ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 29, Number 2, December 2025
Available online at https://ojs.utlib.ee/index.php/ACUTM

On some generalized split problems and their solutions

Mohd Asad and Mohammad Dilshad

Abstract. In this paper, we design some generalized split problems which can be seen as an extended form of the split variational inequality problems. We present several iterative algorithms for solving generalized split problems and demonstrate the weak convergence results under some appropriate assumptions within the context of real Hilbert spaces. Finally, we support these results with the help of numerical examples in both the finite and infinite dimensional spaces. As a result of this work, a new direction will be opened in studying split problems.

1. Introduction

Let E_1 and E_2 be two real Hilbert spaces graced with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$ for the entirety of this work, unless something other is specified. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two non-empty closed and convex subsets of E_1 and E_2 respectively. Let us assume that the symbols $\rightharpoonup$ and $\rightarrow$ represent the weak and strong convergence of a sequence, respectively.

In an excellent paper Censor et al. [8] presented the split variational inequality problem (SVIP), which may be rephrased as follows: identify a solution to variational inequality (VI) such that its image, under a bounded linear operator, solves another (VI). The (SVIP) is defied as follows: find $s \in \mathcal{K}_1$ in such a way that

$$\langle f(s), s - w \rangle \leq 0, \quad \forall w \in \mathcal{K}_1, \tag{1}$$

Received October 23, 2025.
2020 *Mathematics Subject Classification.* 47H10, 47J25, 47H09.
Key words and phrases. Equilibrium problem, split problem, variational inclusion, variational inequality.
https://doi.org/10.12697/ACUTM.2025.29.17
Corresponding author: Mohammad Dilshad

and $u = As \in \mathcal{K}_2$, so that

$$u \quad \text{solves} \quad \langle g(u), u - v\rangle \leq 0, \quad \forall v \in \mathcal{K}_2, \tag{2}$$

where $f : E_1 \to E_1$ and $g : E_2 \to E_2$ are two operators and $A : E_1 \to E_2$ is a bounded linear operator. Problem (1) is classical variational inequality initially introduced and studied by Stampacchia [16, 17]. It includes the the split feasibility problem [3], split zero problem [5] and split minimization problemn [3] as special cases.

Inspired by the work of Censor et al. [8], Moudafi [20] introduced the following split monotone variational inclusion problem (SMVIP): let $B_i : E_i \to E_i$ for $i = 1, 2$ be two multivalued mappings and let $f : E_1 \to E_1$ and $g : E_2 \to E_2$ be two mappings, then the (SMVIP) is to find $s \in E_1$ in such a way that

$$0 \in f(s) + B_1(s) \tag{3}$$

and $v = As \in E_2$, so that

$$v \quad \text{solves} \quad 0 \in g(v) + B_2(v). \tag{4}$$

The split monotone variational inclusion problem (SMVIP) is a fundamental problem in optimization theory, it can be applied to solve problems in many areas of science and applied science, engineering, economics, and medicine such as image processing, machine learning, and modeling intensity-modulated radiation theory treatment planning [4, 11, 7].

When we look separately, problem (3) is the classical monotone variational inclusion problem (MVIP). Later on (SVIP) and (SMVIP) have been studied extensively, see for example [1, 14, 13, 22, 24] and references therein. The (MVIP) has been the subject of extensive research in a wide variety of topics, including mathematical programming, variational inequalities, complementarity problems, optimal control, game theory, finance, economics, and many more.

On the other hand, Moudafi [20] introduced the following split equilibrium problem (SEP): let $F_1 : \mathcal{K}_1 \times \mathcal{K}_1 \to \mathbb{R}$ and $F_2 : \mathcal{K}_2 \times \mathcal{K}_2 \to \mathbb{R}$ be two bi-functions and $A : E_1 \to E_2$ be a bounded linear operator, then in the (SEP) one needs to find $s \in \mathcal{K}_1$ in such a way that

$$F_1(s, w) \geq 0 \quad \forall\, w \in \mathcal{K}_1, \tag{5}$$

and $u = As \in \mathcal{K}_2$, so that

$$u \quad \text{solves} \quad F_2(u, v) \geq 0 \quad \forall\, v \in \mathcal{K}_2. \tag{6}$$

It is well known that problem (5) is the classical equilibrium problem (EP) initially studied by Blum and Oettli [2]. Numerous problems in physics, optimization, and economics are reduced to find a solution of the equilibrium problem, see [19].

As we have seen in all the split problems (1)–(2),(3)–(4) and (5)–(6), the image, under a bounded linear operator, of the solution of problems (VI), (MVIP) and (EP) in one space solves the same problem in another space. Here a natural question arises.

Question. *What will happen if the image, under a bounded linear operator, of the solution of a problem (VI), (MVIP) or (EP) in one space solves a different problem (VI), (MVIP) or (EP) in another space?*

In this article, we answer this question. We define and investigate the generalized split problems in which the solution to one problem from (VI), (MVIP) or (EP) in one space solves another problem from (VI), (MVIP) or (EP) in another space under a bounded linear operator. These generalized split problems can be established as a generalization for finding the common solutions of two distinct problems. Also, we provide generalized iterative techniques and prove the weak convergence theorems to solve the generalized split problems under suitable assumptions. Finally, we provide a numerical justification for these findings. A new path has been opened up in the research on split problems as a result of this work.

The format of this document is as follows. We give some preliminary information in Section 2. We present generalised split problems in Section 3. In section 4, we first describe our approach to solve generalised split problems and present convergence theorems. Lastly, we justify these results numerically.

2. Preliminaries

To support our main results, we provide some auxiliary definitions, helpful lemmas, and fundamental presumptions in this section.

Definition 1 ([23, 25])**.** A mapping $T : E_1 \to E_1$ is said to be
(i) *κ-Lipschitz continuous* if for all $x_1, x_2 \in E_1$ there exists a constant $\kappa \geq 0$ such that

$$\|T(x_1) - T(x_2)\| \leq \kappa \|x_1 - x_2\|,$$

(ii) *nonexpansive* if

$$\|T(x_1) - T(x_2)\| \leq \|x_1 - x_2\|, \quad \forall\, x_1, x_2 \in E_1, \tag{7}$$

(iii) *firmly nonexpansive* if

$$\|T(x_1)-T(x_2)\|^2+\|(I-T)(x_1)-(I-T)(x_2)\|^2 \leq \|x_1-x_2\|^2,\ \forall\, x_1, x_2 \in E_1,$$

(iv) *monotone* if

$$\langle T(x_1) - T(x_2), x_1 - x_2\rangle \geq 0, \quad \forall\, x_1, x_2 \in E_1,$$

(v) *σ-inverse strongly monotone* if for all $x_1, x_2 \in E_1$ there exists a constant $\sigma > 0$ such that

$$\langle x_1 - x_2, T(x_1) - T(x_2)\rangle \geq \sigma\|T(x_1) - T(x_2)\|^2,$$

(vi) *averaged* if

$$T = (1 - \delta)I + \delta S, \tag{8}$$

where $\delta \in (0,\, 1)$ and $S : E_1 \to E_1$ is a nonexpansive mapping.

It is known that, for all $(x_1, x_2) \in E_1 \times E_2$, the nonexpansive mapping $T : E_1 \to E_1$ satisfies the inequality

$$\langle(x_1-T(x_2))-(x_2-T(x_2)), T(x_2)-T(x_1)\rangle \ \leq\ \frac{1}{2}\|(T(x_1)-x_1))-(T(x_2)-x_2)\|^2.$$

Therefore, for all $(x_1, x_2) \in E_1 \times\ Fix\,(T)$, we get

$$\langle x_1 - T(x_1), x_2 - T(x_1)\rangle \leq \frac{1}{2}\|(T(x_1) - x_1)\|^2. \tag{9}$$

Remark 1. We note that

(i) if T is σ-inverse strongly monotone (ism), then T is $\frac{1}{\sigma}$-Lipschitz continuous;

(ii) T is averaged if and only if its complement $I - T$ is σ-ism for some $\sigma > 1/2$.

A mapping $P_{\mathcal{K}_1}$ is called a *metric projection* of E_1 onto $\mathcal{K}_1$ if, for every $x_1 \in E_1$, there exists a unique nearest point in $\mathcal{K}_1$ indicated by $P_{\mathcal{K}_1}x_1$ such that

$$\|x_1 - P_{\mathcal{K}_1}x_1\| \leq \|x_1 - x_2\|, \quad \forall\, x_2 \in \mathcal{K}_1.$$

In addition, $P_{\mathcal{K}_1}$ is firmly nonexpansive and satisfying

$$\langle x_1 - P_{\mathcal{K}_1}x_1, x_2 - P_{\mathcal{K}_1}x_1\rangle \leq 0, \quad \forall\, x_1 \in E_1, x_2 \in \mathcal{K}_1.$$

Lemma 1 ([15])**.** *The metric projection $P_{\mathcal{K}_1}$, has the following properties:*

(i) $P_{\mathcal{K}_1}x_1 \in \mathcal{K}_1, \quad \forall\, x_1 \in E_1$;
(ii) $\|x_1 - x_2\|^2 \geq \|x_1 - P_{\mathcal{K}_1}x_1\|^2 + \|x_2 - P_{\mathcal{K}_1}x_1\|^2, \quad \forall\, x_1 \in E_1$ *and* $x_2 \in \mathcal{K}_1$;
(iii) $\langle P_{\mathcal{K}_1}x_1 - P_{\mathcal{K}_1}x_2, x_1 - x_2\rangle \geq \|P_{\mathcal{K}_1}x_1 - P_{\mathcal{K}_1}x_2\|^2, \quad \forall\, x_1, x_2 \in E_1$.

Lemma 2 ([20])**.** *Let* $M : E_1 \to E_1$ *be an averaged mapping and suppose that the set of all fixed points of* M *is non-empty. Then the sequence* $\{M^n(v_0)\}$ *converges weakly to a fixed point of* M*, where* v_0 *is any initial guess.*

Lemma 3 ([6])**.** *In a real Hilbert space* E_1*, the following inequalities hold:*

(i) $\|x_1 + x_2\|^2 \geq \|x_1\|^2 + 2\langle x_2, x_1 + x_2\rangle \quad \forall\, x_1, x_2 \in E_1$;
(ii) $\|\delta x_1 + (1-\delta)x_2\|^2 = \delta\|x_1\|^2 + (1-\delta)\|x_2\|^2 - \delta(1-\delta)\|x_1 - x_2\|^2, \quad \forall\, x_1, x_2 \in E_1$ *and* $\delta \in [0,\, 1]$;
(iii) $2\langle x_1, x_2\rangle = \|x_1 + x_2\|^2 - \|x_1\|^2 - \|x_2\|^2, \quad \forall\, x_1, x_2 \in E_1$.

Lemma 4 ([21])**.** *Let* $p \in \mathcal{K}_1$ *be a solution of (5) and* x_{n+1} *be the approximate solution obtained from*

$$x_{n+1} = F(x_{n+1}, q) + \frac{1}{r_n}\langle x_{n+1} - x_n, q - x_{n+1}\rangle \quad \forall\, q \in \mathcal{K}_1,$$

where $x_0 \in \mathcal{K}_1$*. If* $F(\cdot,\cdot)$ *is pseudomonotone, then*

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - x_n\|^2.$$

Assumption 1 ([18])**.** Let $F : \mathcal{K}_1 \times \mathcal{K}_1 \to \mathbb{R}$ be a bi-function satisfying the subsequent conditions:
(i) $F(x_1, x_1) \geq 0, \ \forall\, x_1 \in \mathcal{K}_1$;
(ii) F is monotone, i.e., $F(x_1, x_2) + F(x_2, x_1) \leq 0, \ \forall\, x_1, x_2 \in \mathcal{K}_1$;
(iii) F is upper semi continuous, i.e., for each $x_1, x_2, x_3 \in \mathcal{K}_1$,

$$\lim_{t\to 0}\sup F\,(tx_3 + (1-t)x_1, x_2) \leq F(x_1, x_2); \tag{10}$$

(iv) for each fixed $x_1 \in \mathcal{K}_1$, the function $x_2 \mapsto F(x_1, x_2)$ is convex and lower semi continuous.

Lemma 5 ([2])**.** *Assume that a bi-function* $F : \mathcal{K}_1 \times \mathcal{K}_1 \to \mathbb{R}$ *satisfies Assumption* 1*. Then, for fixed* $r > 0$ *and* $x_1 \in E_1$*, there exists* $x_3 \in \mathcal{K}_1$ *such that*

$$F(x_2, x_1) + \frac{1}{r}\langle x_2 - x_1, x_1 - x_3\rangle \geq 0, \quad \forall\, x_2 \in \mathcal{K}_1. \tag{11}$$

Lemma 6 ([12]). *Assume that a bi-function $F : \mathcal{K}_1 \times \mathcal{K}_1 \to \mathbb{R}$ satisfies Assumption 1. For $r > 0$ and $x_1 \in E_1$, define a mapping $T_r^F : E_1 \to \mathcal{K}_1$ as follows:*

$$T_r^F(x_1) = \left\{ x_3 \in \mathcal{K}_1 : F(x_3, x_2) + \frac{1}{r}\langle x_2 - x_3, x_3 - x_1 \rangle \geq 0, \forall\, x_2 \in \mathcal{K}_1 \right\}. \tag{12}$$

Then the following are valid.

(i) *T_r^F is non-empty and single valued.*

(ii) *T_r^F is firmly nonexpansive, i.e.,*

$$\|T_r^F(x_1) - T_r^F(x_2)\|^2 \leq \langle T_r^F(x_1) - T_r^F(x_2), x_1 - x_2 \rangle \quad \forall\, x_1, x_2 \in E_1.$$

(iii) *$Fix(T_r^F) = EP(F)$.*

(iv) *$EP(F)$ is closed and convex.*

Lemma 7 ([9]). *Let $F : \mathcal{K}_1 \times \mathcal{K}_1 \to \mathbb{R}$ be a nonlinear bi-function satisfying Assumption 1 and let T_r^F be defined as above in Lemma 6. If $r > 0$, $x_1, x_2 \in E_1$ and $r_1, r_2 > 0$, then*

$$\|T_{r_2}^F(x_1) - T_{r_1}^F(x_1)\| \leq \|x_1 - x_2\| + \left|\frac{r_2 - r_1}{r_2}\right| \|T_{r_2}^F(x_1) - x_1\|.$$

Let $B : E_1 \to 2^{E_1}$ be a multivalued mapping and let J_λ^B, defined as $J_\lambda^B = (I + \lambda B)^{-1}$, be a backward operator of B for $\lambda > 0$. It is well known that J_λ^B is a single value firmly nonexpansive mapping for all $\lambda > 0$. Also $\mathrm{Dom}(J_\lambda^B) = E_1$ where Dom stands for domain.

Definition 2. Let $B : E_1 \to 2^{E_1}$ be a multivalued mapping with graph $G(B)$. Then B is said to be *monotone* if, for all $x, y \in E_1, x_0 \in Bx$ and $y_0 \in By$,

$$\langle x - y, x_0 - y_0 \rangle \geq 0.$$

If the graph of any other monotone mapping does not properly contain the graph of the monotone mapping B then the mapping B is referred as the *maximal monotone* mapping.

Lemma 8 (Demiclosedness Principle [15]). *Let $\mathcal{K}_1$ be a non empty closed and convex subset of a real Hilbert space E_1 and $T : \mathcal{K}_1 \to \mathcal{K}_1$ be a nonexpansive mapping. If $\{x_n\}$ is a sequence in $\mathcal{K}_1$ which converges weakly to $x \in \mathcal{K}_1$ and $\{(I - T)x_n\}$ strongly converges to $y \in \mathcal{K}_1$, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in \mathrm{Fix}(T)$.*

Definition 3. Let E_1 be a Hilbert space and let $\mathcal{K}_1$ be a non-empty closed and convex subset of E_1. If, for every $p \in \mathcal{K}_1$,

$$\|x_{n+1} - p\| \leq \|x_n - p\| \quad \forall\, n \geq 0,$$

then the sequence $\{x_n\}$ is called *Fejér-monotone.*

Lemma 9 ([8]). *Let $\mathcal{K}_1 \subseteq E_1$ be a non-empty closed and convex subset and $g : E_1 \to E_1$ be an σ-cocoercive operator on E_1. If $\lambda \in [0, 2\sigma]$, then*

(i) *the operator $P_{\mathcal{K}_1}(I - \lambda g)$ is nonexpansive on $\mathcal{K}_1$.*
In addition, if for all $x_2 \in VI(\mathcal{K}_1, g)$,

$$\langle g(x_1), P_{\mathcal{K}_1}(I - \lambda g)(x_1) - x_2 \rangle \geq 0, \quad \forall\, x_1 \in E_1,$$

(ii) $\langle P_{\mathcal{K}_1}(I - \lambda g)(x_1) - x_1, P_{\mathcal{K}_1}(I - \lambda g)(x_1) - p \rangle \leq 0$;
iii) $\|P_{\mathcal{K}_1}(I - \lambda g)(x_1) - p\|^2 \leq \|x_1 - p\|^2 - \|P_{\mathcal{K}_1}(I - \lambda g)(x_1) - x_1\|^2$.

3. Some generalized split problems

In this section, we introduce generalized split problems in real Hilbert spaces.

Problem 1. Let $g : \mathcal{K}_1 \to E_1$ be a mapping, $A : E_1 \to E_2$ be a bounded linear operator and let $F : \mathcal{K}_2 \times \mathcal{K}_2 \to \mathbb{R}$ be a bi-function satisfying Assumption 1. Then the *generalized split problem* (GSP-1) is defined as follows.

$$\text{Find } s \in \mathcal{K}_1 \quad \text{so that} \quad \langle g(s), s - w \rangle \leq 0, \quad \forall w \in \mathcal{K}_1, \tag{13}$$

and

$$u = As \in \mathcal{K}_2 \quad \text{solves} \quad F(u, v) \geq 0, \quad \forall v \in \mathcal{K}_2. \tag{14}$$

The set of all solution of (GSP-1) is denoted by $\Omega_1 := \{s \in \mathcal{K}_1 : s \in VI(\mathcal{K}_1, g) \text{ and } As \in EP(F)\}$ where $VI(\mathcal{K}_1, g)$ is the solution set of problem (13) and $EP(F)$ is the solution set of problem (14).

Remark 2. Let $g : \mathcal{K}_2 \to E_2$ be a mapping, $A : E_1 \to E_2$ be a bounded linear operator and let $F : \mathcal{K}_1 \times \mathcal{K}_1 \to \mathbb{R}$ be a bi-function satisfying Assumption 1. Then the generalized split problem 2 (GSP-2) is defined as follows.

$$\text{Find } s \in \mathcal{K}_1 \quad \text{so that} \quad F(s, w) \geq 0, \quad \forall w \in \mathcal{K}_1, \tag{15}$$

and

$$u = As \in \mathcal{K}_2 \quad \text{solves} \quad \langle g(u), u - v \rangle \leq 0, \quad \forall v \in \mathcal{K}_2. \tag{16}$$

The set of all solutions of (GSP-2) is denoted by $\Omega_2 := \{s \in \mathcal{K}_1 : s \in EP(F) \text{ and } As \in VIP(\mathcal{K}_2, g)\}$ where $EP(F)$ is the solution set of problem (15) and $VIP(\mathcal{K}_2, g)$ is the solution set of problem (16).

Problem 2. Let $g : \mathcal{K}_1 \to E_1$, $f : E_2 \to E_2$ be two mappings, $A : E_1 \to E_2$ be a bounded linear operator and let $B : E_2 \to 2^{E_2}$ be a multivalued mapping. Then the generalized split problem 3 (GSP-3) can be defined as follows.

$$\text{Find } s \in \mathcal{K}_1 \quad \text{so that} \quad \langle g(s), s - w \rangle \leq 0, \quad \forall w \in \mathcal{K}_1, \tag{17}$$

and

$$u = As \in E_2 \quad \text{solves} \quad 0 \in f(u) + B(u). \tag{18}$$

The set of all solution of (GSP-3) is denoted by $\Omega_3 := \{s \in \mathcal{K}_1 : s \in VI(E_1, g) \text{ and } As \in MVIP(B, f)\}$ where $VI(E_1, g)$ is the solution set of problem (17) and $MVIP(B, f)$ is the solution set of problem (18).

Remark 3. Let $f : E_1 \to E_1$, $g : E_2 \to E_2$ be two mappings, $A : E_1 \to E_2$ be a bounded linear operator and let $B : E_1 \to 2^{E_1}$ be a multivalued mapping. Then the generalized split problem 4 (GSP-4) can be defined as follows.

$$\text{Find } s \in E_1 \quad \text{so that} \quad 0 \in f(s) + B(s) \tag{19}$$

and

$$u = As \in E_2 \quad \text{solves} \quad \langle g(u), u - v \rangle \leq 0, \quad \forall v \in E_2. \tag{20}$$

The set of all solutions of (GSP-4) is denoted by $\Omega_4 := \{s \in E_1 : s \in MVIP(B, f) \text{ and } As \in VI(E_2, g)\}$ where $MVIP(B, f)$ is the solution set of problem (19) and $VI(E_2, g)$ is the solution set of problem (20).

Problem 3. Let $f : E_1 \to E_1$ be a mapping, $A : E_1 \to E_2$ be a bounded linear operator and let $B : E_1 \to E_1$ be a multivalued mapping and $F : \mathcal{K}_2 \times \mathcal{K}_2 \to \mathbb{R}$ be a bi-function satisfying Assumption 1. Then generalized split problem 5 (GSP-5) can be defined as follows.

$$\text{Find } s \in E_1 \quad \text{so that} \quad 0 \in f(s) + B(s) \tag{21}$$

and

$$u = As \in \mathcal{K}_2 \quad \text{solves} \quad F(u, v) \geq 0, \quad \forall v \in H \in \mathcal{K}_2. \tag{22}$$

The set of all solutions of (GSP-5) is denoted by $\Omega_5 := \{s \in E_1 : s \in MVIP(B, f) \text{ and } As \in EP(F)\}$ where $MVIP(B, f)$ is the solution set of problem (21) and $EP(F)$ is the solution set of problem (15).

Remark 4. Let $f : E_2 \to E_2$ be a mapping, $A : E_1 \to E_2$ be a bounded linear operator and let $B : E_2 \to E_2$ be a multivalued mapping and $F : \mathcal{K}_1 \times \mathcal{K}_1 \to \mathbb{R}$ be a bi-function satisfying Assumption 1. Then the generalized split problem 6 (GSP-6) can be defined as follows.

$$\text{Find } s \in \mathcal{K}_1 \quad \text{so that} \quad F(s, w) \geq 0, \quad \forall w \in \mathcal{K}_1, \tag{23}$$

and

$$u = As \in E_2 \quad \text{solves} \quad 0 \in f(u) + B(u). \tag{24}$$

The set of all solution of (GSP-6) is denoted by $\Omega_6 := \{s \in \mathcal{K}_1 : s \in EP(F) \text{ and } As \in MVIP(B, f)\}$ where $EP(F)$ is the solution set of problem (23) and $MVIP(B, f)$ is the solution set of problem (24).

4. Iterative algorithms

In this section, we present some iterative algorithms for generalized split problems and discuss their convergence analysis in the of Hilbert spaces.

Theorem 1. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two non-empty, closed and convex subsets of Hilbert spaces E_1 and E_2, respectively. Let $g : \mathcal{K}_1 \to E_1$ be a σ-inverse strongly monotone operator, $A : E_1 \to E_2$ be a bounded linear operator and $F : \mathcal{K}_2 \times \mathcal{K}_2 \to \mathbb{R}$ be a bi-function satisfying Assumption* 1 *such that F is upper semicontinuous in the first argument. Assume that Ω_1 is non-empty and that, for all $u^* \in VI(\mathcal{K}_1, g)$,*

$$\langle g(u), P_{\mathcal{K}_1}(I - \lambda g)(u) - u^* \rangle \geq 0, \quad \forall\, u \in \mathcal{K}_1, \tag{25}$$

where the mapping $P_{\mathcal{K}_1}$ is referred as metric projection of E_1 onto $\mathcal{K}_1$. For $\lambda > 0$ and an arbitrarily given $u_0 \in \mathcal{K}_1$, let the iterative sequence $\{u_n\}$ be generated by

$$u_{n+1} = P_{\mathcal{K}_1}(I - \lambda g)(u_n + \gamma A^*(T^F_{r_n} - I)Au_n), \quad \forall\, n \geq 0, \tag{26}$$

where $r_n \subset (0, \infty)$, $\liminf_{n\to\infty} r_n > 0$, $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A and $A^* : E_2 \to E_1$ is the adjoint operator of A. Let $T^F_{r_n} : E_1 \to \mathcal{K}_1$ be a projection mapping. Then the sequence $\{u_n\}$ converges weakly to $u^* \in \Omega_1$ where Ω_1 is the set of all solutions of GSP-1.

Proof. Let $p \in \Omega_1$, i.e., $p \in VI(\mathcal{K}_1, g)$ and $p \in EP(F)$. We have $p = P_{\mathcal{K}_1}(I - \lambda g)p$ and $p = T^F_{r_n}(Ap)$. By Lemma 9 (i) and from (26), we have

$$\begin{aligned}
\|u_{n+1} - p\|^2 &= \|P_{\mathcal{K}_1}(I - \lambda g)(u_n + \gamma A^*(T^F_{r_n} - I)Au_n) - p\|^2 \\
&= \|P_{\mathcal{K}_1}(I - \lambda g)(u_n + \gamma A^*(T^F_{r_n} - I)Au_n) - P_{\mathcal{K}_1}(I - \lambda g)p\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq \|u_n + \gamma A^*(T^F_{r_n} - I)Au_n - p\|^2 \\
&\leq \|u_n - p\|^2 + \gamma^2\|A^*(T^F_{r_n} - I)Au_n\|^2 \\
&\quad + 2\gamma\langle u_n - p, A^*(T^F_{r_n} - I)Au_n\rangle \\
&\leq \|u_n - p\|^2 + \gamma^2\langle (T^F_{r_n} - I)Au_n, AA^*(T^F_{r_n} - I)Au_n\rangle \\
&\quad + 2\gamma\langle u_n - p, A^*(T^F_{r_n} - I)Au_n\rangle. \qquad (27)
\end{aligned}$$

Now we have

$$\begin{aligned}
\gamma^2\langle (T^F_{r_n} - I)Au_n, AA^*(T^F_{r_n} - I)Au_n\rangle &\leq L\gamma^2\langle (T^F_{r_n} - I)Au_n, (T^F_{r_n} - I)Au_n\rangle \\
&= L\gamma^2\|(T^F_{r_n} - I)Au_n\|^2. \qquad (28)
\end{aligned}$$

Denoting $\Lambda := 2\gamma\langle u_n - p, A^*(T^F_{r_n} - I)Au_n\rangle$ and using (9), we have

$$\begin{aligned}
\Lambda &= 2\gamma\langle u_n - p, A^*(T^F_{r_n} - I)Au_n\rangle \\
&= 2\gamma\langle Au_n - Ap, (T^F_{r_n} - I)Au_n\rangle \\
&= 2\gamma\langle Au_n - Ap + (T^F_{r_n} - I)Au_n - (T^F_{r_n} - I)Au_n, (T^F_{r_n} - I)Au_n\rangle \\
&= 2\gamma\left\{\langle T^F_{r_n}Au_n - Ap, (T^F_{r_n} - I)Au_n\rangle - \|(T^F_{r_n} - I)Au_n\|^2\right\} \\
&\leq 2\gamma\left\{\frac{1}{2}\|(T^F_{r_n} - I)Au_n\|^2 - \|(T^F_{r_n} - I)Au_n\|^2\right\} \\
&\leq -\gamma\|(T^F_{r_n} - I)Au_n\|^2. \qquad (29)
\end{aligned}$$

Using (27),(28) and (29), we obtain

$$\|u_{n+1} - p\|^2 \leq \|u_n - p\|^2 + \gamma(L\gamma - 1)\|(T^F_{r_n} - I)Au_n\|^2. \qquad (30)$$

From the definition of γ, we have

$$\|u_{n+1} - p\|^2 \leq \|u_n - p\|^2. \qquad (31)$$

Therefore, the sequence $\{u_n\}$ is Fejér-monotone. It follows from (31) that the sequence $\{\|u_n - p\|\}_{n=0}^{\infty}$ is monotonically decreasing and therefore, convergent, which shows by (30), that

$$\lim_{n\to\infty} \|(T^F_{r_n} - I)Au_n\| = 0. \qquad (32)$$

Otherwise, if $T^F_{r_n}Ap \neq Ap$ for some $r > 0$, Fejér-monotonicity implies that the sequence $\{u_n\}$ is bounded, so it has a weakly convergent subsequence $\{u_{n_j}\}$ such that $u_{n_j} \rightharpoonup u^*$, then by Opial's condition, Lemma 7 and (27), we obtain

$$\lim_{j\to\infty} \inf\|Au_{n_j} - Au^*\|$$

$$
\begin{aligned}
&< \liminf_{j\to\infty} \|Au_{n_j} - T^F_{r_{n_j}} Au^*\| \\
&\le \liminf_{j\to\infty} \left\{ \|Au_{n_j} - T^F_{r_{n_j}} Au_{n_j}\| + \|T^F_{r_{n_j}} Au_{n_j} - T^F_{r_{n_j}} Au^*\| \right\} \\
&= \lim_{j\to\infty} \|T^F_{r_{n_j}} Au_{n_j} - T^F_{r_{n_j}} Au^*\| \\
&\le \lim_{j\to\infty} \left(\|Au_{n_j} - Au^*\| + \left|\frac{r_{n_j} - r}{r}\right| \|T^F_{r_{n_j}} Au_{n_j} - Au_{n_j}\| \right) \\
&= \lim_{j\to\infty} \|Au_{n_j} - Au^*\|,
\end{aligned}
$$

which is a contradiction. Therefore, $T^F_{r_n} Au^* = Au^*$ for some $r > 0$, i.e., $Au^* \in EP(F)$.
Denote

$$v_n := u_n + \gamma A^*(T^F_{r_n} - I)Au_n. \tag{33}$$

Then

$$v_{n_j} := u_{n_j} + \gamma A^*(T^F_{r_{n_j}} - I)Au_{n_j}. \tag{34}$$

Since $u_{n_j} \rightharpoonup u^*$, (32) implies that $v_{n_j} \rightharpoonup u^*$ too. Now it remains to show that $u^* \in VI(\mathcal{K}_1, g)$. Assume, on contrary, that $u^* \notin VI(\mathcal{K}_1, g)$, i.e., $P_{\mathcal{K}_1}(I - \lambda g)u^* \neq u^*$. Since $P_{\mathcal{K}_1}(I - \lambda g)$ is nonexpansive, $P_{\mathcal{K}_1}(I - \lambda g) - I$ is demiclosed at 0. The contrary assumption must lead to

$$\lim_{j\to\infty} \|P_{\mathcal{K}_1}(I - \lambda g)v_{n_j} - v_{n_j}\| \neq 0. \tag{35}$$

Therefore, there exists an $\epsilon > 0$ and a subsequence $\{v_{n_{j_s}}\}$ of $\{v_{n_j}\}$ such that

$$\lim_{j\to\infty} \|P_{\mathcal{K}_1}(I - \lambda g)v_{n_{j_s}} - v_{n_{j_s}}\| > \epsilon \quad \forall\, s \ge 0. \tag{36}$$

Condition (25) justifies the use of Lemma 9. Therefore, we have

$$
\begin{aligned}
&\|P_{\mathcal{K}_1}(I - \lambda g)v_{n_{j_s}} - P_{\mathcal{K}_1}(I - \lambda g)p\|^2 = \|P_{\mathcal{K}_1}(I - \lambda g)v_{n_{j_s}} - p\|^2 \\
&\le \|v_{n_{j_s}} - p\|^2 - \|P_{\mathcal{K}_1}(I - \lambda g)v_{n_{j_s}} - v_{n_{j_s}}\|^2 \\
&< \|v_{n_{j_s}} - p\|^2 - \epsilon^2.
\end{aligned} \tag{37}
$$

By arguments similar to those above we have

$$\|v_n - p\| = \|u_n + \gamma A^*(T^F_{r_n} - I)Au_n - p\| \le \|u_n - p\|. \tag{38}$$

Since $P_{\mathcal{K}_1}(I - \lambda g)$ is nonexpansive,

$$\|u_{n+1} - p\| = \|P_{\mathcal{K}_1}(I - \lambda g)v_n - p\| \le \|v_n - p\|. \tag{39}$$

Combining (38) and (39), we get

$$\|u_{n+1} - p\| \leq \|v_n - p\| \leq \|u_n - p\|, \tag{40}$$

which means that the sequence $\{u_1, v_1, u_2, v_2, \cdots\}$ is Fejér-monotone w.r.t. Ω_1. Since $u_{n_{j_{s+1}}} = P_{\mathcal{K}_1}(I - \lambda g)v_{n_{j_s}}$, we obtain

$$\|v_{n_{j_{s+1}}} - p\|^2 \leq \|v_{n_{j_s}} - p\|^2. \tag{41}$$

Hence the sequence $\{v_{n_{j_s}}\}$ is also Fejér-monotone w.r.t. Ω_1. Now (37) and (40) imply that

$$\|v_{n_{j_{s+1}}} - p\|^2 \leq \|v_{n_{j_s}} - p\|^2 - \epsilon^2, \tag{42}$$

which leads to a contradiction. Therefore, $u^* \in VI(\mathcal{K}_1, g)$ and finally $u^* \in \Omega_1$. Since the subsequence $\{u_{n_j}\}$ was arbitrary, we get that $u_n \rightharpoonup u^*$. □

Theorem 2. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two non-empty, closed and convex subsets of Hilbert spaces E_1 and E_2, respectively. Let $g : \mathcal{K}_2 \to E_2$ be a σ-inverse strongly monotone operator, $A : E_1 \to E_2$ be a bounded linear operator and $F : \mathcal{K}_1 \times \mathcal{K}_1 \to \mathbb{R}$ be a bi-function satisfying Assumption 1. Assume that Ω_2 is non-empty. For $\lambda > 0$ and an arbitrarily given $u_0 \in \mathcal{K}_1$, let the iterative sequence $\{u_n\}$ be generated by*

$$u_{n+1} = T^F_{r_n}(u_n + \gamma A^*(P_{\mathcal{K}_2}(I - \lambda g) - I)Au_n), \quad \forall\, n \geq 0, \tag{43}$$

where $r_n \subset (0, \infty)$, $\lim_{n\to\infty} \inf r_n > 0$, $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A and $A^* : E_2 \to E_1$ is the adjoint operator of A. The projection mappings are $P_{\mathcal{K}_2} : E_2 \to \mathcal{K}_2$ and $T^F_{r_n} : E_1 \to \mathcal{K}_1$. Then the sequence $\{u_n\}$ converges weakly to $u^* \in \Omega_2$ where Ω_2 is the set of all solutions of GSP-2.

Proof. Let $p \in \Omega_2$. Then $p \in EP(F)$ and $p \in VI(\mathcal{K}_2, g)$. Therefore, by using similar arguments as in Theorem 1, we obtain from (43) that

$$\|u_{n+1} - p\|^2 \leq \|u_n - p\|^2 + \gamma(L\gamma - 1)\|(P_{\mathcal{K}_2}(I - \lambda g) - I)Au_n\|^2. \tag{44}$$

From the definition of γ, we have

$$\|u_{n+1} - p\|^2 \leq \|u_n - p\|^2. \tag{45}$$

Therefore, the sequence $\{u_n\}$ is Fejér-monotone. It follows from (45) that the sequence $\{\|u_n - p\|\}_{n=0}^{\infty}$ is monotonically decreasing and therefore, convergent, which shows by (44), that

$$\lim_{n\to\infty} \|(P_{\mathcal{K}_2}(I - \lambda g) - I)Au_n\| = 0. \tag{46}$$

Fejér-monotonicity implies that the sequence $\{u_n\}$ is bounded, so it has a weakly convergent subsequence $\{u_{n_j}\}$ such that $u_{n_j} \rightharpoonup u^*$. Since $P_{\mathcal{K}_2}(I-\lambda g)$ is nonexpansive by Lemma 9, applying the demiclosedness of $P_{\mathcal{K}_2}(I-\lambda g) - I$ at 0 to (46), we obtain

$$P_{\mathcal{K}_2}(I-\lambda g)Au^* = Au^*, \tag{47}$$

which means that $Au^* \in VI(\mathcal{K}_2, g)$. It remains to be shown that $u^* \in EP(F)$. Since $u_{n+1} = T_{r_n}^F(v_n)$ where $v_n := u_n + \gamma A^*(T_{r_n}^F - I)Au_n$, we have

$$F(u_{n+1}, v) + \frac{1}{r_n}\langle u_{n+1} - v_n, v - u_{n+1}\rangle \geq 0, \quad \forall\, v \in \mathcal{K}_1. \tag{48}$$

Applying Lemma 4 to (43), we obtain

$$\|u_{n+1} - p\|^2 \leq \|v_n - p\|^2 - \|u_{n+1} - v_n\|^2. \tag{49}$$

Using (46) and (49), we obtain

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{50}$$

Using monotonicity of F, we have

$$\frac{1}{r_n}\langle u_{n+1} - v_n, v - u_{n+1}\rangle \geq -F(u_{n+1}, v), \quad \forall\, v \in \mathcal{K}_1.$$

Hence

$$\frac{1}{r_{n_j}}\langle u_{n_j+1} - v_{n_j}, v - u_{n_j+1}\rangle \geq -F(u_{n_j+1}, v), \quad \forall\, v \in \mathcal{K}_1.$$

Let $v_t = (1-t)w + tv$ for all $t \in (0,1]$. Since $v \in \mathcal{K}_1$ and $w \in \mathcal{K}_1$, we get $v_t \in \mathcal{K}_1$. Therefore, for all $v_t \in \mathcal{K}_1$, we have

$$0 \leq F(u_{n_j+1}, v_t) + \left\langle u_{n_j+1} - v_t, \frac{u_{n_j+1} - u_{n_j}}{r_{n_j}} + \gamma A^*\left(\frac{(P_{\mathcal{K}_2}(I-\lambda g) - I)Au_{n_j}}{r_{n_j}}\right)\right\rangle.$$

Since A^* is a bounded linear operator, from (46), (50) and $\lim_{n\to\infty} \inf r_n > 0$ we obtain by taking $t \to \infty$,

$$F(v, u^*) \geq 0, \quad \forall\, v \in \mathcal{K}_1.$$

This implies that $u^* \in EP(F)$. Therefore, $u^* \in \Omega_5$. Since the subsequence $\{u_{n_j}\}$ was arbitrary, we get that $u_n \rightharpoonup u^*$. □

Theorem 3. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two non-empty, closed and convex subsets of Hilbert spaces E_1 and E_2 respectively. Let $A : E_1 \to E_2$ be a bounded linear operator. Let $g : E_1 \to E_1$ and $f : E_2 \to E_2$ be two σ_1- and σ_2-inverse strongly monotone operators, respectively, and*

$B : E_2 \to 2^{E_2}$ be a maximal monotone operator. Assume that Ω_3 is non-empty and the mapping $J^B_\lambda(I-\lambda f)$ is nonexpansive. For an arbitrarily given $u_0 \in \mathcal{K}_1$, let the iterative sequence $\{u_n\}$ be generated by

$$u_{n+1} = P_{\mathcal{K}_1}(I-\lambda g)(u_n + \gamma A^*(J^B_\lambda(I-\lambda f) - I)Au_n), \quad \forall\, n \geq 0, \tag{51}$$

where $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A and $A^* : E_2 \to E_1$ is the adjoint operator of A and $\lambda \in (0, 2\sigma)$ where $\sigma := \min\{\sigma_1, \sigma_2\}$. The projection mappings are $P_{\mathcal{K}_1} : E_1 \to \mathcal{K}_1$ and $J^B_\lambda = (I + \lambda B)^{-1}$. Then the sequence $\{u_n\}$ converges weakly to $u^* \in \Omega_2$ where Ω_2 is the set of all solutions of GSP-3.

Proof. Let $p \in \Omega_3$. Then $p \in VI(\mathcal{K}_1, g)$ and $p \in MVIP(B, f)$. Therefore, by using similar arguments as in Theorem 1, we obtain from (51) that

$$\|u_{n+1} - p\|^2 \leq \|u_n - p\|^2 + \gamma(L\gamma - 1)\|(J^B_\lambda(I-\lambda f) - I)Au_n\|^2. \tag{52}$$

From the definition of γ, we have

$$\|u_{n+1} - p\|^2 \leq \|u_n - p\|^2. \tag{53}$$

Therefore, the sequence $\{u_n\}$ is Fejér-monotone. It follows from (53) that the sequence $\{\|u_n - p\|\}_{n=0}^{\infty}$ is monotonically decreasing and therefore, convergent, which shows by (52), that

$$\lim_{n\to\infty} \|(J^B_\lambda(I-\lambda f) - I)Au_n\| = 0. \tag{54}$$

Fejér-monotonicity implies that the sequence $\{u_n\}$ is bounded, so it has a weakly convergent subsequence $\{u_{n_j}\}$ such that $u_{n_j} \rightharpoonup u^*$. Since $J^B_\lambda(I-\lambda f)$ is nonexpansive, applying the demiclosedness of $J^B_\lambda(I-\lambda f) - I$ at 0 to (54), we obtain

$$J^B_\lambda(I-\lambda f) - I)Au^* = Au^*, \tag{55}$$

which means that $Au^* \in MVIP(B, f)$. It remains to be shown that $u^* \in VI(\mathcal{K}_1, g)$. By the similar arguments as in Theorem 1, we can easily conclude that $u^* \in VI(\mathcal{K}_1, g)$. Therefore, $u^* \in \Omega_2$. Since the subsequence $\{u_{n_j}\}$ was arbitrary, we get that $u_n \rightharpoonup u^*$. □

Theorem 4. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two non-empty, closed and convex subsets of Hilbert spaces E_1 and E_2 respectively. Let $A : E_1 \to E_2$ be a bounded linear operator. Let $f : E_1 \to E_1$ and $g : E_2 \to E_2$ be two σ_1- and σ_2-inverse strongly monotone operators, respectively, and $B : E_1 \to 2^{E_1}$ be a maximal monotone operator. Assume that Ω_4 is* *non-empty and the operator* $J_\lambda^B(I-\lambda f)$ *is averaged. For an arbitrarily given* $u_0 \in \mathcal{K}_1$*, let the iterative sequence* $\{u_n\}$ *be generated by*

$$u_{n+1} = J_\lambda^B(I-\lambda f)(u_n + \gamma A^*(P_{\mathcal{K}_2}(I-\lambda g) - I)Au_n), \quad \forall\, n \geq 0, \tag{56}$$

where $\gamma \in (0, 1/L)$*,* L *is the spectral radius of the operator* A^*A *and* $A^* : E_2 \to E_1$ *is the adjoint operator of* A *and* $\lambda \in (0, 2\sigma)$ *where* $\sigma := \min\{\sigma_1, \sigma_2\}$*. The projection mapping is* $P_{\mathcal{K}_2} : E_2 \to \mathcal{K}_2$ *and* $J_\lambda^B = (I+\lambda B)^{-1}$*. Then the sequence* $\{u_n\}$ *converges weakly to* $u^* \in \Omega_3$ *where* Ω_3 *is the set of all solutions of SMVIPVIP.*

Proof. Let $p \in \Omega_4$. Then $p \in VI(\mathcal{K}_2, g)$ and $p \in MVIP(B, f)$. Therefore, by using similar arguments as in Theorem 1, we obtain from (56) that

$$\|u_{n+1} - p\|^2 \leq \|u_n - p\|^2 + \gamma(L\gamma - 1)\|(P_{\mathcal{K}_2}(I-\lambda g) - I)Au_n\|^2. \tag{57}$$

From the definition of γ, we have

$$\|u_{n+1} - p\|^2 \leq \|u_n - p\|^2. \tag{58}$$

Therefore, the sequence $\{u_n\}$ is Fejér-monotone. It follows from (58) that the sequence $\{\|u_n - p\|\}_{n=0}^{\infty}$ is monotonically decreasing and therefore, convergent, which shows by (57), that

$$\lim_{n\to\infty} \|(P_{\mathcal{K}_2}(I-\lambda g) - I)Au_n\| = 0. \tag{59}$$

Fejér-monotonicity implies that the sequence $\{u_n\}$ is bounded, so it has a weakly convergent subsequence $\{u_{n_j}\}$ such that $u_{n_j} \rightharpoonup u^*$. Since $P_{\mathcal{K}_2}(I-\lambda g)$ is nonexpansive by Lemma 9 (i), applying demiclosedness of $P_{\mathcal{K}_2}(I-\lambda g) - I$ at 0 to (59), we obtain

$$P_{\mathcal{K}_2}(I-\lambda g) - I)Au^* = Au^*, \tag{60}$$

which means that $Au^* \in VI(\mathcal{K}_2, g)$. It remains to be shown that $u^* \in MVIP(B, f)$. Indeed, by Remark 1 (ii), we have

$$\begin{aligned}
&\langle A^*(I - P_{\mathcal{K}_2}(I-\lambda g))Au - A^*(I - P_{\mathcal{K}_2}(I-\lambda g))Av, u - v\rangle \\
&= \langle (I - P_{\mathcal{K}_2}(I-\lambda g))Au - (I - P_{\mathcal{K}_2}(I-\lambda g))Av, Au - Av\rangle \\
&\geq \|(I - P_{\mathcal{K}_2}(I-\lambda g))Au - (I - P_{\mathcal{K}_2}(I-\lambda g))Av\|^2 \\
&\geq \frac{\sigma}{L}\|(I - P_{\mathcal{K}_2}(I-\lambda g))Au - (I - P_{\mathcal{K}_2}(I-\lambda g))Av\|^2.
\end{aligned}$$

Hence $\gamma A^*(I - P_{\mathcal{K}_2}(I-\lambda g))A$ is $\dfrac{\sigma}{L}$-ism, because of the condition $\gamma \in (0,\, 1/L)$, the complement $I - \gamma A^*(I - P_{\mathcal{K}_2}(I-\lambda g))A$ is averaged and therefore, $M = J_\lambda^B(I-\lambda f)(I - \gamma A^*(I - P_{\mathcal{K}_2}(I-\lambda g))A)$ is averaged.

Applying Lemma 2, we can conclude that the sequence $\{u_n\}$ converges weakly to a fixed point u^* of M which must be a fixed point of $J_\lambda^B(I - \lambda f)$. From the equality we must have $u^* \in MVIP(B, f)$. Therefore, $u^* \in \Omega_3$. Since the subsequence $\{u_{n_j}\}$ was arbitrary, we get that $u_n \rightharpoonup u^*$. □

Theorem 5. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two non-empty, closed and convex subsets of Hilbert spaces E_1 and E_2, respectively. Let $f : E_1 \to E_1$ be a σ-inverse strongly monotone operator and $B : E_1 \to 2^{E_1}$ be a maximal monotone operator. Let $A : E_1 \to E_2$ be a bounded linear operator and $F : \mathcal{K}_2 \times \mathcal{K}_2 \to \mathbb{R}$ be a bi-function satisfying Assumption 1 such that F is upper semicontinuous in the first argument. Assume that Ω_5 is non-empty. For an arbitrarily given $u_0 \in \mathcal{K}_1$, let the iterative sequence $\{u_n\}$ be generated by*

$$u_{n+1} = J_\lambda^B(I - \lambda f)(u_n + \gamma A^*(T_{r_n}^F - I)Au_n), \quad \forall\, n \geq 0, \tag{61}$$

where $r_n \subset (0, \infty)$, $\liminf_{n\to\infty} r_n > 0$, $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A and $A^* : E_2 \to E_1$ is the adjoint operator of A and $\lambda \in (0, 2\sigma)$. The projection mapping is $T_{r_n}^F : E_2 \to \mathcal{K}_2$ and $J_\lambda^B = (I + \lambda B)^{-1}$. Then the sequence $\{u_n\}$ converges weakly to $u^* \in \Omega_5$ where Ω_5 is the set of all solution of GSP-5.

Proof. The proof follows directly the proofs of Theorem 1 and Theorem 4. □

Theorem 6. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two non-empty, closed and convex subsets of Hilbert spaces E_1 and E_2, respectively. Let $f : E_2 \to E_2$ be a σ-inverse strongly monotone operator and $B : E_2 \to 2^{E_2}$ be a maximal monotone operator. Let $A : E_1 \to E_2$ be a bounded linear operator and $F : \mathcal{K}_1 \times \mathcal{K}_1 \to \mathbb{R}$ be a bi-function satisfying Assumption 1. Assume that Ω_6 is non-empty and the operator $J_\lambda^B(I - \lambda f)$ is averaged. For an arbitrarily given $u_0 \in \mathcal{K}_1$, let the iterative sequence $\{u_n\}$ be generated by*

$$u_{n+1} = T_{r_n}^F(u_n + \gamma A^*(J_\lambda^B(I - \lambda f)Au_n), \quad \forall\, n \geq 0, \tag{62}$$

where $r_n \subset (0, \infty)$, $\liminf_{n\to\infty} r_n > 0$, $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A and $A^* : E_2 \to E_1$ is the adjoint operator of A and $\lambda \in (0, 2\sigma)$. The projection mapping is $T_{r_n}^F : E_1 \to \mathcal{K}_1$ and $J_\lambda^B = (I + \lambda B)^{-1}$. Then the sequence $\{u_n\}$ converges weakly to $u^* \in \Omega_6$ where Ω_6 is the set of all solution of GSP-6.

Proof. The proof follows directly the proofs of Theorem 2 and Theorem 3. □

5. Analytical discussion

In this section, we provide numerical examples to demonstrate the validity of suggested algorithms. All numerical computations were carried out using Matlab version R2021a in Asus Core i5 8th Gen Laptop with Nvidia 1650 Geforce GTX graphics card. The stopping criterion used for our computation is $E_n = \|u_{n+1} - u_n\| \leq 10^{-6}$.

Example 1. (Finite dimension) Set $E_1 = E_2 = \mathbb{R}^3$, $\mathcal{K}_1 = \mathcal{K}_2 = \mathbb{R}^3$. Let $F : \mathcal{K}_1 \times \mathcal{K}_1 \to \mathbb{R}$ be defined by $F(u, v) = v^2 - u^2$ for all $u = \{u^i\}_{i=1}^3, v = \{v^i\}_{i=1}^3 \in \mathcal{K}_1$ be a nonlinear bi-function satisfying Assumption 1. Using (6), we have

$$T^F_{r_n}(u) = \frac{u}{1 + 2r_n}, \quad \text{for all } u = \{u^i\}_{i=1}^3 \in R^3.$$

Let a bounded linear operator $A : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $A(u) = \frac{u}{2}$ for all $u = \{u^i\}_{i=1}^3 \in \mathbb{R}^3$ with its adjoint operator $A^* : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $A^*(v) = \frac{v}{2}$ for all $v = \{v^i\}_{i=1}^3 \in \mathbb{R}^3$ and $B : \mathbb{R}^3 \to 2^{\mathbb{R}^3}$ be a multivalued maximal monotone mapping defined by $B(u) = -\left\{\frac{4u}{5}\right\}$ for all $u = \{u^i\}_{i=1}^3 \in \mathbb{R}^3$. Let $f, g : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $f(u) = \frac{3u}{2}$ and $g(u) = \frac{u}{3}$ for all $u = \{u^i\}_{i=1}^3 \in \mathbb{R}^3$ be $\frac{2}{3}$- and 3-inverse strongly monotone mappings respectively. Choose $r_n = \frac{1}{4}$, $\gamma = \frac{1}{10}$ and $\lambda = \frac{1}{4}$, then all the requirements of Theorems 1–6 are satisfied. Then the sequence $\{u_n\}$ induced by (26), (43), (51), (56), (61), and (62) converges weakly to a fixed point $\{0\}$ in Ω_1, Ω_5, Ω_2, Ω_3, Ω_4, and Ω_6, respectively.

Example 2. (Infinite dimension) Set $E_1 = E_2 = L^2[0, 1]$, $\mathcal{K}_1 = \mathcal{K}_2 = L^2[0, 1]$ with standard inner product defined by

$$\langle u, v \rangle = \int_0^1 u(t)v(t)dt, \quad \forall\, u, v \in L^2[0, 1],\, t \in [0, 1],$$

and usual norm defined by

$$\|u\| = \left(\int_0^1 |u(t)|^2 dt\right)^{1/2}, \quad \forall\, u \in L^2[0, 1],\, t \in [0, 1].$$

Let $F : \mathcal{K}_1 \times \mathcal{K}_1 \to \mathbb{R}$ be defined by $F(u, v) = 4v^2 - 3u^2 + 2uv$ for all $u, v \in \mathcal{K}_1$ be a nonlinear bi-function satisfying Assumption 1. Using (6), we have

$$T^F_{r_n}(u(t)) = \frac{u(t)}{1 + 2r_n}, \quad \text{for all } u \in \mathcal{K}_1,\, t \in [0,\, 1].$$

Let a bounded linear operator $A : L^2[0,\, 1] \to L^2[0,\, 1]$ be defined by $A(u(t)) = \dfrac{u(t)}{2}$ for all $u \in L^2[0,\, 1]$, $t \in [0,\, 1]$ with its adjoint operator $A^* : L^2[0,\, 1] \to L^2[0,\, 1]$ defined by $A^*(v(t)) = \dfrac{v(t)}{2}$ for all $v \in L^2[0,\, 1]$, $t \in [0,\, 1]$ and $B : L^2[0,\, 1] \to 2^{L^2[0,\, 1]}$ be a multivalued maximal monotone mapping defined by $B(u(t)) = \left\{\dfrac{3u(t)}{2}\right\}$ for all $u \in L^2[0,\, 1]$, $t \in [0,\, 1]$. Let $f, g : L^2[0,\, 1] \to L^2[0,\, 1]$, defined by $f(u(t)) = \dfrac{u(t)}{2}$ and $g(u(t)) = \dfrac{u(t)}{3}$ for all $u \in L^2[0,\, 1]$, $t \in [0,\, 1]$, be 2- and 3-inverse strongly monotone mappings, respectively. Choose $r_n = \dfrac{1}{4}$, $\gamma = \dfrac{1}{10}$ and $\lambda = \dfrac{1}{4}$, then all the requirements of Theorems 1–6 are satisfied. Then the sequence $\{u_n\}$ induced by (26), (43), (51), (56), (61), and (62) converges weakly to a fixed point $\{0\}$ in Ω_1, Ω_5, Ω_2, Ω_3, Ω_4, and Ω_6, respectively.

We illustrate these examples using the following cases.
For Example 1, we investigate the following initial values of u_0:
Case I. $u_0 = (0.5,\, 0.25,\, 0.125)$;
Case II. $u_0 = (5,\, 0,\, 0.2)$.

For Example 2, we examine the following initial value of u_0:
Case I. $u_0 = 1 + t^2$;
Case II. $u_0 = te^t$.

In Table 1 and Table 2, for a given stopping criterion and randomly chosen initial points, we collect data of the number of iterations and time required to execute algorithms (26), (43), (51), (56), (61), and (62) for Example 1. For the same Example 1, we plot the graphs of errors against the number of iterations for randomly chosen different initial points in Figure 1.

TABLE 1. Example 1, the initial value $u_0 = (0.5, 0.25, 0.125)$.

Numerical results for **Case I.**			
Algorithms	Iterations	E_n	CPU time
26	92	$9 \times e^{-6}$	0.0228
43	27	$9 \times e^{-6}$	0.0125
51	124	$9 \times e^{-6}$	0.0464
56	43	$9 \times e^{-6}$	0.0248
61	39	$8 \times e^{-6}$	0.0298
62	27	$8 \times e^{-6}$	0.0281

TABLE 2. Example 1, the initial value $u_0 = (5, 0, 0.2)$.

Numerical results for **Case II.**			
Algorithms	Iterations	E_n	CPU time
26	114	$9 \times e^{-6}$	0.0319
43	33	$8 \times e^{-6}$	0.0134
51	156	$9 \times e^{-6}$	0.0501
56	53	$9 \times e^{-6}$	0.0246
61	47	$9 \times e^{-6}$	0.0183
62	33	$7 \times e^{-6}$	0.0160

In Table 3 and Table 4, for a given stopping criterion and randomly selected initial points, we determine the duration and number of iterations needed to run algorithms (26), (43), (51), (56), (61), and (62) for Example 2.

TABLE 3. Example 2, the initial value $u_0 = 1 + t^2$.

Numerical results for **Case I.**			
Algorithms	Iterations	E_n	CPU time
26	121	$9 \times e^{-6}$	3.4887
43	19	$8 \times e^{-6}$	1.0411
51	168	$8 \times e^{-6}$	5.0685
56	29	$8 \times e^{-6}$	1.3528
61	27	$8 \times e^{-6}$	1.3095
62	19	$7 \times e^{-6}$	1.0536

In Figure 2, we illustrate the errors against the number of iterations for various randomly selected initial values for Example 2.

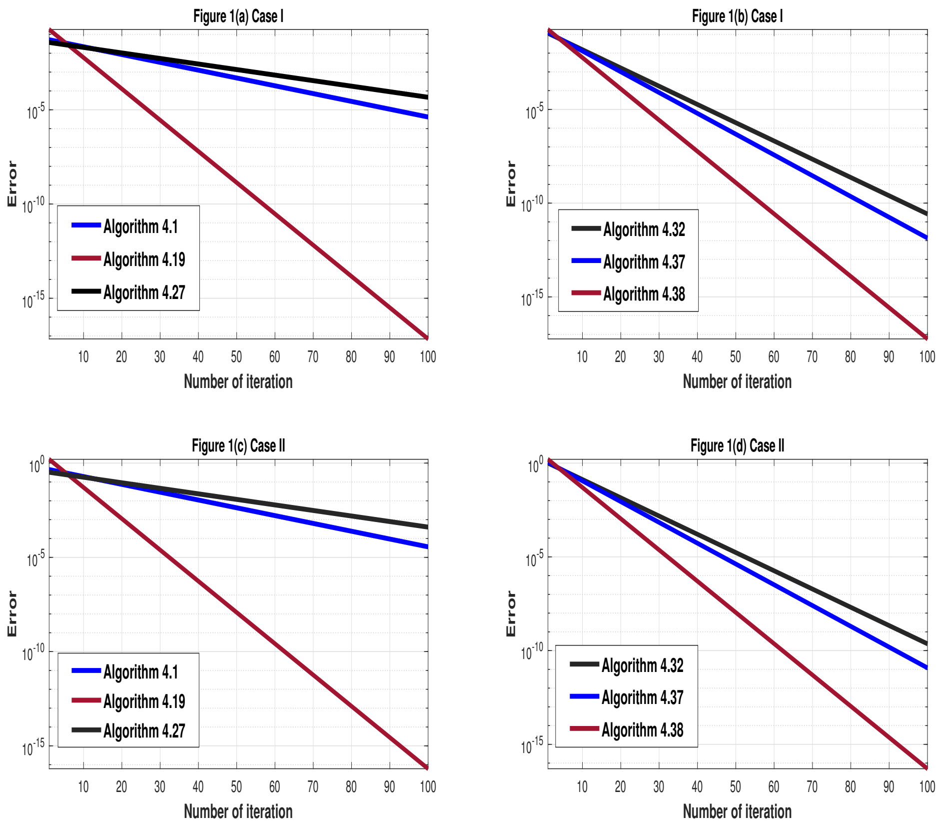

FIGURE 1. Graphical representations of the error.

TABLE 4. Example 2, the initial value $u_0 = te^t$.

Numerical results for **Case II.**			
Algorithms	Iterations	E_n	CPU time
26	118	$9 \times e^{-6}$	4.4373
43	19	$6 \times e^{-6}$	1.5509
51	164	$9 \times e^{-6}$	5.7670
56	28	$9 \times e^{-6}$	1.8612
61	26	$9 \times e^{-6}$	1.8664
62	19	$5 \times e^{-6}$	1.5741

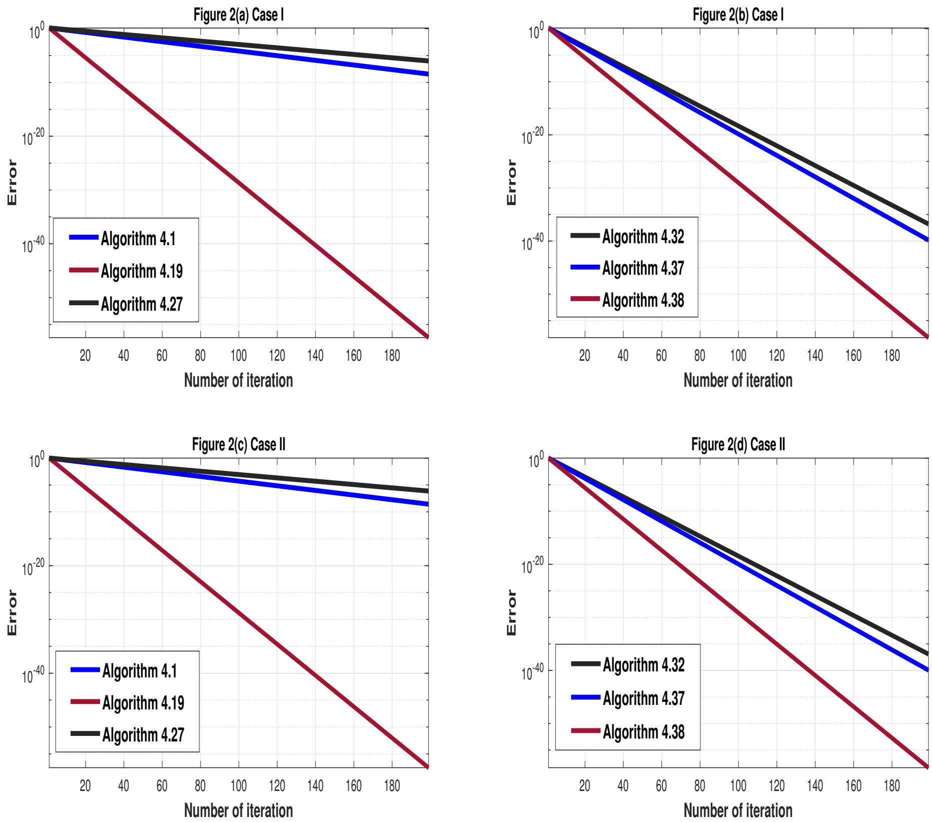

FIGURE 2. Graphical representations of the error.

6. Conclusion

In this paper, we define and study some generalized split problems. These problems can be seen as a generalization for finding a common solution of two distinct problems. In these problems, the solution to one problem (VI, MVIP, EP) in one space under the image of a bounded linear operator solves another problem (VI, MVIP, EP) in a different space. We also provide iterative techniques to tackle generalized split problems in real Hilbert spaces, and under the right circumstances, we provide weak convergence results. In conclusion, we provide numerical examples in finite and infinite spaces to back up our findings. A new path has been opened up in the research on split problems as a result of this work.

References

[1] A. Alamer and M. Dilshad, *Halpern-type inertial iteration methods with self-adaptive step size for split common null point problem*, Mathematics **12**(5) (2024), 16 pp. DOI

[2] E. Blum, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123–145.

[3] C. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Problems **18**(2) (2002), 441.

[4] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Problems **20**(1) (2003), 103. DOI

[5] C. Byrne, Y. Censor, A. Gibali, and S. Reich, *The split common null point problem*, J. Nonlinear Convex Anal. **13** (2012), 759–775.

[6] L.-C. Ceng, Q. H. Ansari, and J.-C. Yao, *Some iterative methods for finding fixed points and for solving constrained convex minimization problems*, Nonlinear Anal. Theory Methods Appl. **74** (2011), 5286–5302. DOI

[7] Y. Censor, T. Bortfeld, B. Martin, and A. Trofimov, *A unified approach for inversion problems in intensity-modulated radiation therapy*, Phys. Med. Biol. **51**(10) (2006), 2353.

[8] Y. Censor, A. Gibali, and S. Reich, *Algorithms for the split variational inequality problem*, Numer. Algorithms **59** (2012), 301–323.

[9] F. Cianciaruso, G. Marino, L. Muglia, and Y. Yao, *A hybrid projection algorithm for finding solutions of mixed equilibrium problem and variational inequality problem*, Fixed Point Theory Appl. **2010** (2009), 1–19.

[10] P. L. Combettes, *Solving monotone inclusions via compositions of nonexpansive averaged operators*, Optim. **53** (2004), 475–504.

[11] P. L. Combettes and V. R. Wajs, *Signal recovery by proximal forward-backward splitting*, Multiscale Model. Simul. **4** (2005), 1168–1200.

[12] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005), 117–136.

[13] M. Dilshad, A. F. Aljohani, and M. Akram, *Iterative scheme for split variational inclusion and a fixed-point problem of a finite collection of nonexpansive mappings*, J. Funct. Spaces 2020, Article ID 3567648. DOI

[14] M. Dilshad, M. Akram, M. Nasiruzzaman, D. Filali, and A. Khidir, *Adaptive inertial Yosida approximation iterative algorithms for split variational inclusion and fixed point problems*, AIMS Math. **8** (2023), 12922–12942. DOI

[15] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, 1990.

[16] D. Kinderlehrer and G. Stampacchia, *An Iteration to Variational Inequalities and Their Applications*, Academic Press, New York, 1990.

[17] J.-L. Lions and G. Stampacchia, *Variational inequalities*, Commun. Pure Appl. Math. **20** (1967), 493–519.

[18] H. Mahdioui and O. Chadli, *On a system of generalized mixed equilibrium problems involving variational-like inequalities in Banach spaces: existence and algorithmic aspects*, Adv. Oper. Res. 2012 (2012).

[19] A. Moudafi and M. Théra, *Proximal and dynamical approaches to equilibrium problems.* In: *Ill-posed Variational Problems and Regularization Techniques*, Springer, 1999, 187–201.

[20] A. Moudafi, *Split monotone variational inclusions*, J. Optim. Theory Appl. **150** (2011), 275–283. DOI

[21] M. A. Noor and K. I. Noor, *On equilibrium problems*, Appl. Math. E-Notes **4** (2004), 125–132.

[22] S. Reich and A. Taiwo, *Fast hybrid iterative schemes for solving variational inclusion problems*, Math. Methods Appl. Sci. **46** (2023), 17177–17198. DOI

[23] S. Suwannaut, S. Suantai, and A. Kangtunyakarn, *The method for solving variational inequality problems with numerical results*, Afrika Mat. **30** (2019), 311–334.

[24] Y. Tang and Y. J. Cho, *Convergence theorems for common solutions of split variational inclusion and systems of equilibrium problems*, Mathematics **7**(3) (2019), 255.

[25] H.-K. Xu, *Averaged mappings and the gradient-projection algorithm*, J. Optim. Theory Appl. **150** (2011), 360–378.

Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh 202002, India
E-mail address: masad19932015@gmail.com

Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk-71491, Saudi Arabia
E-mail address: mdilshaad@gmail.com, mdilshad@ut.edu.sa