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Characterizations of ρ-Einstein solitons on LP-Sasakian manifolds admitting the Zamkovoy connection

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Abstract. The purpose of this research is to investigate ρ-Einstein solitons on LP-Sasakian manifolds under certain curvature conditions. The novelty of our research lies in the fact that we characterize ρ-Einstein solitons on LP-Sasakian manifolds equipped with the Zamkovoy connection when the structure vector field is considered as the potential vector field. We obtain some significant results on classifications of ρ-Einstein solitons in regard to the W_8-curvature tensor and the Zamkovoy connection. In extension, we build a non-trivial example of a three dimensional LP-Sasakian manifold endowed with the Zamkovoy connection.

1. Introduction

A Ricci soliton is a natural generalization of an Einstein metric, which was introduced by Hamilton [13] as the fixed point of the Hamilton's Ricci flow

$$\frac{\partial g}{\partial t} = -2S, \; g(0) = g_0,$$

where g is a Riemannian metric, S is the Ricci curvature tensor and t is time. The solitons for the Ricci flow are the solutions of the above equation, where the metrics at different times differ by a diffeomorphism of the manifold. A Ricci soliton is represented by a triple (g, V, σ), where V is a vector field and σ is a scalar, which satisfies the equation

$$L_V g + 2S + 2\sigma g = 0,$$

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in which $L_V g$ denotes the Lie derivative of g with respect to the vector field V, and S again stands for the Ricci curvature tensor. Depending on the sign of the constant σ, the Ricci soliton is called *shrinking*, *steady*, or *expanding* if $\sigma < 0$, $\sigma = 0$, or $\sigma > 0$, respectively. The vector field V is referred to as the *potential vector field*. In the particular case when V is the gradient of a smooth function, the Ricci soliton (g, V, σ) is termed a *gradient Ricci soliton*. Extensive investigations on Ricci solitons have been carried out by numerous authors; see, for example, [21, 24, 27, 28, 29, 31] and the references therein.

As a generalization of the concept of Ricci flow, the Ricci–Bourguignon flow [6, 7, 8] has been described by a partial differential equation

$$\frac{\partial g}{\partial t} = -2\left[S - \rho r g\right], \ g(0) = g_0,$$

where ρ is a constant, S and r are, respectively, the Ricci curvature tensor and the scalar curvature tensor regarding g.

Let (M, g) be an n-dimensional semi-Riemannian manifold and ρ be a real number. The *ρ-Einstein soliton* [26] of (M, g) is a self similar solution of Ricci–Bourguignon flow equation satisfying

$$L_V g + 2S + 2(\lambda - \rho r)g = 0, \tag{1}$$

where λ is a real number. Similarly to Ricci solitons, a ρ-Einstein soliton (g, V, λ) is called *shrinking* if $\lambda < 0$, *steady* if $\lambda = 0$, and *expanding* if $\lambda > 0$. If $V = gradF$, then the soliton is called a *gradient ρ-Einstein soliton*. Hence (1) takes the form

$$HessF + S + (\lambda - \rho r)g = 0,$$

where $HessF$ denotes the Hessian of the smooth function F.

For specific values of ρ, the ρ-Einstein soliton is called
(i) *an Einstein soliton*, if $\rho = \frac{1}{2}$,
(ii) *a traceless soliton*, if $\rho = \frac{1}{n}$,
(iii) *a Schouten soliton*, if $\rho = \frac{1}{2(n-1)}$,
(iv) *a Ricci soliton*, if $\rho = 0$.

Recently, ρ-Einstein solitons have been studied by several researchers, in papers such as [8, 14, 23, 33].

Recent studies on ρ-Einstein solitons have explored diverse geometric settings: Patra [23] characterizes such solitons on Sasakian manifolds, showing that the soliton field is a Jacobi field along the Reeb flow; Ahmad et al. [1] investigate the soliton types in Lorentzian para-Kenmotsu manifolds under various curvature conditions; Venkatesha and Kumara [33] prove that gradient ρ-Einstein solitons on almost Kenmotsu manifolds are either Einstein or have a potential function collinear with the Reeb field; and Azami and Fasihi-Ramandi [2] demonstrate that Ricci ρ-solitons on 3-dimensional η-Einstein almost Kenmotsu manifolds force the structure to be Kenmotsu with constant curvature -1 and an expanding soliton. Moreover, several related works have contributed to the understanding of ρ-Einstein solitons and their generalizations in various geometric contexts [8, 14].

A special linear connection on paracontact manifolds, known as the Zamkovoy connection was first introduced by Zamkovoy [37]. This connection is characterized by having a torsion tensor that measures how far a paracontact manifold deviates from being para-Sasakian. In their seminal work, Biswas and Baishya [3, 4] investigated this connection in several geometric settings, namely generalized pseudo-Ricci symmetric and almost pseudo-symmetric Sasakian manifolds. The Zamkovoy connection has been the focus of extensive research by numerous scholars. For instance, the works [5, 15, 16, 17, 18] have contributed to our understanding of the subject.

The Zamkovoy connection ∇^* for an n-dimensional almost contact metric manifold M equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, is defined as

$$\nabla^*_{\theta_1}\theta_2 = \nabla_{\theta_1}\theta_2 + \left(\nabla_{\theta_1}\eta\right)(\theta_2)\,\xi - \eta(\theta_2)\,\nabla_{\theta_1}\xi + \eta(\theta_1)\,\phi\theta_2,$$

for all $\theta_1, \theta_2 \in \chi(M)$, where $\chi(M)$ denotes the set of all vector fields on M.

Yano [34] introduced the notion of torse-forming vector field as a common generalization of concircular, concurrent and parallel vector fields. On a Riemannian manifold M, a non-zero vector field θ_t is said to be torse forming if it satisfies

$$\nabla_{\theta_1}\theta_t = l\theta_1 + \pi(\theta_1)\theta_t, \tag{2}$$

where $l \in C^\infty(M)$, the set of all smooth functions on M, and π denotes a 1-form. Moreover, if $\pi = 0$, then the vector field θ_t is concircular [9, 35]. If $l = 1$ and $\pi = 0$, then θ_t is concurrent [36], also if $l = \pi = 0$, then the vector field θ_t is parallel.

In 1989, Matsumoto [19] first introduced the notion of LP-Sasakian manifold. In this context it may be mentioned that Mihai and Rosca [20] also introduced independently the notion of LP-Sasakian manifold in 1992. The generalized recurrent manifold was introduced by Dubey [12] and it was studied by De and Guha [10]. The ϕ-recurrent LP-Sasakian manifold was studied by Shaikh [25]. Moreover, the ϕ-concircularly flat LP-Sasakian manifold was studied by Taleshian [30]. Apart from this, the properties of LP-Sasakian manifolds were studied by several authors. For instance, see [11, 22] and their references.

On an n-dimensional semi-Riemannian manifold (M, g), Tripathi and Gupta [32] introduced the τ-curvature tensor, defined by

$$\begin{aligned}\tau(\theta_1,\theta_2)\theta_3 \;=\;& a_0 R(\theta_1,\theta_2)\theta_3 + a_1 S(\theta_2,\theta_3)\theta_1 + a_2 S(\theta_1,\theta_3)\theta_2 \\ &+ a_3 S(\theta_1,\theta_2)\theta_3 + a_4 g(\theta_2,\theta_3) Q\theta_1 + a_5 g(\theta_1,\theta_3) Q\theta_2 \\ &+ a_6 g(\theta_1,\theta_2) Q\theta_3 + r a_7\left[g(\theta_2,\theta_3)\theta_1 - g(\theta_1,\theta_3)\theta_2\right],\end{aligned}$$

for all $\theta_1, \theta_2, \theta_3 \in \chi(M)$ and for suitable real parameters a_i, where R, S, Q and r are the Riemannian curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature tensor, respectively.

For specific choice $a_0 = 1, a_1 = -a_3 = -\frac{1}{n-1}$, and $a_2 = a_4 = a_5 = a_6 = a_7 = 0$, the τ-curvature tensor reduces to W_8-curvature tensor.

As a special case of the τ-curvature tensor [32], the W_8-curvature tensor on M is defined as

$$W_8(\theta_1, \theta_2)\theta_3 = R(\theta_1, \theta_2)\theta_3 - \frac{1}{n-1}\left[S(\theta_2, \theta_3)\theta_1 - S(\theta_1, \theta_2)\theta_3\right],$$

for all $\theta_1, \theta_2, \theta_3 \in \chi(M)$, where R and S are the Riemannian curvature tensor and the Ricci curvature tensor, respectively. The W_8-curvature tensor captures specific trace relations between the Riemann and Ricci curvatures. In the present work, we study this tensor in the context of LP-Sasakian manifolds equipped with the Zamkovoy connection, where it plays a central role in the curvature conditions considered in Sections 4 and 6. Algebraic conditions such as $W_8 \cdot S = 0$, $S \cdot W_8 = 0$, and $W_8 \cdot R = 0$ are structures frequently encountered in the study of generalized symmetries (e.g., pseudosymmetry). These conditions impose strong constraints on the curvature structure of the manifold, enabling the derivation of explicit, computable formulas for scalar quantities, particularly for the soliton constant.

Definition 1 ([15])**.** An n-dimensional LP -Sasakian manifold M is said to be a *generalized η-Einstein manifold* if the Ricci tensor of type (0,2) is of the form

$$S(\theta_2, \theta_3) = k_1 g(\theta_2, \theta_3) + k_2 \eta(\theta_2)\eta(\theta_3) + k_3 \omega(\theta_2, \theta_3)$$

for all $\theta_2, \theta_3 \in \chi(M)$, where k_1, k_2 and k_3 are scalars and ω is a 2-form.

The primary objective of the present paper is to study ρ-Einstein solitons on LP-Sasakian manifolds equipped with the Zamkovoy connection, with the structure vector field ξ as the potential vector field. Specifically, we aim to: (i) characterize the Ricci tensor of an LP-Sasakian manifold admitting such a soliton and show that the manifold is necessarily generalized η-Einstein; (ii) classify the soliton types (shrinking, steady, expanding) under various curvature conditions involving the W_8-curvature tensor with respect to the Zamkovoy connection; and (iii) analyze the soliton when the potential vector field is torse-forming. The results are illustrated by a concrete three-dimensional example.

This paper is structured, after necessary preliminaries, as follows.

Section 3 establishes the fundamental soliton equation with respect to the Zamkovoy connection when the structure vector field ξ is taken as the potential vector field. We show that any LP-Sasakian manifold admitting such a ρ-Einstein soliton is necessarily a generalized η-Einstein manifold

(Theorem 2), and we derive an explicit formula for the Ricci tensor in terms of g, η, and the fundamental 2-form ω.

Section 4 investigates the curvature condition $W_8^*(\xi,\theta_1)\cdot S^*=0$, which expresses a symmetry requirement between the W_8-curvature tensor and the Ricci tensor. Under this condition, we derive the explicit formula $\lambda=\rho(r-n+1+3\psi^2)$ for the soliton constant (Theorem 3), and classify the resulting Ricci, Einstein, traceless, and Schouten solitons as shrinking, steady, or expanding.

Section 5 studies the condition $S^*(\xi,\theta_1)\cdot R^*=0$, which encodes a symmetry between the Ricci and Riemann curvature tensors. We prove that under this condition the soliton is steady if and only if the scalar curvature satisfies $r=n-1-3\psi^2$ (Theorem 4).

Section 6 considers the condition $S^*(\xi,\theta_1)\cdot W_8^*=0$, which leads to a different expression for the soliton constant involving r and ψ (Theorem 5), and yields a classification in terms of explicit inequalities on the scalar curvature.

Section 7 examines the case when the potential vector field is torse-forming. We express λ in terms of r, n, ρ, and the function l, and we discuss the special sub-cases of parallel, concurrent, and concircular potential vector fields.

Section 8 presents a concrete three-dimensional example explicitly verifying the formulas and conditions established in the preceding sections.

2. Preliminaries

An n-dimensional differentiable manifold is called an *LP-Sasakian manifold* if it admits a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Lorentzian metric g which satisfy:

$$\begin{aligned}
\phi^2\theta_2 &= \theta_2+\eta\left(\theta_2\right)\xi,\eta(\xi)=-1,\eta\left(\phi\theta_1\right)=0,\ \phi\xi=0,\\
g(\phi\theta_1,\phi\theta_2) &= g(\theta_1,\theta_2)+\eta(\theta_1)\eta(\theta_2),\\
g(\theta_1,\phi\theta_2) &= g(\phi\theta_1,\theta_2),\eta(\theta_2)=g(\theta_2,\xi),\\
\nabla_{\theta_1}\xi &= \phi\theta_1,\quad g(\theta_1,\xi)=\eta(\theta_1),\\
\left(\nabla_{\theta_1}\phi\right)\theta_2 &= g(\theta_1,\theta_2)\xi+\eta(\theta_2)\theta_1+2\eta(\theta_1)\eta(\theta_2)\xi,
\end{aligned}$$

for all vector fields θ_1 and θ_2 on M, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Let us introduce a symmetric $(0,2)$ tensor field ω, such that

$$\omega\left(\theta_1,\theta_2\right)=g(\theta_1,\phi\theta_2).$$

Also, since the vector field η is closed in an LP-Sasakian manifold M, we have

$$\left(\nabla_{\theta_1}\eta\right)\theta_2=\omega\left(\theta_1,\theta_2\right),\omega\left(\theta_1,\xi\right)=0,\forall\theta_1,\theta_2\in\chi\left(M\right).$$

For an LP-Sasakian manifold, the following relations also hold:

$$\begin{aligned}
\eta\left(R\left(\theta_1,\theta_2\right)\theta_3\right) &= g\left(\theta_2,\theta_3\right)\eta\left(\theta_1\right)-g\left(\theta_1,\theta_3\right)\eta\left(\theta_2\right),\\
R\left(\theta_1,\theta_2\right)\xi &= \eta\left(\theta_2\right)\theta_1-\eta\left(\theta_1\right)\theta_2,\\
R\left(\xi,\theta_2\right)\theta_3 &= g\left(\theta_2,\theta_3\right)\xi-\eta\left(\theta_3\right)\theta_2,\\
R\left(\xi,\theta_2\right)\xi &= \eta\left(\theta_2\right)\xi+\theta_2,\\
S\left(\theta_1,\xi\right) &= (n-1)\,\eta\left(\theta_1\right),\\
S\left(\phi\theta_1,\phi\theta_2\right) &= S\left(\theta_1,\theta_2\right)+(n-1)\,\eta\left(\theta_1\right)\eta\left(\theta_2\right),\\
S\left(\theta_1,\theta_2\right) &= g\left(Q\theta_1,\theta_2\right), Q\xi=(n-1)\,\xi, Q\phi=\phi Q,
\end{aligned}$$

for all $\theta_1,\theta_2,\theta_3\in\chi(M)$.

Lemma 1 ([15]). *Let M be an n-dimensional LP-Sasakian manifold, then relation between the Zamkovoy connection and the Levi-Civita connection on M is given by*

$$\nabla^*_{\theta_1}\theta_2=\nabla_{\theta_1}\theta_2+g\left(\theta_1,\phi\theta_2\right)\xi-\eta\left(\theta_2\right)\phi\theta_1+\eta\left(\theta_1\right)\phi\theta_2, \tag{3}$$

where the torsion tensor of the Zamkovoy connection is given by

$$T^*\left(\theta_1,\theta_2\right)=2\left[\eta\left(\theta_1\right)\phi\theta_2-\eta\left(\theta_2\right)\phi\theta_1\right], \tag{4}$$

for all vector fields θ_1,θ_2 on M. Here, ∇^* denotes the Zamkovoy connection and ∇ denotes the Levi-Civita connection.

Proposition 1 ([15]). *For an n-dimensional LP-Sasakian manifold M with the structure (ϕ,ξ,η,g) we have*

$$\left(\nabla^*_{\theta_1}g\right)\left(\theta_2,\theta_3\right)=-2g\left(\theta_2,\phi\theta_3\right)\eta\left(\theta_1\right),\ \nabla^*_{\theta_1}\xi=2\phi\theta_1, \tag{5}$$

for any vector field θ_1 on M, where ∇^* denotes the Zamkovoy connection on M.

Proposition 2 ([15]). *The integral curve of ξ with respect to the Zamkovoy connection is a geodesic in an LP-Sasakian manifold.*

Let R^* be the Riemannian curvature tensor with respect to the Zamkovoy connection and let it be defined as

$$\begin{aligned}
R^*\left(\theta_1,\theta_2\right)\theta_3 &= R\left(\theta_1,\theta_2\right)\theta_3+3g\left(\theta_1,\theta_3\right)\eta\left(\theta_2\right)\xi\\
&\quad -3g\left(\theta_2,\theta_3\right)\eta\left(\theta_1\right)\xi+3g\left(\theta_2,\phi\theta_3\right)\phi\theta_1-3g\left(\theta_1,\phi\theta_3\right)\phi\theta_2\\
&\quad -\eta\left(\theta_1\right)\eta\left(\theta_3\right)\theta_2+\eta\left(\theta_2\right)\eta\left(\theta_3\right)\theta_1,
\end{aligned} \tag{6}$$

where $R^*\left(\theta_1,\theta_2\right)\theta_3=\nabla^*_{\theta_1}\nabla^*_{\theta_2}\theta_3-\nabla^*_{\theta_2}\nabla^*_{\theta_1}\theta_3-\nabla^*_{[\theta_1,\theta_2]}\theta_3$.

Lemma 2 ([15]). *Let M be an n-dimensional LP-Sasakian manifold, then the following relations hold:*

$$S^*\left(\theta_2,\theta_3\right) = S\left(\theta_2,\theta_3\right)+(n-1)\,\eta\left(\theta_2\right)\eta\left(\theta_3\right)+3\psi g\left(\theta_2,\phi\theta_3\right), \tag{7}$$

$$\begin{aligned} S^*(\xi,\theta_3) &= S^*(\theta_3,\xi)=0, & (8)\\ Q^*\theta_2 &= Q\theta_2+(n-1)\eta(\theta_2)\xi+3\psi\phi\theta_2, \\ Q^*\xi &= 0, & (9)\\ r^* &= r-(n-1)+3\psi^2, & (10)\\ R^*(\theta_1,\theta_2)\xi &= 0, & (11)\\ R^*(\xi,\theta_2)\theta_3 &= 4g(\phi\theta_2,\phi\theta_3)\xi, & (12)\\ R^*(\theta_1,\xi)\theta_3 &= -4g(\phi\theta_1,\phi\theta_3)\xi, & (13)\end{aligned}$$

for all $\theta_1,\theta_2,\theta_3\in\chi(M)$, *where* S^*, Q^* *and* r^* *are the Ricci curvature tensor, the Ricci operator and the scalar curvature tensor of* M *with respect to the Zamkovoy connection, and* $\psi=trace(\phi)$.

3. ρ-Einstein solitons on LP-Sasakian manifolds with respect to the Zamkovoy connection

In this section, we study the ρ-Einstein soliton with respect to the Zamkovoy connection on an LP-Sasakian manifold $M(\phi,\xi,\eta,g)$. We also derive the Ricci tensor of M corresponding to the ρ-Einstein soliton with respect to the Zamkovoy connection. Before introducing the ρ-Einstein soliton equation with respect to the Zamkovoy connection on M, we recall the explicit form of the Lie derivative of the metric with respect to the potential vector field V. In the present framework, this derivative is computed using the Zamkovoy connection ∇^*. Referring to (4) and (5), we get

$$\begin{aligned}\left(L_V^*g\right)(\theta_1,\theta_2) &= g(\nabla^*_{\theta_1}V,\theta_2)+g(\theta_1,\nabla^*_{\theta_2}V)-2g(\theta_1,\phi\theta_2)\eta(V)\\ &\quad -2g(\theta_1,\phi V)\eta(\theta_2)-2g(\theta_2,\phi V)\eta(\theta_1). \qquad (14)\end{aligned}$$

Setting $V=\xi$ in (14) and using (5), we have

$$\left(L_\xi^*g\right)(\theta_1,\theta_2)=g(\nabla^*_{\theta_1}\xi,\theta_2)+g(\theta_1,\nabla^*_{\theta_2}\xi)-2g(\theta_1,\phi\theta_2). \qquad (15)$$

This expression does not coincide with the classical definition based on the Levi-Civita connection ∇, its behavior differs significantly because ∇^* is a non-metric connection; in particular, $\left(\nabla^*_{\theta_1}g\right)(\theta_2,\theta_3)\neq 0$ for arbitrary vector fields θ_1,θ_2 and θ_3. Consequently, L_V^*g contains additional terms arising from both the non-symmetric torsion and the non-metricity of ∇^*, and these contributions play an essential role in the structure of the soliton equation.

Let (g,V,λ) be a ρ-Einstein soliton with respect to the Zamkovoy connection on an n-dimensional LP-Sasakian manifold $M(\phi,\xi,\eta,g)$. Then writing the equation (1) with respect to ∇^*, we have

$$0=\left(L_V^*g\right)(\theta_1,\theta_2)+2S^*(\theta_1,\theta_2)+2(\lambda-\rho r^*)g(\theta_1,\theta_2), \qquad (16)$$

for all $\theta_1,\theta_2\in\chi(M)$.

Utilizing (3), (7), (10) and (14) in (16), we get

$$\begin{aligned}0 &= g(\nabla^*_{\theta_1}V,\theta_2)+g(\theta_1,\nabla^*_{\theta_2}V)-2g(\theta_1,\phi\theta_2)\eta(V)\\ &\quad -2g(\theta_1,\phi V)\eta(\theta_2)-2g(\theta_2,\phi V)\eta(\theta_1)\\ &\quad +2S^*(\theta_1,\theta_2)+2(\lambda-\rho r^*)g(\theta_1,\theta_2)\\ &= (L_V g)(\theta_1,\theta_2)+2S(\theta_1,\theta_2)+2(\lambda-\rho r)g(\theta_1,\theta_2)\\ &\quad +6\psi g(\theta_1,\phi\theta_2)+2\rho\left[n-1-3\psi^2\right]g(\theta_1,\theta_2)\\ &\quad +2(n-1)\eta(\theta_1)\eta(\theta_2).\end{aligned}$$

Hence we get the following theorem.

Theorem 1. *A ρ-Einstein soliton on an n-dimensional LP-Sasakian manifold $M(\phi,\xi,\eta,g)$ is invariant under the Zamkovoy connection if and only if*

$$0=3\psi g(\theta_1,\phi\theta_2)+\rho\left[n-1-3\psi^2\right]g(\theta_1,\theta_2)+(n-1)\eta(\theta_1)\eta(\theta_2),$$

holds for all $\theta_1,\theta_2\in\chi(M)$.

Setting $V=\xi$ in (16) and using (15), we have

$$S^*(\theta_1,\theta_2)=-(\lambda-\rho r^*)g(\theta_1,\theta_2)-g(\theta_1,\phi\theta_2). \tag{17}$$

Replacing θ_2 by ξ in (17), we obtain

$$S^*(\theta_1,\xi)=-(\lambda-\rho r^*)\eta(\theta_1). \tag{18}$$

Referring to (7) and (15), we get

$$\begin{aligned}S(\theta_1,\theta_2) &= \left[\rho\left(r-n+1+3\psi^2\right)-\lambda\right]g(\theta_1,\theta_2)\\ &\quad -(n-1)\eta(\theta_1)\eta(\theta_2)-(3\psi+1)g(\theta_1,\phi\theta_2).\end{aligned}$$

This implies the following theorem.

Theorem 2. *If an n-dimensional LP-Sasakian manifold $M(\phi,\xi,\eta,g)$ contains a ρ-Einstein soliton (g,ξ,λ) with respect to the Zamkovoy connection, then M becomes generalized η-Einstein.*

4. ρ-Einstein solitons on LP-Sasakian manifolds satisfying $W_8^*(\xi,\theta_1).S^*=0$

The W_8-curvature tensor of M with respect to the Zamkovoy connection is given by

$$W_8^*(\theta_1,\theta_2)\theta_3=R^*(\theta_1,\theta_2)\theta_3-\frac{1}{n-1}\left[S(\theta_2,\theta_3)\theta_1-S(\theta_1,\theta_2)\theta_3\right], \tag{19}$$

for all θ_1,θ_2 and $\theta_3\in\chi(M)$.

Taking into the equations (8), (11), (12), (13) and (19), we derive the expressions

$$W_8^*(\theta_1,\theta_2)\xi = \frac{1}{n-1}S^*(\theta_1,\theta_2)\xi, W_8^*(\theta_1,\xi)\theta_3=-4g(\phi\theta_1,\phi\theta_3)\xi$$

and

$$W_8^*(\xi,\theta_2)\theta_3 \;=\; 4g(\phi\theta_2,\phi\theta_3)\xi - \frac{1}{n-1}S^*(\theta_2,\theta_3)\xi.$$

The curvature tensor S^* must obey the constraint

$$S^*\left(W_8^*(\xi,\theta_1)\theta_2,\theta_3\right) + S^*\left(\theta_2,W_8^*(\xi,\theta_1)\theta_3\right) = 0. \tag{20}$$

Now, with the help of (18) and (20), we get

$$\begin{aligned} S^*\left(W_8^*(\xi,\theta_1)\theta_2,\theta_3\right) &= \left[4g(\phi\theta_1,\phi\theta_2) - \frac{1}{n-1}S^*(\theta_1,\theta_2)\right]S^*(\xi,\theta_3) \\ &= (\rho r^* - \lambda)\left[4g(\phi\theta_1,\phi\theta_2) - \frac{S^*(\theta_1,\theta_2)}{n-1}\right]\eta(\theta_3), \end{aligned} \tag{21}$$

and

$$\begin{aligned} S^*\left(\theta_2,W_8^*(\xi,\theta_1)\theta_3\right) &= \left[4g(\phi\theta_1,\phi\theta_3) - \frac{1}{n-1}S^*(\theta_1,\theta_3)\right]S^*(\xi,\theta_2) \\ &= (\rho r^* - \lambda)\left[4g(\phi\theta_1,\phi\theta_3) - \frac{S^*(\theta_1,\theta_3)}{n-1}\right]\eta(\theta_2). \end{aligned} \tag{22}$$

Using (21) and (22) in (20), we get

$$\begin{aligned} 0 &= (\lambda - \rho r^*)\left[4g(\phi\theta_1,\phi\theta_2)\eta(\theta_3) - \frac{1}{n-1}S^*(\theta_1,\theta_2)\eta(\theta_3)\right] \\ &\quad +(\lambda - \rho r^*)\left[4g(\phi\theta_1,\phi\theta_3)\eta(\theta_2) - \frac{1}{n-1}S^*(\theta_1,\theta_3)\eta(\theta_2)\right]. \end{aligned} \tag{23}$$

Setting $\theta_1 = \xi$ in (23) and then using (10), (18) in it, we get

$$\lambda = \rho\left(r - n + 1 + 3\psi^2\right). \tag{24}$$

Thus we have the following theorem.

Theorem 3. *Let an n-dimensional LP-Sasakian manifold $M(\phi,\xi,\eta,g)$ contain a ρ-Einstein soliton (g,ξ,λ) with respect to the Zamkovoy connection. If M satisfies $W_8^*(\xi,\theta_1)\cdot S^* = 0$, then the soliton constant is given by the equation* (24).

Corollary 1. *A Ricci soliton (g,ξ,λ) on $M(\phi,\xi,\eta,g)$, satisfying $W_8^*(\xi,\theta_1)\cdot S^* = 0$, is always steady.*

Corollary 2. *An Einstein soliton, a traceless soliton or a Schouten soliton on M satisfying $W_8^*(\xi,\theta_1)\cdot S^* = 0$, is*

i) shrinking, if $r < n - 3\psi^2 - 1$,

ii) steady, if $r = n - 3\psi^2 - 1$,

iii) expanding, if $r > n - 3\psi^2 - 1$.

5. ρ-Einstein solitons on LP-Sasakian manifolds satisfying $S^*(\xi,\theta_1)\cdot R^*=0$

Let $M(\phi,\xi,\eta,g)$ be an n-dimensional LP-Sasakian manifold containing a ρ-Einstein soliton (g,ξ,λ) with respect to the Zamkovoy connection. The condition that must be satisfied by S^* is

$$\begin{aligned} 0 &= S^*(\theta_1,R^*(\theta_2,\theta_3)\theta_4)\xi-S^*(\xi,R^*(\theta_2,\theta_3)\theta_4)\theta_1\\ &\quad+S(\theta_1,\theta_2)R^*(\xi,\theta_3)\theta_4-S^*(\xi,\theta_2)R^*(\theta_1,\theta_3)\theta_4\\ &\quad+S^*(\theta_1,\theta_3)R^*(\theta_2,\xi)\theta_4-S^*(\xi,\theta_3)R^*(\theta_2,\theta_1)\theta_4\\ &\quad+S^*(\theta_1,\theta_4)R^*(\theta_2,\theta_3)\xi-S^*(\xi,\theta_4)R^*(\theta_2,\theta_3)\theta_1, \end{aligned} \tag{25}$$

for all $\theta_1,\theta_2,\theta_3$ and $\theta_4\in\chi(M)$. Taking inner product with ξ the equation (25) becomes

$$\begin{aligned} 0 &= -S^*(\theta_1,R^*(\theta_2,\theta_3)\theta_4)-S^*(\xi,R^*(\theta_2,\theta_3)\theta_4)\eta(\theta_1)\\ &\quad+S^*(\theta_1,\theta_2)\eta(R^*(\xi,\theta_3)\theta_4)-S^*(\xi,\theta_2)\eta(R^*(\theta_1,\theta_3)\theta_4)\\ &\quad+S^*(\theta_1,\theta_3)\eta(R^*(\theta_2,\xi)\theta_4)-S^*(\xi,\theta_3)\eta(R^*(\theta_2,\theta_1)\theta_4)\\ &\quad+S^*(\theta_1,\theta_4)\eta(R^*(\theta_2,\theta_3)\xi)-S^*(\xi,\theta_4)\eta(R^*(\theta_2,\theta_3)\theta_1). \end{aligned} \tag{26}$$

Setting $\theta_4=\xi$ in (26), we obtain

$$\begin{aligned} 0 &= -S^*(\theta_1,R^*(\theta_2,\theta_3)\xi)-S^*(\xi,R^*(\theta_2,\theta_3)\xi)\eta(\theta_1)\\ &\quad+S^*(\theta_1,\theta_2)\eta(R^*(\xi,\theta_3)\xi)-S^*(\xi,\theta_2)\eta(R^*(\theta_1,\theta_3)\xi)\\ &\quad+S^*(\theta_1,\theta_3)\eta(R^*(\theta_2,\xi)\xi)-S^*(\xi,\theta_3)\eta(R^*(\theta_2,\theta_1)\xi)\\ &\quad+S^*(\theta_1,\xi)\eta(R^*(\theta_2,\theta_3)\xi)-S^*(\xi,\xi)\eta(R^*(\theta_2,\theta_3)\theta_1). \end{aligned} \tag{27}$$

Using (6), (11) and (18) in (27), we get

$$0=(\lambda-\rho r^*)\eta(R(\theta_2,\theta_3)\theta_1). \tag{28}$$

Since $\eta(R(\theta_2,\theta_3)\theta_1)\neq 0$, on M, we must have

$$\lambda=\rho\left(r-n+1+3\psi^2\right),$$

which gives

$$\lambda=0$$

if $\rho=0$ or $r=n-1-3\psi^2$.

Theorem 4. *Let an n-dimensional LP-Sasakian manifold $M(\phi,\xi,\eta,g)$ contain a ρ-Einstein soliton (g,ξ,λ) with respect to the Zamkovoy connection. If M satisfies $S^*(\xi,\theta_1)\cdot R^*=0$, then the soliton is steady if and only if $r=n-1-3\psi^2$, provided $\rho\neq 0$.*

Corollary 3. *A Ricci soliton (g,ξ,λ) on $M(\phi,\xi,\eta,g)$, satisfying $S^*(\xi,\theta_1)\cdot R^*=0$, is always steady.*

6. ρ-Einstein solitons on LP-Sasakian manifolds satisfying $S^{*}(\xi,\theta_1)\cdot W_8^{*}=0$

Referring to (6), (8) and (19), we have

$$\begin{aligned}\eta(W_8^*(\theta_1,\theta_2)\theta_3) &= 4\left[g(\theta_2,\theta_3)\eta(\theta_1)-g(\theta_1,\theta_3)\eta(\theta_2)\right]\\ &\quad -\frac{1}{n-1}\left[S^*(\theta_2,\theta_3)\eta(\theta_1)-S^*(\theta_1,\theta_2)\eta(\theta_3)\right], \quad (29)\\ \eta(W_8^*(\theta_1,\theta_2)\xi) &= -\frac{1}{n-1}S^*(\theta_1,\theta_2), \quad (30)\end{aligned}$$

for all $\theta_1,\theta_2,\theta_3\in\chi(M)$. The condition that must be satisfied by S^* is

$$\begin{aligned}0 &= S^*(\theta_1,W_8^*(\theta_2,\theta_3)\theta_4)\xi-S^*(\xi,W_8^*(\theta_2,\theta_3)\theta_4)\theta_1\\ &\quad +S(\theta_1,\theta_2)W_8^*(\xi,\theta_3)\theta_4-S^*(\xi,\theta_2)W_8^*(\theta_1,\theta_3)\theta_4\\ &\quad +S^*(\theta_1,\theta_3)W_8^*(\theta_2,\xi)\theta_4-S^*(\xi,\theta_3)W_8^*(\theta_2,\theta_1)\theta_4\\ &\quad +S^*(\theta_1,\theta_4)W_8^*(\theta_2,\theta_3)\xi-S^*(\xi,\theta_4)W_8^*(\theta_2,\theta_3)\theta_1, \quad (31)\end{aligned}$$

for all $\theta_1,\theta_2,\theta_3$ and $\theta_4\in\chi(M)$. Taking inner product with ξ the equation (31) becomes

$$\begin{aligned}0 &= -S^*(\theta_1,W_8^*(\theta_2,\theta_3)\theta_4)-S^*(\xi,W_8^*(\theta_2,\theta_3)\theta_4)\eta(\theta_1)\\ &\quad +S^*(\theta_1,\theta_2)\eta(W_8^*(\xi,\theta_3)\theta_4)-S^*(\xi,\theta_2)\eta(W_8^*(\theta_1,\theta_3)\theta_4)\\ &\quad +S^*(\theta_1,\theta_3)\eta(W_8^*(\theta_2,\xi)\theta_4)-S^*(\xi,\theta_3)\eta(W_8^*(\theta_2,\theta_1)\theta_4)\\ &\quad +S^*(\theta_1,\theta_4)\eta(W_8^*(\theta_2,\theta_3)\xi)-S^*(\xi,\theta_4)\eta(W_8^*(\theta_2,\theta_3)\theta_1). \quad (32)\end{aligned}$$

Setting $\theta_4=\xi$ in (32), we obtain

$$\begin{aligned}0 &= -S^*(\theta_1,W_8^*(\theta_2,\theta_3)\xi)-S^*(\xi,W_8^*(\theta_2,\theta_3)\xi)\eta(\theta_1)\\ &\quad +S^*(\theta_1,\theta_2)\eta(W_8^*(\xi,\theta_3)\xi)-S^*(\xi,\theta_2)\eta(W_8^*(\theta_1,\theta_3)\xi)\\ &\quad +S^*(\theta_1,\theta_3)\eta(W_8^*(\theta_2,\xi)\xi)-S^*(\xi,\theta_3)\eta(W_8^*(\theta_2,\theta_1)\xi)\\ &\quad +S^*(\theta_1,\xi)\eta(W_8^*(\theta_2,\theta_3)\xi)-S^*(\xi,\xi)\eta(W_8^*(\theta_2,\theta_3)\theta_1). \quad (33)\end{aligned}$$

Using (18), (29) and (30) in (33), we get

$$\begin{aligned}0 &= 4\left[g(\theta_1,\theta_3)\eta(\theta_2)-g(\theta_1,\theta_2)\eta(\theta_3)\right]\\ &\quad +\frac{1}{n-1}\left[(\lambda-\rho r^*)\,g(\theta_1,\theta_3)\eta(\theta_2)+g(\theta_1,\phi\theta_3)\eta(\theta_2)\right]. \quad (34)\end{aligned}$$

Contracting (34) over θ_1 and θ_3, we get

$$\lambda=\rho\left(r-n+1+3\psi^2\right)-\frac{1}{n}\left[4(n-1)^2+\psi\right]. \quad (35)$$

Therefore, we have the following theorem.

Theorem 5. *Let an n-dimensional LP-Sasakian manifold $M(\phi,\xi,\eta,g)$ contain a ρ-Einstein soliton (g,ξ,λ) with respect to the Zamkovoy connection. If M satisfies $S^*(\xi,\theta_1)\cdot W_8^*=0$, then the soliton is*

$$\begin{aligned} i)\ \textit{shrinking, if}\quad & r<\tfrac{1}{n\rho}\left[4(n-1)^2+\psi\right]+(n-1)-3\psi^2,\\ ii)\ \textit{steady, if}\quad & r=\tfrac{1}{n\rho}\left[4(n-1)^2+\psi\right]+(n-1)-3\psi^2,\\ iii)\ \textit{expanding, if}\quad & r>\tfrac{1}{n\rho}\left[4(n-1)^2+\psi\right]+(n-1)-3\psi^2. \end{aligned}$$

Setting $\rho=0,\frac{1}{2},\frac{1}{n}$ and $\frac{1}{2(n-1)}$ in (35), we obtain the following corollaries.

Corollary 4. *A Ricci soliton (g,ξ,λ) on $M(\phi,\xi,\eta,g)$, satisfying $W_8^*(\xi,\theta_1)\cdot S^*=0$, is always shrinking, provided trace$(\phi)=0$.*

Corollary 5. *An Einstein soliton (g,ξ,λ) on $M(\phi,\xi,\eta,g)$, satisfying $W_8^*(\xi,\theta_1)\cdot S^*=0$, is*

$$\begin{aligned} i)\ \textit{shrinking, if}\quad & r<\tfrac{1}{n}\left[(n-1)(9n-8)-\psi(3n\psi+2)\right],\\ ii)\ \textit{steady, if}\quad & r=\tfrac{1}{n}\left[(n-1)(9n-8)-\psi(3n\psi+2)\right],\\ iii)\ \textit{expanding, if}\quad & r>\tfrac{1}{n}\left[(n-1)(9n-8)-\psi(3n\psi+2)\right]. \end{aligned}$$

Corollary 6. *A traceless soliton (g,ξ,λ) on $M(\phi,\xi,\eta,g)$, satisfying $W_8^*(\xi,\theta_1)\cdot S^*=0$, is*

$$\begin{aligned} i)\ \textit{shrinking, if}\quad & r<(n-1)(4n-3)-\psi(3\psi+1),\\ ii)\ \textit{steady, if}\quad & r=(n-1)(4n-3)-\psi(3\psi+1),\\ iii)\ \textit{expanding, if}\quad & r>(n-1)(4n-3)-\psi(3\psi+1). \end{aligned}$$

Corollary 7. *A Schouten soliton (g,ξ,λ) on $M(\phi,\xi,\eta,g)$, satisfying $W_8^*(\xi,\theta_1)\cdot S^*=0$, is*

$$\begin{aligned} i)\ \textit{shrinking, if}\quad & r<\tfrac{1}{n}(n-1)\left[8(n-1)^2+n+2\psi\right]-3\psi^2,\\ ii)\ \textit{steady, if}\quad & r=\tfrac{1}{n}(n-1)\left[8(n-1)^2+n+2\psi\right]-3\psi^2,\\ iii)\ \textit{expanding, if}\quad & r>\tfrac{1}{n}(n-1)\left[8(n-1)^2+n+2\psi\right]-3\psi^2. \end{aligned}$$

7. ρ-Einstein solitons on LP-Sasakian manifolds admitting torse forming vector fields with respect to the Zamkovoy connection

Let (g,V,λ) be a ρ-Einstein soliton with respect to the Zamkovoy connection on $M(\phi,\xi,\eta,g)$, where the potential vector field V is a torse forming vector field. Then, in view of (2) and (3), we have

$$\nabla^*_{\theta_1}V=l\theta_1+\pi(\theta_1)V+g(\theta_1,\phi V)\xi-\eta(V)\phi\theta_1+\eta(\theta_1)\phi V. \tag{36}$$

Expanding L_V^* and using (36) in (14), we get

$$0 \;=\; 2lg(\theta_1,\theta_2)+g(V,\theta_2)\pi(\theta_1)+g(V,\theta_1)\pi(\theta_2)$$

$$+2S^*(\theta_1,\theta_2)+2(\lambda-\rho r^*)g(\theta_1,\theta_2). \tag{37}$$

Contracting (37) over θ_1 and θ_2, and utilizing (10), we get

$$n\lambda=(n\rho-1)\left(r-n+1+3\psi^2\right)-\pi(V)-nl. \tag{38}$$

If V is parallel, then $l=\pi=0$, and hence (38) gives

$$\lambda=\frac{1}{n}(n\rho-1)\left(r-n+1+3\psi^2\right). \tag{39}$$

For traceless soliton ($\rho=\frac{1}{n}$), (39) yields

$$\lambda=0.$$

Thus we have the following theorem.

Theorem 6. *A traceless soliton* (g,V,λ) *under the Zamkovoy connection on* $M(\phi,\xi,\eta,g)$ *with* V *being parallel is always steady.*

If V is concurrent, then $l=1,\pi=0$, and hence (38) gives

$$\lambda=\frac{1}{n}(n\rho-1)\left(r-n+1+3\psi^2\right)-1. \tag{40}$$

This implies the following theorem.

Theorem 7. *An* ρ*-Einstein soliton* (g,V,λ) *under the Zamkovoy connection on* $M(\phi,\xi,\eta,g)$ *with* V *being concurrent is*

$$\begin{aligned} i)\ \textit{shrinking, if}\quad & r<\tfrac{n}{n\rho-1}+n-3\psi^2-1,\\ ii)\ \textit{steady, if}\quad & r=\tfrac{n}{n\rho-1}+n-3\psi^2-1,\\ iii)\ \textit{expanding, if}\quad & r>\tfrac{n}{n\rho-1}+n-3\psi^2-1. \end{aligned}$$

Setting $\rho=0,\frac{1}{2},\frac{1}{n}$ and $\frac{1}{2(n-1)}$ in (40), we have the following corollaries.

Corollary 8. *A Ricci soliton* (g,V,λ) *under the Zamkovoy connection on* $M(\phi,\xi,\eta,g)$ *with* V *being concurrent is*

$$\begin{aligned} i)\ \textit{shrinking, if}\quad & r<-\left(3\psi^2+1\right),\\ ii)\ \textit{steady, if}\quad & r=-\left(3\psi^2+1\right),\\ iii)\ \textit{expanding, if}\quad & r>-\left(3\psi^2+1\right). \end{aligned}$$

Corollary 9. *An Einstein soliton* (g,V,λ) *under the Zamkovoy connection on* $M(\phi,\xi,\eta,g)$ *with* V *being concurrent is*

$$\begin{aligned} i)\ \textit{shrinking, if}\quad & r<\tfrac{n^2}{n-2}-\left(3\psi^2+1\right),\\ ii)\ \textit{steady, if}\quad & r=\tfrac{n^2}{n-2}-\left(3\psi^2+1\right),\\ iii)\ \textit{expanding, if}\quad & r>\tfrac{n^2}{n-2}-\left(3\psi^2+1\right). \end{aligned}$$

Corollary 10. *A traceless soliton* (g,V,λ) *under the Zamkovoy connection on* $M(\phi,\xi,\eta,g)$ *with* V *being concurrent is always shrinking.*

Corollary 11. *A Schouten soliton (g, V, λ) under the Zamkovoy connection on $M(\phi, \xi, \eta, g)$ with V being concurrent is*

$$\begin{aligned} i)\ \textit{shrinking, if} \quad & r < -\tfrac{2n(n-1)}{n-2} + n - \left(3\psi^2 + 1\right), \\ ii)\ \textit{steady, if} \quad & r = -\tfrac{2n(n-1)}{n-2} + n - \left(3\psi^2 + 1\right), \\ iii)\ \textit{expanding, if} \quad & r > -\tfrac{2n(n-1)}{n-2} + n - \left(3\psi^2 + 1\right). \end{aligned}$$

Again, if V is concircular, then $\pi = 0$, and hence (38) gives

$$\lambda = \frac{1}{n}(n\rho - 1)\left(r - n + 1 + 3\psi^2\right) - l. \tag{41}$$

Therefore, we get the following theorem.

Theorem 8. *A ρ-Einstein soliton (g, V, λ) under the Zamkovoy connection on $M(\phi, \xi, \eta, g)$ with V being concircular is*

$$\begin{aligned} i)\ \textit{shrinking, if} \quad & r < \tfrac{nl}{n\rho-1} + n - 3\psi^2 - 1, \\ ii)\ \textit{steady, if} \quad & r = \tfrac{nl}{n\rho-1} + n - 3\psi^2 - 1, \\ iii)\ \textit{expanding, if} \quad & r > \tfrac{nl}{n\rho-1} + n - 3\psi^2 - 1. \end{aligned}$$

Setting $\rho = 0, \frac{1}{2}, \frac{1}{n}$ and $\frac{1}{2(n-1)}$ in (41), we have the following corollaries.

Corollary 12. *A Ricci soliton (g, V, λ) under the Zamkovoy connection on $M(\phi, \xi, \eta, g)$ with V being concircular is*

$$\begin{aligned} i)\ \textit{shrinking, if} \quad & r < -\left[n(l-1) + 3\psi^2 + 1\right], \\ ii)\ \textit{steady, if} \quad & r = -\left[n(l-1) + 3\psi^2 + 1\right], \\ iii)\ \textit{expanding, if} \quad & r > -\left[n(l-1) + 3\psi^2 + 1\right]. \end{aligned}$$

Corollary 13. *An Einstein soliton (g, V, λ) under the Zamkovoy connection on $M(\phi, \xi, \eta, g)$ with V being concircular is*

$$\begin{aligned} i)\ \textit{shrinking, if} \quad & r < \tfrac{1}{n-2}\left[(n-1)(n-2) + 2nl\right] - 3\psi^2, \\ ii)\ \textit{steady, if} \quad & r = \tfrac{1}{n-2}\left[(n-1)(n-2) + 2nl\right] - 3\psi^2, \\ iii)\ \textit{expanding, if} \quad & r > \tfrac{1}{n-2}\left[(n-1)(n-2) + 2nl\right] - 3\psi^2. \end{aligned}$$

Corollary 14. *A traceless soliton (g, V, λ) under the Zamkovoy connection on $M(\phi, \xi, \eta, g)$ with V being concircular is shrinking if $l > 0$, steady if $l = 0$, or expanding if $l < 0$.*

Corollary 15. *A Schouten soliton (g, V, λ) under the Zamkovoy connection on $M(\phi, \xi, \eta, g)$ with V being concurrent is*

$$\begin{aligned} i)\ \textit{shrinking, if} \quad & r < \tfrac{n-1}{n-2}\left(n - 2nl - 2\right) - 3\psi^2, \\ ii)\ \textit{steady, if} \quad & r = \tfrac{n-1}{n-2}\left(n - 2nl - 2\right) - 3\psi^2, \\ iii)\ \textit{expanding, if} \quad & r > \tfrac{n-1}{n-2}\left(n - 2nl - 2\right) - 3\psi^2. \end{aligned}$$

8. Example of a 3-dimensional LP-Sasakian manifold admitting the Zamkovoy connection

We consider the manifold $\mathbb{R}^3$ endowed with the pseudo-Euclidean metric of signature $(+,+,-)$. Let $\{\varrho_1,\varrho_2,\varrho_3\}$ be a basis of $\chi\left(\mathbb{R}^3\right)$, given by

$$\varrho_1 = e^z\frac{\partial}{\partial x},\ \varrho_2 = e^{z-x}\frac{\partial}{\partial y},\ \varrho_3 = \frac{\partial}{\partial z}.$$

Let g be a Lorentzian metric defined by

$$\begin{aligned} g(\varrho_1,\varrho_1) &= g(\varrho_2,\varrho_2) = 1,\ g(\varrho_3,\varrho_3) = -1, \\ g(\varrho_1,\varrho_2) &= g(\varrho_1,\varrho_3) = g(\varrho_2,\varrho_3) = 0, \end{aligned}$$

and η be a 1-form defined by

$$\begin{aligned} \eta(X) &= g(X,\xi) = g(X,\varrho_3), \\ \eta(\varrho_1) &= 0,\ \eta(\varrho_2) = 0,\ \eta(\varrho_3) = -1, \end{aligned}$$

for all $X \in \chi\left(\mathbb{R}^3\right)$. Let ϕ be a $(1,1)$ tensor field given by

$$\phi(\varrho_1) = -\varrho_1,\ \phi(\varrho_2) = -\varrho_2,\ \phi(\varrho_3) = 0.$$

For $X, Y, Z \in \chi\left(\mathbb{R}^3\right)$,

$$\begin{aligned} X &= x_1\varrho_1 + x_2\varrho_2 + x_3\varrho_3, \\ Y &= y_1\varrho_1 + y_2\varrho_2 + y_3\varrho_3, \\ Z &= z_1\varrho_1 + z_2\varrho_2 + z_3\varrho_3, \end{aligned}$$

where x_i, y_i, z_i, $i = 1,2,3$ are non-negative real numbers.

Using linearity of ϕ and g we have

$$\begin{aligned} g(X,Y) &= x_1y_1 + x_2y_2 - x_3y_3, \\ \eta(X) &= g(x_1\varrho_1 + x_2\varrho_2 + x_3\varrho_3, \xi) = -x_3, \\ \phi^2 Y &= Y + \eta(Y)\xi,\ for\ \varrho_3 = \xi. \end{aligned}$$

Therefore, $\mathbb{R}^3$ is an LP-Sasakian manifold of dimension three. Also, we have

$$[\varrho_1,\varrho_2] = -\varrho^z\varrho_2,\ \ [\varrho_1,\varrho_3] = -\varrho_1,\ \ [\varrho_2,\varrho_3] = -\varrho_2.$$

Let ∇ be the Levi-Civita connection with respect to a Lorentzian metric g and R be the Riemannian curvature tensor of g. Using the Koszul formula for g, we have

$$\begin{aligned} \nabla_{\varrho_1}^{\varrho_1} &= -\varrho_3,\ \nabla_{\varrho_1}^{\varrho_2} = 0,\ \ \nabla_{\varrho_1}^{\varrho_3} = -\varrho_1, \\ \nabla_{\varrho_2}^{\varrho_1} &= \varrho^z\varrho_2,\ \nabla_{\varrho_2}^{\varrho_2} = -\varrho^z\varrho_1 - \varrho_3,\ \ \nabla_{\varrho_2}^{\varrho_3} = -\varrho_2, \\ \nabla_{\varrho_3}^{\varrho_1} &= 0,\ \nabla_{\varrho_3}^{\varrho_2} = 0,\ \ \nabla_{\varrho_3}^{\varrho_3} = 0. \end{aligned}$$

Using the above relations we can compute the non-vanishing components of R as

$$R(\varrho_2,\varrho_3)\varrho_3 = -\varrho_2, R(\varrho_1,\varrho_3)\varrho_3 = -\varrho_1, R(\varrho_1,\varrho_2)\varrho_2 = \left(1-\varrho^{2z}\right)\varrho_1,$$

$$R\left(\varrho_2,\varrho_3\right)\varrho_2 \;=\; -\varrho^z\varrho_1-\varrho_3, R\left(\varrho_1,\varrho_3\right)\varrho_1=-\varrho_3, R\left(\varrho_1,\varrho_2\right)\varrho_1=-\left(1-\varrho^{2z}\right)\varrho_2.$$

The non-vanishing components of the Ricci tensor are given by

$$S\left(\varrho_1,\varrho_1\right)=-e^{2z}, S\left(\varrho_2,\varrho_2\right)=-e^{2z}, S\left(\varrho_3,\varrho_3\right)=-2.$$

Using the above results, we have

$$\begin{aligned} g\left(X,\phi Y\right) &= -x_1y_1-x_2y_2,\\ S\left(X,Y\right) &= -x_1y_1e^{2z}-x_2y_2e^{2z}-2x_3y_3,\\ r &= -2e^{2z}-2. \end{aligned}$$

Using (3), we can easily obtain the followings equalities:

$$\begin{aligned} \nabla^{*}_{\varrho_1}{}^{\varrho_1} &= -2\varrho_3,\ \nabla^{*}_{\varrho_1}{}^{\varrho 2}=0,\ \nabla^{*}_{\varrho_1}{}^{\varrho_3}=-2\varrho_1,\\ \nabla^{*}_{\varrho_2}{}^{\varrho_1} &= e^z\varrho_2,\ \nabla^{*}_{\varrho_2}{}^{\varrho_2}=-e^z\varrho_1-2\varrho_3,\ \ \nabla^{*}_{\varrho_2}{}^{\varrho_3}=-2\varrho_2,\\ \nabla^{*}_{\varrho_3}{}^{\varrho_1} &= \varrho_1,\ \ \nabla^{*}_{\varrho_3}{}^{\varrho_2}=\varrho_2,\ \ \nabla^{*}_{\varrho_3}{}^{\varrho_3}=0. \end{aligned}$$

The non-zero components of the Riemannian curvature tensor with respect to the Zamkovoy connection are given by

$$\begin{aligned} R^*\left(\varrho_2,\varrho_3\right)\varrho_3 &= 0, R^*\left(\varrho_1,\varrho_3\right)\varrho_3=0, R^*\left(\varrho_2,\varrho_3\right)\varrho_1=\varrho_1,\\ R^*\left(\varrho_3,\varrho_2\right)\varrho_2 &= 4\varrho_3+e^z\varrho_1, R^*\left(\varrho_3,\varrho_1\right)\varrho_1=4\varrho_3,\\ R^*\left(\varrho_1,\varrho_1\right)\varrho_2 &= 0, R^*\left(\varrho_1,\varrho_1\right)\varrho_3=0, R^*\left(\varrho_1,\varrho_3\right)\varrho_2=0,\\ R^*\left(\varrho_2,\varrho_2\right)\varrho_1 &= 0, R^*\left(\varrho_2,\varrho_2\right)\varrho_3=0, R^*\left(\varrho_3,\varrho_3\right)\varrho_1=0,\\ R^*\left(\varrho_1,\varrho_2\right)\varrho_2 &= 4\varrho_1-e^{2z}\varrho_1, R^*\left(\varrho_2,\varrho_1\right)\varrho_1=4\varrho_2-e^{2z}\varrho_2. \end{aligned}$$

The components of the Ricci curvature with respect to the Zamkovoy connection are given by

$$S^*\left(\varrho_1,\varrho_1\right)=4-e^{2z}, S^*\left(\varrho_2,\varrho_2\right)=4-e^{2z},\ S^*\left(\varrho_3,\varrho_3\right)=0.$$

The scalar curvature with respect to the Zamkovoy connection is given by

$$\begin{aligned} r^* &= S^*\left(\varrho_1,\varrho_1\right)+S^*\left(\varrho_2,\varrho_2\right)+S^*\left(\varrho_3,\varrho_3\right)\\ &= -2e^{2z}+8\\ &= -2e^{2z}-2-(3-1)+3(-2)^2\\ &= r-(n-1)+3\psi^2. \end{aligned}$$

Therefore, the relation (10) is satisfied in $\mathbb{R}^3$.

We now indicate which results from the preceding sections are verified by this example. In Theorem 2, it is established that an LP-Sasakian manifold admitting a ρ-Einstein soliton (g,ξ,λ) with respect to the Zamkovoy connection must be a generalized η-Einstein manifold. One may verify using the values of $S^*(\varrho_i,\varrho_j)$ computed above that the Ricci tensor S^* is indeed expressible in the form $S^*(\theta_1,\theta_2)=k_1g(\theta_1,\theta_2)+k_2\eta(\theta_1)\eta(\theta_2)+k_3\omega(\theta_1,\theta_2)$, confirming the generalized η-Einstein character of the manifold.

In Theorem 3, under the condition $W_8^*(\xi,\theta_1)\cdot S^* = 0$, the soliton constant satisfies $\lambda = \rho(r - n + 1 + 3\psi^2)$. For the present example with $n = 3$, $r = -2e^{2z}-2$, and $\psi = \text{trace}(\phi) = -2$, this gives $\lambda = \rho(-2e^{2z}-2-2+12) = \rho(-2e^{2z}+8)$, which is consistent with the computed value of r^*.

The scalar curvature formula $r^* = r-(n-1)+3\psi^2$, established in Lemma 2 as equation (10), is directly verified by the computation above, which yields $r^* = -2e^{2z}+8$.

Similarly, the relations $R^*(\theta_1,\theta_2)\xi = 0$ and $S^*(\xi,\theta_3) = 0$, stated in (11) and (8), respectively, can be read off from the computed components of R^* and S^*.

9. Conclusion

From the results obtained in the paper, the following conclusions can be made.

- It is shown that, if M is an LP-Sasakian manifold admitting a ρ-Einstein soliton with respect to the Zamkovoy connection, then M becomes a generalized η-Einstein manifold.
- The Ricci, Einstein, traceless, and Schouten solitons are classified as shrinking, steady, or expanding according to the sign of λ.
- Under the condition $W_8^*\cdot S^* = 0$, an explicit expression for λ is obtained and the corresponding soliton classification is given.
- Under the condition $S^*\cdot W_8^* = 0$, a new relation between λ, r, and ρ is derived, and the types of solitons are characterized.
- When the potential vector field is a torse forming vector field, then the soliton constant λ is obtained in terms of the scalar curvature r, the dimension m, and the parameter ρ.
- In all these cases, the results show that the geometric behavior of the soliton depends essentially on the sign of λ.

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