *for all* $n \geq 1$, *such that* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ *and*

$$\sum_{n=1}^{\infty} \Theta_n \|\mathsf{y}_n - \mathsf{y}_{n-1}\| < \infty.$$

*Then the approximate solutions* $\mathsf{x}_n$ *and* $\mathsf{y}_n$, *obtained from the iterative scheme defined in Algorithm 6 converge strongly to the unique solution* $\mathsf{x}^*$ *and* $\mathsf{y}^*$ *of the SVID* (5), *respectively.*

## 5. Numerical experiments

To verify the theoretical convergence and computational efficiency of the proposed inertial method (Algorithm 4), we perform numerical tests on the quasi-variational inequality system (2) in $\mathbb{R}^2$. The performance of the inertial scheme is compared with that of the non-inertial scheme, highlighting the acceleration effect of the inertial term. To that end, we will consider

$$\begin{aligned} \mathtt{T}_1(\mathsf{x},\mathsf{y}) &= A\mathsf{y} + B\mathsf{x} + a, \\ \mathtt{T}_2(\mathsf{x},\mathsf{y}) &= C\mathsf{x} + D\mathsf{y} + b, \end{aligned} \tag{24}$$

TABLE 1. Inertial method (Algorithm 4)

| Iter | $\Theta$ | $x_1$ | $x_2$ | $y_1$ | $y_2$ | Error |
|---|---|---|---|---|---|---|
| 1 | 0.000 | -0.7093 | -0.4332 | -0.7288 | -0.6847 | 1.5275 |
| 2 | 0.250 | 0.4046 | -0.1388 | 0.6529 | 0.2448 | 1.1522 |
| 3 | 0.400 | -0.7387 | -0.4642 | -0.3681 | -0.1909 | 1.1888 |
| 4 | 0.500 | 0.1814 | -0.1213 | -0.5193 | -0.3926 | 0.98195 |
| 5 | 0.571 | -0.3730 | -0.4484 | 0.7108 | 0.0410 | 0.64365 |
| 6 | 0.625 | -0.0246 | -0.2092 | -0.0796 | -0.0001 | 0.42259 |
| 7 | 0.667 | -0.5846 | -0.3713 | -0.9294 | -0.3692 | 0.58293 |
| 8 | 0.700 | 0.2517 | -0.1885 | 0.4584 | -0.1823 | 0.85599 |
| 9 | 0.727 | -0.3611 | -0.4406 | 0.9497 | 0.1162 | 0.6626 |
| 10 | 0.750 | -0.3236 | -0.2145 | -0.9889 | -0.1483 | 0.2292 |

TABLE 2. Non-inertial method

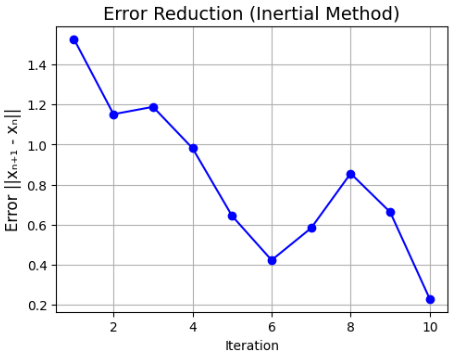
| Iter | $\Theta$ | $x_1$ | $x_2$ | $y_1$ | $y_2$ | Error |
|---|---|---|---|---|---|---|
| 1 | 0 | -0.709295 | -0.433188 | -0.728840 | -0.684684 | 1.5275 |
| 2 | 0 | 0.590673 | -0.247569 | 0.899604 | 0.436707 | 1.3132 |
| 3 | 0 | -0.827729 | -0.178357 | -0.906880 | -0.421389 | 1.4201 |
| 4 | 0 | 0.679473 | -0.292891 | 0.990876 | 0.134775 | 1.5115 |
| 5 | 0 | -0.805285 | -0.247155 | -0.969936 | -0.243361 | 1.4855 |
| 6 | 0 | 0.611362 | -0.143553 | 0.999609 | 0.027962 | 1.4204 |
| 7 | 0 | -0.769836 | -0.360022 | -0.979112 | -0.203323 | 1.3981 |
| 8 | 0 | 0.558042 | -0.033177 | 0.999540 | 0.030329 | 1.3675 |
| 9 | 0 | -0.742125 | -0.419622 | -0.975478 | -0.220096 | 1.3564 |
| 10 | 0 | 0.527178 | 0.006137 | 0.998261 | 0.058946 | 1.3388 |

and we will use the following matrices and vectors:

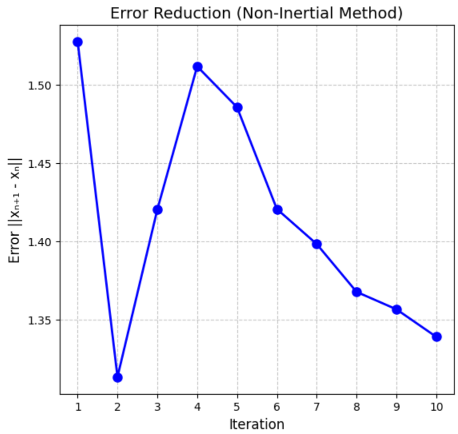
$$\begin{aligned} A &= \begin{pmatrix} 2 & 0.5 \\ 0.5 & 1 \end{pmatrix}, & B &= \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \\ C &= \begin{pmatrix} 2 & 0.3 \\ 0.3 & 1 \end{pmatrix}, & D &= \begin{pmatrix} 4 & 0.2 \\ 0.2 & 3 \end{pmatrix}, \\ a &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & b &= \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}. \end{aligned} \tag{25}$$

We implemented Algorithm 4 in Python with the following parameters

- Step sizes: $\rho_1 = \rho_2 = 0.5$.
- Relaxation parameter: $\alpha = 0.9$.
- Initial conditions: $x_0 = y_0 = (0.5, 0.5)^T$.




(A) Error reduction of Algorithm 4




(B) Error reduction of Algorithm 4 with $\Theta_n = 0$

FIGURE 1. Convergence behavior of Algorithm 4 showing (a) error reduction with inertial term and (b) error reduction with $\Theta_n = 0$

- Inertial parameter $\Theta_n = \frac{n-1}{n+2}$.
- Constraint set: $\mathcal{C} = \{\mathsf{x} \in \mathbb{R}^2 : \|\mathsf{x}\| \leq 1\}$.

Under these circumstances, the convergence behavior of Algorithm 4 is described in Table 1. Meanwhile, the convergence behavior of Algorithm 4 in its non-inertial version ($\Theta_n = 0$) is presented in Table 2. The performance of both approaches is depicted in Figure 1. Obviously, the tables and graphs verify empirically that the inertial method 4 is superior to the non-inertial method, which is consistent with the theoretical analysis results. It is a more efficient method for solving a system of quasi-variational inequalities because the inclusion of inertial term speeds up and stabilizes convergence.

## 6. Conclusion

In this study, an inertial algorithm for solving the system of generalized quasi-variational inequalities (SGQVI) in a Hilbert space is developed. SGQVI is reformulated as a fixed point problem by introducing a projection operator. Inertial terms are introduced to accelerate convergence. Theoretical results show that Algorithm 1 converges strongly under relaxed co-coercivity and Lipschitz continuity conditions. Numerical experiments in $\mathbb{R}^2$ show that the inertial method (Algorithm 4) outperforms the non-inertial method in terms of speed and stability. The proposed method extends existing variational inequality methods and provides a unified framework applicable to optimization and related areas. Future work will explore the effectiveness of inertial techniques in solving non-smooth and non-convex problems.

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